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Permutahedron Triangulations via Total Linear Stability and the Dual Braid Group

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Abstract. For each finite Coxeter group W and each standard Coxeter element of W, we construct a triangulation of the W-permutahedron. For particular realizations of the W-permutahedron, we show that this is a regular triangulation induced by a height function coming from the theory of total linear stability for Dynkin quivers. We also explore the combinatorial properties of these triangulations, relating the Bruhat order, the noncrossing partition lattice, and Cambrian congruences.

Keywords: permutahedron, triangulation, braid group, Bruhat order, noncrossing partition lattice, total linear stability

1 Introduction

The complexified complements of finite real simplicial hyperplane arrangements and the complements of hyperplane arrangements for finite well-generated complex reflection groups share the remarkable property of being $K(\pi, 1)$ spaces: the fundamental group—called the *pure braid group*—is the only nontrivial homotopy group of the space.

Building on Garside's single mathematical paper [17] ("que je suis souvent de très près" [15]), Deligne proved that the complexified complement of a real finite simplicial hyperplane arrangement is a $K(\pi, 1)$ space [15].

Building on the structure of Deligne's proof [15] ("The general architecture of our proof is borrowed from Deligne's original approach, but the details are quite different" [3]), Bessis proved that the complement of the hyperplane arrangement of a finite complex reflection group is $K(\pi, 1)$ [3].

The proofs that these spaces are $K(\pi, 1)$ rely on two different combinatorial models for the hyperplane complements.

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The complexified hyperplane complement of a real simplicial arrangement has a combinatorial model called the *Salvetti complex*. In [27], by choosing a real basepoint for the fundamental group, Salvetti used the real chamber geometry of any real finite hyperplane arrangement—not just simplicial arrangements—to construct a CW complex that is a deformation retract of the complexified hyperplane complement.

The hyperplane complement for a finite well-generated complex reflection group has a combinatorial model we call the *Bessis–Brady–Watt complex*. In [3], by choosing an eigenvector for a Coxeter element as the basepoint for the fundamental group, Bessis used the noncrossing partition lattice as a replacement for the chamber geometry for real arrangements (see also Brady and Brady–Watt [9, 10, 12]).

An arrangement that lies at the intersection of simplicial arrangements and well-generated complex reflection arrangements—that is, a finite Coxeter arrangement—has *both* combinatorial models: that is, the hyperplane complement is homotopic to both the Salvetti complex and the Bessis–Brady–Watt complex.

Associated to quotients of the Salvetti and Bessis–Brady–Watt complexes are the Artin and dual presentations for the braid group of the arrangement, as we now explain. Let W be a finite Coxeter group acting in its reflection representation on a real vector space V of dimension r. Write $\mathcal{H}$ for the set of its N reflecting hyperplanes and $V^{\mathrm{reg}} := V \setminus \bigcup \mathcal{H}$ for the hyperplane complement. Choose a connected component of V^{reg} to call the *base region* $\mathbb{B}$; this induces a bijection from W to the set of connected components of V^{reg} given by $w \mapsto w\mathbb{B}$. Let $V^{\mathrm{reg}}_{\mathbb{C}}$ be the complexification of V^{reg}. The *pure braid group* of type W is the fundamental group $\mathbf{P}_W := \pi_1(V^{\mathrm{reg}})$. The *braid group* of type W is $\mathbf{B}_W := \pi_1(V^{\mathrm{reg}}_{\mathbb{C}}/W)$, the fundamental group of the quotient of $V^{\mathrm{reg}}_{\mathbb{C}}$ by the natural action of W. These fit into a natural short exact sequence

$$0 \to \mathbf{P}_W \to \mathbf{B}_W \to W \to 0.$$

The group W is generated by its set S of r *simple reflections*, which act on V via reflections through the walls of $\mathbb{B}$. The *length* $\ell_S(w)$ of an element $w \in W$ is the minimum number of simple reflections required to write w. Equivalently, $\ell_S(w)$ is the number of hyperplanes in $\mathcal{H}$ separating $\mathbb{B}$ from $w\mathbb{B}$.

The *long element* $w_\circ$ is the unique element of W with $\ell_S(w_\circ) = N$. For $\mathfrak{y} \in \mathbb{B}$, the *$W$-permutahedron* $\mathrm{Perm}_{\mathfrak{y}} = \mathrm{Perm}_{\mathfrak{y}}(W) \subseteq V$ is the polytope defined as the convex hull of the W-orbit $W\mathfrak{y}$. The 1-skeleton of $\mathrm{Perm}_{\mathfrak{y}}$

The group W is generated by its set T of N *reflections*, which act on V via reflections through the hyperplanes in $\mathcal{H}$. The *absolute length* $\ell_T(w)$ of an element $w \in W$ is the minimum number of reflections required to write w. Equivalently (as W is real), $\ell_T(w)$ is the dimension of the image of $w - 1$.

