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Changing Bases with Pipe Dream Combinatorics

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Abstract. Lascoux and Schützenberger introduced Schubert and Grothendieck polynomials to study the cohomology and K-theory of the complete flag variety. We present explicit combinatorial rules for expressing Grothendieck polynomials in the basis of Schubert polynomials, and vice versa, using the bumpless pipe dreams (BPDs) of Lam, Lee, and Shimozono. A key advantage of BPDs is that they are naturally back stable, which allows us to give a combinatorial formula for expanding back stable Grothendieck polynomials in terms of back stable Schubert polynomials. We also provide pipe dream interpretations for the rules originally given by Lenart (Grothendieck to Schubert) and Lascoux (Schubert to Grothendieck), which were previously formulated in terms of binary triangular arrays. We give new proofs of these results, relying on Knutson's co-transition recurrences. The key connection between the pipe dream and BPD change of basis formulas is the canonical bijection of Gao and Huang. We show that co-permutations are preserved by this map.

Keywords: Schubert polynomials, Grothendieck polynomials, bumpless pipe dreams, pipe dreams, Schubert calculus

1 Introduction

The ***complete flag variety*** $\mathsf{Flags}(n)$ is the space of nested sequences of vector subspaces of $\mathbb{C}^n$ of the form $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$, where $\dim(V_i) = i$ for all i. The flag variety has distinguished subvarieties called *Schubert varieties*, which are indexed by permutations in the *symmetric group* S_n. Each Schubert variety determines a class $\sigma_w \in \mathrm{H}^*(\mathsf{Flags}(n))$ in the cohomology ring of $\mathsf{Flags}(n)$. These *Schubert classes* form a linear basis for $\mathrm{H}^*(\mathsf{Flags}(n))$. A central problem in Schubert calculus is to find a combinatorial rule for the structure constants $c_{u,v}^w$ in the product $\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$. The Borel isomorphism identifies $\mathrm{H}^*(\mathsf{Flags}(n))$ with $\mathbb{Z}[x_1, \ldots, x_n]/I$, where I is the ideal generated by the nonconstant elementary symmetric polynomials. Lascoux and Schützenberger [26] introduced *Schubert polynomials* $\mathfrak{S}_w$, which are representatives for the Schubert classes. There is an analogous story in K-theory. Here, *Grothendieck polynomials* $\mathfrak{G}_w$ serve as representatives for the classes of structure sheaves in the K-theory of $\mathsf{Flags}(n)$ [27].

In this article, we present combinatorial formulas for expanding Grothendieck polynomials in the basis of Schubert polynomials, and vice versa[1]. We give formulas for

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[1]This extended abstract summarizes [37]. We refer the reader there for further details.

changing bases between Schubert and Grothendieck polynomials as sums over the *bumpless pipe dreams (BPDs)* of Lam, Lee, and Shimozono [22]. We also translate the change of basis formulas of Lenart [29] and Lascoux [25], originally stated in terms of binary triangular arrays, into sums over *pipe dreams*. Pipe dreams and BPDs are certain tilings of the $n \times n$ grid, both of which are used in formulas for computing the monomial expansions of Schubert and Grothendieck polynomials [3, 1, 10, 8, 9, 22, 36]. BPDs were originally developed to study back stable Schubert calculus [22, 23]. Since then, BPDs have become an important tool in Schubert calculus, see, e.g., [4, 13, 14, 16, 28].

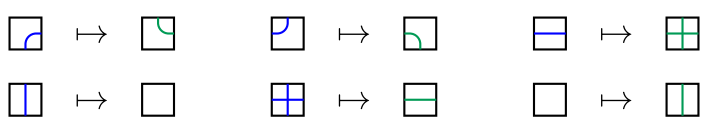
Certain properties of Schubert and Grothendieck polynomials appear more transparently in terms of pipe dreams or bumpless pipe dreams. For instance, the transition recurrence on Schubert and Grothendieck polynomials has a simple bijective explanation in terms of BPDs (see [24, 36]), whereas the co-transition recurrence of Knutson [19] is compatible with pipe dreams. Similarly, pipe dreams index components in antidiagonal Gröbner degenerations of *matrix Schubert varieties* [20], while BPDs govern certain diagonal degenerations [21, 12, 17, 18]. We show that the BPD and pipe dream change of basis formulas are closely connected via the canonical bijection from pipe dreams to bumpless pipe dreams of Gao and Huang [11].