A *standard Coxeter element* is an element c of W obtained as a product of the simple reflections in some order. We have $\ell_T(c) = r = \ell_S(c)$. The order of c is denoted h. The *c-noncrossing partition lattice* is the in-is the *weak order*, the interval $\mathsf{Weak}(W) := [e, w_\circ]_S$ between the identity e and the long element in the right Cayley graph of W generated by S (with edges labeled by simple reflections).

Write $\mathbf{S} := \{\mathbf{s} : s \in S\}$ for a formal copy of the set S of simple reflections. The *Artin presentation* of the braid group is

$$\mathbf{B}_{W,w_\circ} = \langle \mathbf{S} : [\mathrm{Red}_S(w_\circ)] \rangle,$$

where $[\mathrm{Red}_S(w_\circ)]$ stands for the relations setting equal all reduced S-words for $w_\circ$ (replacing each $s \in S$ by the $\mathbf{s} \in \mathbf{S}$).

terval $\mathsf{NC}(W, c) := [e, c]_T$ between the identity e and the Coxeter element c in the right Cayley graph of W generated by T (with edges labeled by reflections). Elements of $\mathsf{NC}(W, c)$ are called *c-noncrossing partitions*.

Write $\mathbf{T}_c := \{\mathbf{t}_c : t \in T\}$ for a formal copy of the set T of reflections. Bessis's *dual presentation* of the braid group is

$$\mathbf{B}_{W,c} = \langle \mathbf{T}_c : [\mathrm{Red}_T(c)] \rangle,$$

where $[\mathrm{Red}_T(c)]$ stands for the relations setting equal all reduced T-words for c (replacing each $t \in T$ by the $\mathbf{t}_c \in \mathbf{T}_c$).

These two different presentations for the braid group come from the two different combinatorial models for the complexified hyperplane complement. In more detail:

For a finite Coxeter group W, the Salvetti complex $\mathsf{Sal}(W)$ can be taken to be invariant with respect to the action of the group, giving a homotopy equivalence between the quotient complex and the orbit space [28].

The Artin presentation for $\mathbf{B}_W$ is associated to the *quotient Salvetti complex*: informally, this complex is formed by gluing together oriented copies of $\mathrm{Perm}_{\mathfrak{y}}(W)$—one for each element of W—and then quotienting by the action of W. The edges of this complex can be viewed as oriented edges of the W-permutahedron, so they are labeled by simple reflections, while relations come from higher-dimensional faces of the permutahedron.

For a Coxeter element c of W, the Bessis–Brady–Watt complex $\mathsf{BBW}(W, c)$ is a simplicial complex built from the noncrossing partition lattice [11, Examples 2.2 & 2.3]; it carries an action of W.

Bessis's dual presentation for $\mathbf{B}_W$ is associated to the *quotient Bessis–Brady–Watt complex*: informally, this complex is formed by gluing together oriented copies of the order complex of the noncrossing partition lattice $\mathsf{NC}(W, c)$—one for each element of W—and then quotienting by the action of W. The edges of this complex can be viewed as oriented edges in the noncrossing partition lattice, while relations come from longer chains.

Our triangulation of the permutahedron (depending on a Coxeter element c) subdivides $\mathsf{Sal}(W)$ into simplices used in the Bessis–Brady–Watt complex.

Remark 1. Since one can identify $\mathbf{S}$ as a subset of $\mathbf{T}_c$, it is reasonable to restrict the Cayley graph of the *dual braid monoid* to the image of the *positive Artin monoid*. After drawing some highly suggestive low-rank pictures, the third author was led to search the internet for the term "triangulation of the permutahedron" in the summer of 2021. This search yielded some excellent slides by Brady, Delucchi, and Watt [8] that appeared to have the same construction as these low-rank drawings and a similar construction to the one we present here for bipartite c. (In this bipartite case, the height function is particularly nice to describe.) After some number of years, it was suggested to us by several parties that we write up our version—this is the second step after [14] to proving certain conjectures of the third author giving explicit, elegant, type-uniform presentations for the pure braid group of W, relying on the interaction of Salvetti's presentation with noncrossing Coxeter–Catalan combinatorics.

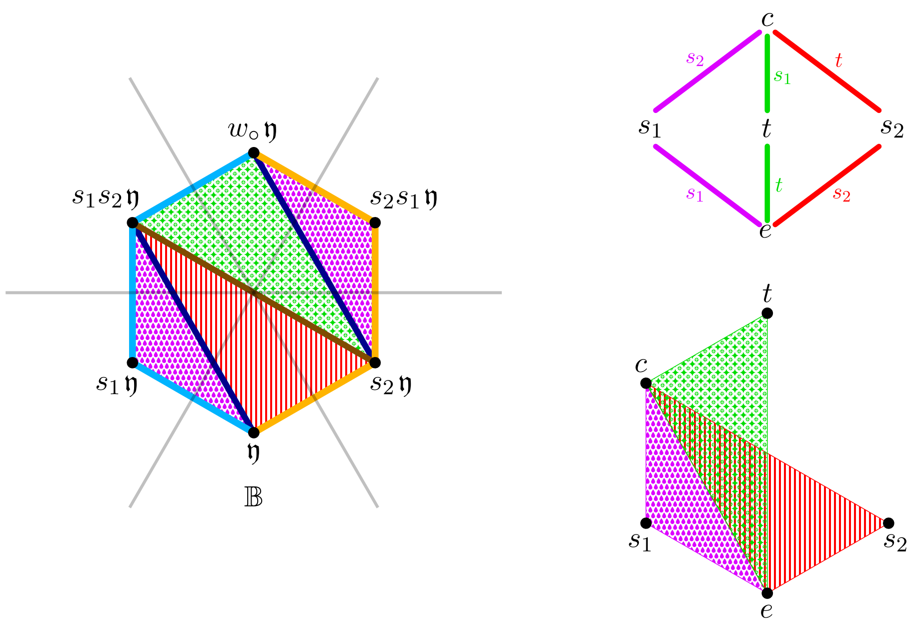

Figure 1: On the top right is the c-noncrossing partition lattice of $\mathfrak{S}_3$, where $c = s_1s_2$. On the bottom right is its order complex. On the left is the permutahedron $\mathrm{Perm}_{\mathfrak{y}}(\mathfrak{S}_3)$, triangulated by $\mathsf{SBDW}_{\mathfrak{y}}(\mathfrak{S}_3, c)$. Each simplex $\mathrm{conv}(w\vec{\pi}\mathfrak{y})$ is drawn on the left in the same color as the edges in the chain $\vec{\pi}$ on the right. There are two ways of walking from $\mathfrak{y}$ up to $w_\circ\mathfrak{y}$ along the 1-skeleton of $\mathrm{Perm}_{\mathfrak{y}}$; these paths are colored differently.

2 Triangulations of the permutahedron and linear stability

Recall that a *subdivision* of a polytope P is a collection Θ of polytopes of the same dimension as P such that $\bigcup\Theta = P$ and such that for all $P_1, P_2 \in \Theta$, the intersection $P_1 \cap P_2$ is a (possibly empty) common face of P_1 and P_2. A subdivision Θ is a *triangulation* if all polytopes in Θ are simplices.

Fix a standard Coxeter element c of W and a point $\mathfrak{y} \in \mathbb{B}$. The *Bruhat order* is the partial order $\leq_{\mathrm{B}}$ on W with a cover relation $u \lessdot_{\mathrm{B}} v$ if and only if $\ell_S(v) = \ell_S(u) + 1$ and

$v = ut$ for some $t \in T$. We will construct a triangulation of $\mathrm{Perm}_{\mathfrak{y}}$ by leveraging an interplay between the noncrossing partition lattice $\mathsf{NC}(W, c)$ and the Bruhat order. Let

$$W_c^+ := \{u \in W : \ell_S(uc) = \ell_S(u) + \ell_S(c)\}. \tag{2.1}$$

(One can show that W_c^+ is the set of elements less than or equal to $w_\circ c^{-1}$ in left weak order.) For each $u \in W_c^+$, the Bruhat interval $[u, uc]_{\mathrm{B}}$—consisting of those elements between u and uc in the Bruhat order—is necessarily isomorphic to a subposet of $\mathsf{NC}(W, c)$ under the map $x \mapsto u^{-1}x$.

A maximal chain $\{u = u_0 \lessdot_{\mathrm{B}} u_1 \lessdot_{\mathrm{B}} \cdots \lessdot_{\mathrm{B}} u_r = uc\}$ in the interval $[u, uc]_{\mathrm{B}}$ now defines a polytope $\mathrm{conv}\big(\{u_0\mathfrak{y}, u_1\mathfrak{y}, \ldots, u_r\mathfrak{y}\}\big)$, where $\mathrm{conv}(\cdot)$ denotes convex hull. This polytope is in fact an r-dimensional simplex.

Definition 1. Fix a standard Coxeter element c of W and a point $\mathfrak{y} \in \mathbb{B}$. We define the *Salvetti–Brady–Delucchi–Watt triangulation* (or *SBDW triangulation*) $\mathsf{SBDW}_{\mathfrak{y}}(W, c)$ to be the set of simplices in $\mathrm{Perm}_{\mathfrak{y}}(W)$ of the form $\mathrm{conv}(\{u_0\mathfrak{y}, u_1\mathfrak{y}, \ldots, u_r\mathfrak{y}\})$, where $u \in W_c^+$ and $\{u = u_0 \lessdot_{\mathrm{B}} u_1 \lessdot_{\mathrm{B}} \cdots \lessdot_{\mathrm{B}} u_r = uc\}$ is a maximal chain of the Bruhat interval $[u, uc]_{\mathrm{B}}$.

Example 1. Figure 1 illustrates Definition 1 for the symmetric group $\mathfrak{S}_3$, with Coxeter element $c = s_1s_2$ (where $s_i = (i\ i+1)$).

It is not obvious that $\mathsf{SBDW}_{\mathfrak{y}}(W, c)$ is a triangulation of the W-permutahedron $\mathrm{Perm}_{\mathfrak{y}}$. Indeed, it is not clear that the union of the simplices is the entire W-permutahedron, nor that two simplices do not intersect in their interior, nor that intersections occur along common faces. Remarkably, these simplices coming from the interaction of $\mathsf{NC}(W, c)$ and the Bruhat order do form a triangulation of $\mathrm{Perm}_{\mathfrak{y}}$—to steal a phrase from Bessis, the noncrossing partition lattice is "somehow real" [4].

Theorem 1. *Let W be a finite Coxeter group. For each standard Coxeter element c of W and each point $\mathfrak{y} \in \mathbb{B}$, $\mathsf{SBDW}_{\mathfrak{y}}(W, c)$ is a triangulation of the W-permutahedron $\mathrm{Perm}_{\mathfrak{y}}(W)$.*

To prove Theorem 1, we first assume that W is simply-laced (i.e., of type A, D, or E). Under this assumption, we provide a type-uniform argument that relies on recent developments in quiver representation theory. Specifically, we invoke *totally stable linear stability functions*, the existence of which was established in [1, 18, 21] in type A and in [13] in general. Geometrically, specifying a totally stable linear stability function amounts to finding a point $\mathfrak{y}^\star \in \mathbb{B}$ and a direction γ so that shooting a laser from $\mathfrak{y}^\star$ in the direction of γ crosses a certain piece of each hyperplane in the arrangement $\mathcal{H}$ in a certain order (more precisely, the laser crosses the *c-noncrossing shards* in the order specified by the inversion sequence of the c-sorting word for $w_\circ$). The time at which the hyperplane piece is crossed gives a height function, and the existence of such a function allows us to show that $\mathsf{SBDW}_{\mathfrak{y}^\star}(W, c)$ is a *regular* triangulation of $\mathrm{Perm}_{\mathfrak{y}^\star}$ (for the particular $\mathfrak{y}^\star \in \mathbb{B}$ used in the stability function)—we refer the reader to an interactive visualization [31].

We also establish a result that allows us to deform $\mathrm{Perm}_{\mathfrak{y}^\star}$ into $\mathrm{Perm}_{\mathfrak{y}}$ for an arbitrary choice of $\mathfrak{y}$, thereby proving Theorem 1 in general for simply-laced Coxeter groups. We use *folding* to deduce the theorem for the remaining finite Coxeter groups.

3 Combinatorics of the SBDW triangulation

While our height function gives us a coarse global understanding of the SBDW triangulation, we actually have very fine combinatorial control of the dual graph (see [30] for an interactive animation, and Example 2 and Figure 2 for a static visualization for $\mathfrak{S}_4$).

To analyze the combinatorics of the SBDW triangulation, it is helpful to use the following alternative description. Let $\mathsf{MC}(W,c)$ denote the set of maximal chains in the c-noncrossing partition lattice $\mathsf{NC}(W,c)$. There is a straightforward bijection from $\mathsf{MC}(W,c)$ to the set $\mathrm{Red}_T(c)$ of reduced T-words for c that sends the maximal chain $\vec{\pi} = \{\pi_0 \lessdot_T \pi_1 \lessdot_T \cdots \lessdot_T \pi_r\}$ to the word $\mathrm{rw}(\vec{\pi}) = t_1 \cdots t_r$, where $t_i = \pi_{i-1}^{-1}\pi_i$.

Definition 2. We say that a chain $\vec{\pi} = \{\pi_0 \lessdot_T \pi_1 \lessdot_T \cdots \lessdot_T \pi_r\} \in \mathsf{MC}(W,c)$ and an element $u \in W_c^+$ are *concordant* if $u\pi_0 \leq_{\mathrm{B}} u\pi_1 \leq_{\mathrm{B}} \cdots \leq_{\mathrm{B}} u\pi_r$. Let $\Omega(W,c)$ denote the set of pairs $(u,\vec{\pi}) \in W_c^+ \times \mathsf{MC}(W,c)$ such that $\vec{\pi}$ and u are concordant.

In this language, we have $\mathsf{SBDW}_{\mathfrak{y}}(W,c) = \{\mathrm{conv}(u\vec{\pi}\mathfrak{y}) : (u,\vec{\pi}) \in \Omega(W,c)\}$, where $\mathrm{conv}(u\vec{\pi}\mathfrak{y}) = \mathrm{conv}(\{u\pi\mathfrak{y} : \pi \in \vec{\pi}\}) \subseteq \mathrm{Perm}_{\mathfrak{y}}$.

We say two chains $\vec{\pi}, \vec{\pi}' \in \mathsf{MC}(W,c)$ are *commutation equivalent* if their corresponding reduced T-words $\mathrm{rw}(\vec{\pi})$ and $\mathrm{rw}(\vec{\pi}')$ are related by commutation moves (equivalently, these words use the same set of reflections). Write CC_c for the set of commutation classes of maximal chains in $\mathsf{MC}(W,c)$, and let $\mathcal{C}_{\vec{\pi}} \in \mathsf{CC}_c$ denote the commutation equivalence class of $\vec{\pi}$. For $u \in W_c^+$, let $\mathsf{CC}_c(u)$ denote the set of commutation equivalence classes of chains in $\mathsf{MC}(W,c)$ that are concordant with u. We say u and $\mathcal{C}$ are concordant if $\mathcal{C} \in \mathsf{CC}_c(u)$.

There is a partial order $\preceq_c$ on T coming from the heap of the c-sorting word for the long element $w_\circ$. We say a chain $\vec{\pi} \in \mathsf{MC}(W,c)$ with $\mathrm{rw}(\vec{\pi}) = t_1 \cdots t_r$ is *$\preceq_c$-increasing* (respectively, *$\preceq_c$-decreasing*) if there do not exist $1 \leq i < j \leq r$ such that $t_j \preceq_c t_i$ (respectively, $t_i \preceq_c t_j$). We say the class $\mathcal{C}_{\vec{\pi}}$ is *$\preceq_c$-increasing* (respectively, *$\preceq_c$-decreasing*) if $\vec{\pi}$ is (this is well defined). It is known that there is a unique $\preceq_c$-increasing commutation class, which we denote by $\mathcal{I}_c$. When W is irreducible, it is well known that the number of $\preceq_c$-decreasing commutation classes is the *positive W-Catalan number*

$$\mathsf{Cat}^+(W) := \prod_{i=1}^{r} \frac{h-1+e_i}{d_i}, \tag{3.1}$$

where h is the Coxeter number (the order of c), $d_1, \ldots, d_r$ are the degrees, and $e_1, \ldots, e_r$ are the exponents. In particular, the number of such classes is independent of c.

Theorem 2. *For each $u \in W_c^+$, the set $\mathsf{CC}_c(u)$ contains the $\preceq_c$-increasing class $\mathcal{I}_c$ and contains a unique $\preceq_c$-decreasing class, denoted $\mathcal{D}_c(u)$.*

Say two Bruhat intervals $[u_1, v_1]_{\mathrm{B}}$ and $[u_2, v_2]_{\mathrm{B}}$ are *reflection isomorphic* if there is a poset isomorphism $\varphi\colon [u_1, v_1]_{\mathrm{B}} \to [u_2, v_2]_{\mathrm{B}}$ such that for every cover relation $x \lessdot_{\mathrm{B}} y$ in $[u_1, v_1]_{\mathrm{B}}$, we have $x^{-1}y = \varphi(x)^{-1}\varphi(y)$. If $u_1, u_2 \in W_c^+$, then $[u_1, u_1c]_{\mathrm{B}}$ and $[u_2, u_2c]_{\mathrm{B}}$ are reflection isomorphic if and only if $\mathsf{CC}_c(u_1) = \mathsf{CC}_c(u_2)$. The next theorem uses the c^{-1}*-Cambrian congruence*, a lattice congruence of the right weak order of W due to Reading [25, 26].

Theorem 3. *Let $u_1, u_2 \in W_c^+$. The following are equivalent: $\mathcal{D}_c(u_1) = \mathcal{D}_c(u_2)$; $\mathsf{CC}_c(u_1) = \mathsf{CC}_c(u_2)$; $[u_1, u_1c]_{\mathrm{B}}$ and $[u_2, u_2c]_{\mathrm{B}}$ are reflection isomorphic; and u_1^{-1} and u_2^{-1} belong to the same c^{-1}-Cambrian class.*

Although the number of simplices in the SBDW triangulation and the number of elements in W_c^+ both depend on the choice of c, we have the following corollary.

Corollary 1. *When W is irreducible, the number of reflection isomorphism classes of Bruhat intervals of the form $[u, uc]_{\mathrm{B}}$ with $u \in W_c^+$ is the positive W-Catalan number $\mathsf{Cat}^+(W)$. In particular, the number of these classes is independent of c.*

We now use the positive cluster complex to interpret W_c^+ and $\mathsf{MC}(W, c)$ in a common geometric setting. The positive c^{-1}-cluster complex $\mathrm{Clus}^+(W, c^{-1})$ is a certain flag simplicial complex on the vertex set $[N]$. Let $\mathrm{MClus}^+(W, c^{-1})$ be the set of maximal simplices in $\mathrm{Clus}^+(W, c^{-1})$; these maximal simplices are naturally in bijection with the $\preceq_c$-decreasing commutation classes of chains in $\mathsf{MC}(W, c)$. In particular, $|\mathrm{MClus}^+(W, c^{-1})| = \mathsf{Cat}^+(W)$. For $A \in \mathrm{MClus}^+(W, c^{-1})$, write $W_c^+(A)$ for the collection of elements $u \in W_c^+$ such that $\mathcal{D}_c(u)$ is the $\preceq_c$-decreasing class corresponding to A (so that $W_c^+(A)$ is the c^{-1}-Cambrian congruence class containing u^{-1}).

Each $t \in T$ acts on V by reflecting through some hyperplane $H_t \in \mathcal{H}$; let H_t^- be the closed half-space determined by H_t that contains the region $\mathbb{B}$. Let $\beta_t^\vee \in V^*$ be the coroot of W that is normal to H_t and lies in H_t^-, and write $(\Phi^+)^\vee = \{\beta_t^\vee : t \in T\}$. Let $T_{\mathrm{L}}(c) = \{t \in T : \ell_S(tc) < \ell_S(c)\}$ denote the set of left inversions of c. Our constructions will live in the simplicial cone given by any of the following equivalent definitions:

$$\Delta_c^+ := \bigcap_{t \in T_{\mathrm{L}}(c)} H_t^- = \bigcup_{u \in W_c^+} \overline{u^{-1}\mathbb{B}} = (1 - c^{-1})^{-1}\mathrm{span}_{\geq 0}(\Phi^+)^\vee \subseteq V.$$

(Here, $\mathrm{span}_{\geq 0}$ denotes nonnegative span.) For each $t \in T$, there is a distinguished point $\delta_t = (1 - c^{-1})^{-1}\beta_t^\vee$ inside Δ_c^+, and $\Delta_c^+ = \mathrm{span}_{\geq 0}\{\delta_s : s \in S\}$. We will always draw Δ_c^+ as a simplex by deleting the origin and passing to the positive projectivization (identifying points that have the same positive span). We sometimes refer to the points δ_t for $t \in T$ as *vertices* since they naturally correspond to vertices of the positive c^{-1}-cluster complex.

The next table summarizes our geometric interpretations of W_c^+, $\mathrm{MClus}^+(W,c^{-1})$, $\mathsf{MC}(W,c)$, and CC_c by describing maps (by abuse of notation, each denoted Δ) from each object to a simplicial cone inside of Δ_c^+.

Set	Element	Description	Image under Δ
W_c^+	u	$\ell_S(uc) = \ell_S(u) + r$	$\Delta(u) := \overline{u^{-1}\mathbb{B}}$
$\mathrm{MClus}^+(W,c^{-1})$	A	positive c^{-1}-cluster, or $\preceq_c$-decreasing T-word for c	$\Delta(A) := \bigcup_{u\in W_c^+(A)} \Delta(u)$
$\mathsf{MC}(W,c)$	$\vec{\pi}$	maximal chain in $\mathsf{NC}(W,c)$	
$\mathrm{Red}_T(c)$	$\mathrm{rw}(\vec{\pi}) = t_1\cdots t_r$	reduced T-word for c	$\Delta(\vec{\pi}) := \mathrm{span}_{\geq 0}\{\delta_{t_1},\ldots,\delta_{t_r}\}$
CC_c	$\mathcal{C}_{\vec{\pi}}$	commutation class of $\vec{\pi}$	

We note the following fundamental facts about these maps Δ:

- For $u \in W_c^+$, $A \in \mathrm{MClus}^+(W,c^{-1})$, and $\vec{\pi} \in \mathsf{MC}(W,c)$, each of the cones $\Delta(u)$, $\Delta(A)$, and $\Delta(\vec{\pi})$ is r-dimensional; it is the nonnegative span of some r-subset (determined from u, A, or $\vec{\pi}$) of the set $\{\delta_t : t \in T\}$.
- The images under Δ of both W_c^+ and $\mathrm{MClus}^+(W,c^{-1})$ triangulate Δ_c^+, and the image of W_c^+ refines the image of $\mathrm{MClus}^+(W,c^{-1})$ (specifically, $\Delta(u) \subseteq \Delta(A)$ if u^{-1} belongs to the c^{-1}-Cambrian class $W_c^+(A)$).
- For each $\vec{\pi} \in \mathsf{MC}(W,c)$, the cone $\Delta(\vec{\pi})$ is the union of some of the cones in the image of $\mathrm{MClus}^+(W,c^{-1})$ under Δ.

We use the preceding geometric interpretations to characterize concordancy.

Theorem 4. *An element $u \in W_c^+$ and a chain $\vec{\pi} \in \mathsf{MC}(W,c)$ are concordant if and only if $\Delta(u) \subseteq \Delta(\vec{\pi})$.*

Theorem 4 characterizes the maximal chains appearing in $[u,uc]_{\mathrm{B}}$ (for $u \in W_c^+$), so it specifies the subposet of the $\mathsf{NC}(W,c)$ isomorphic to this interval.

Example 2. Let $W = A_3 = \mathfrak{S}_4$, and let $c = s_1s_2s_3$ (where $s_i = (i\ i+1)$). We have $W_c^+ = \{e, s_2, s_3, s_2s_3, s_3s_2, s_2s_3s_2\}$. Figure 2 is a visualization of the adjacency structure of our triangulation. It shows:

- a **huge** triangle, representing the cone Δ_c^+, subdivided into
- six **large** triangles, representing the cones $\Delta(u)$ for elements $u \in W_c^+$ (the exterior tip of $\Delta(u)$ is labeled by u^{-1}). Each **large** triangle contains

- some medium triangles, each representing another copy of Δ_c^+, each subdivided into
- five small triangles, representing the cones $\Delta(A)$ for $A \in \mathrm{MClus}^+(W, c^{-1})$.

Some small triangles in each medium triangle are colored. A medium triangle and the coloring of its small triangles together represent a concordant pair $(u, \mathcal{C})$; equivalently, this pair represents the collection $\{u\vec{\pi}\mathfrak{y} : \vec{\pi} \in \mathcal{C}\}$ of commutation equivalent SBDW simplices with the same base point. The element u in the pair $(u, \mathcal{C})$ is recorded in two ways: the medium triangle is contained in the large triangle indexed by u^{-1}, and $\Delta(u)$ is indicated in the medium triangle by a white star.

Two medium triangles M_1 and M_2 corresponding to concordant pairs $(u_1, \mathcal{C}_1)$ and $(u_2, \mathcal{C}_2)$ are connected by an edge if and only if some SBDW simplex in the collection $\{u_1\vec{\pi}\mathfrak{y} : \vec{\pi} \in \mathcal{C}_1\}$ shares a facet with an SBDW simplex in the collection $\{u_2\vec{\pi}\mathfrak{y} : \vec{\pi} \in \mathcal{C}_2\}$.

Figure 2 also illustrates Theorem 3 as follows. Let $w = s_2$ and $u = s_3 s_2$. The large triangles labeled by w^{-1} and u^{-1} contain the same medium triangles (ignoring the white star). This indicates that the intervals $[w, wc]_{\mathrm{B}}$ and $[u, uc]_{\mathrm{B}}$ are reflection isomorphic, which corresponds to the fact that u^{-1} and w^{-1} lie in the same c^{-1}-Cambrian class. This, in turn, is encoded by the fact that the white stars marking $\Delta(w)$ and $\Delta(u)$ both lie in the blue small triangle.

Each SBDW simplex $u\vec{\pi}\mathfrak{y}$ is labeled by the chain $\vec{\pi} \in \mathsf{MC}(W, c)$, but $\vec{\pi}$ may occur as the label of multiple simplices (for example, those chains whose corresponding reduced T-words use only simple reflections label a simplex for *every* $u \in W_c^+$). We prove that the set $\{u \in W_c^+ : (u, \vec{\pi}) \in \Omega(W, c)\}$ of elements concordant with a particular $\vec{\pi}$ is a nonempty interval in the left weak order, so this set has a lowest element $\omega_\downarrow(\vec{\pi})$.

With this in mind, it is natural to restrict our triangulation to the simplices corresponding to pairs of the form $(\omega_\downarrow(\vec{\pi}), \vec{\pi})$. For this subset of simplices, each $\vec{\pi}$ is the label of exactly one simplex. We prove that the vertices of the permutahedron used by this subset of simplices correspond exactly to the c^{-1}-sortable elements of W.

4 Past and present work

Let us briefly survey some related previous and ongoing work. Research on *Bruhat interval polytopes*, their subdivisions, and connections to the *permutahedral variety* has been conducted by Kodama and Williams [23], Tsukerman and Williams [29], and Knutson, Sanchez, and Sherman-Bennett [22], with further connections to the tropical positive flag variety explored by others [6, 20, 7]. In a related vein, the study of the quasisymmetric flag variety by Bergeron, Gagnon, Nadeau, Spink, and Tewari [2, 24] links toric subvarieties to the combinatorics of noncrossing partitions. Additionally, the topic of weighted chains

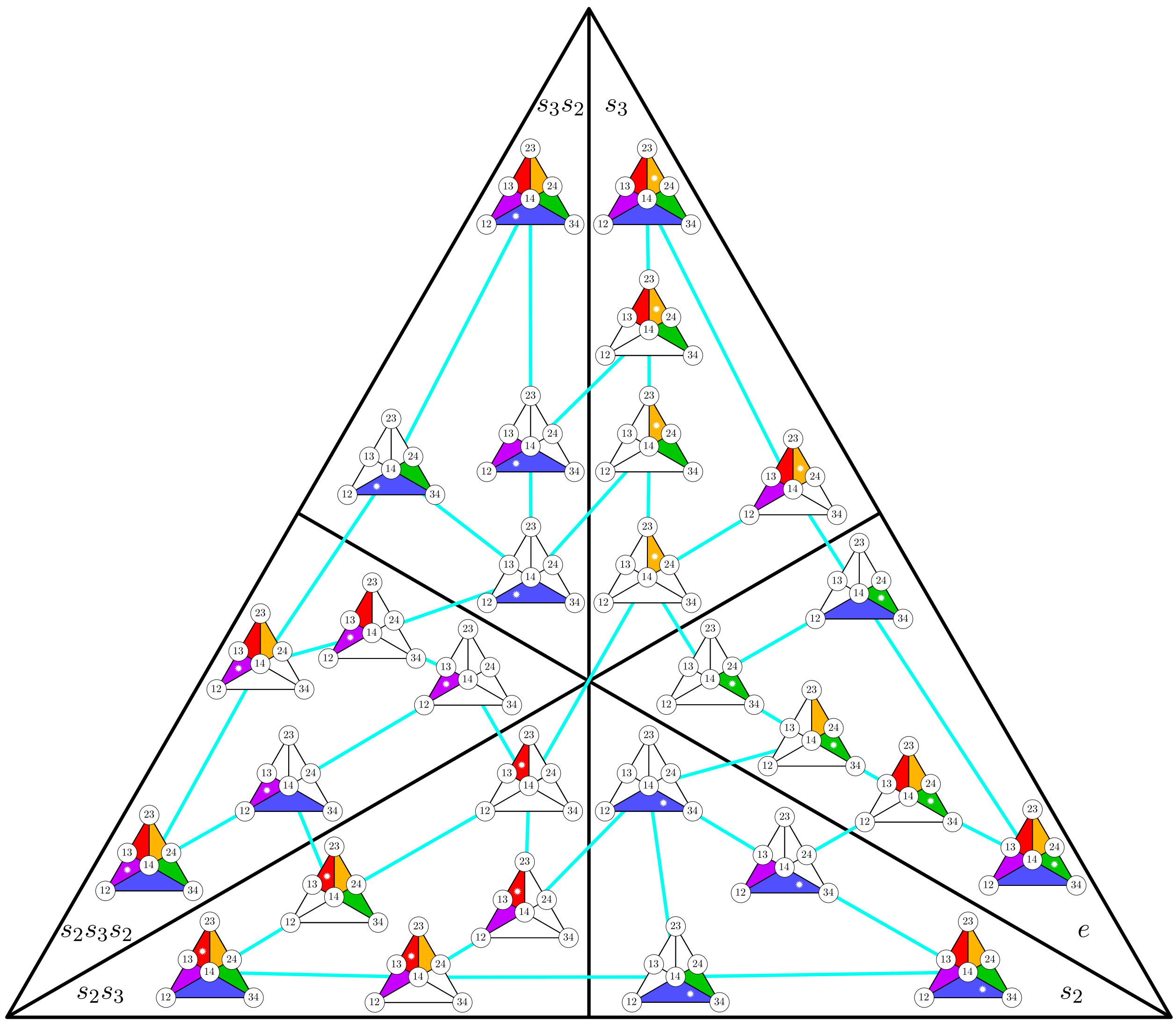

Figure 2: A representation of the pairs $(u, \mathcal{C})$ with $\mathcal{C} \in \mathsf{CC}_c(u)$, where $W = A_3 = \mathfrak{S}_4$ and $c = s_1s_2s_3$. Edges represent adjacencies between corresponding collections of simplices in the SBDW triangulation.

in the noncrossing partition lattice, which merges absolute order with Bruhat order, has been developed by Josuat-Vergès, Biane, and Douvropoulos [5, 16, 19].

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