2 Changing bases with bumpless pipe dreams

A ***bumpless pipe dream (BPD)*** of size n is a tiling of the $n \times n$ grid using the tiles

(2.1)

so that we get a network of n pipes, with each pipe starting at the bottom edge of the grid and ending at the right. A ***co-BPD*** is an upside down BPD. To each BPD, we associate a co-BPD $\check{\mathcal{B}}$ by making the tile-by-tile replacements pictured below.



Remark 2.1. *BPDs are in direct bijection with states of the six-vertex model. As explained in [36], one can obtain a BPD $\mathcal{B}$ from a state of the six-vertex model by drawing pipe segments corresponding to the arrows that point left and down. To obtain the corresponding co-BPD $\check{\mathcal{B}}$, instead select the arrows that point left and up.*

We assign a permutation to each BPD (or co-BPD) by following the pipes and ignoring crossings whenever two pipes have previously crossed. If $\mathcal{B}$ is a BPD (or co-BPD) write $\delta(\mathcal{B})$ for its associated permutation. We call a BPD (or co-BPD) ***reduced*** if each pair of pipes crosses at most one time. For each permutation w, we denote the set of associated BPDs by $\overline{\mathsf{BPD}}(w)$, and write $\mathsf{BPD}(w)$ for the subset of reduced BPDs.

Our first main theorem states that we may expand the Grothendieck polynomial $\mathfrak{G}_w$ as a signed sum over Schubert polynomials, indexed by permutations associated to the reduced co-BPDs arising from BPDs of w.

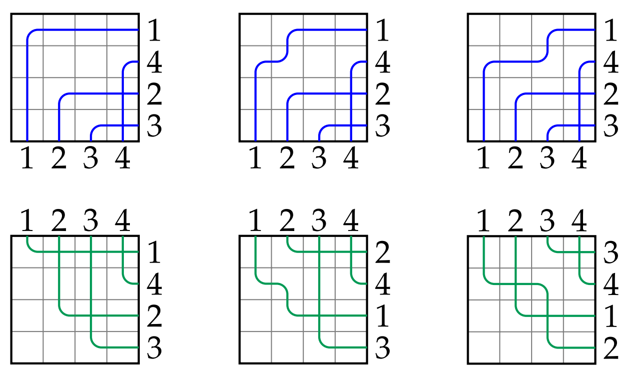
Theorem 2.2. *Given $w \in S_n$, we have* $\mathfrak{G}_w = \sum_{\substack{\mathcal{B} \in \overline{\mathsf{BPD}}(w) \\ \check{\mathcal{B}} \text{ is reduced}}} (-1)^{\ell(\delta(\check{\mathcal{B}})) - \ell(w)} \mathfrak{S}_{\delta(\check{\mathcal{B}})}$.

We also show that each Schubert polynomial $\mathfrak{S}_w$ expands as a positive sum of Grothendieck polynomials, indexed by permutations arising from the co-BPDs associated to the reduced BPDs of w.

Theorem 2.3. *Given $w \in S_n$, we have* $\mathfrak{S}_w = \sum_{\mathcal{B} \in \mathsf{BPD}(w)} \mathfrak{G}_{\delta(\check{\mathcal{B}})}$.

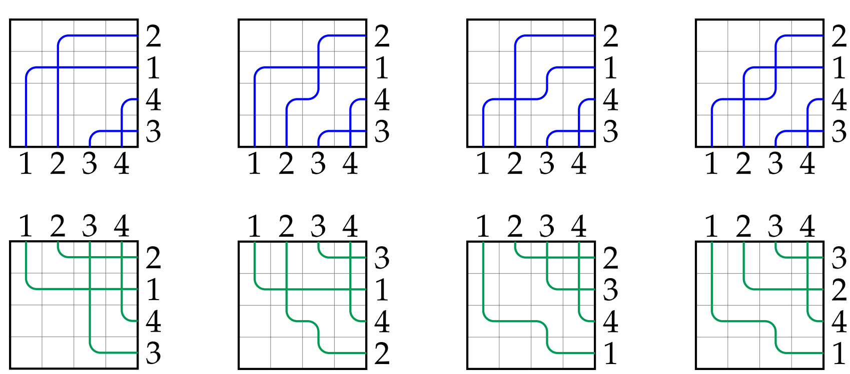
We now give examples to illustrate the main theorems.

Example 2.4. Let $w = 1423$. All BPDs of w are reduced. We list them below, followed by their associated co-BPDs.



The first two co-BPDs are reduced, but the third is not. Thus, Theorem 2.2 says that $\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$. Also, Theorem 2.3 says that $\mathfrak{S}_{1423} = \mathfrak{G}_{1423} + \mathfrak{G}_{2413} + \mathfrak{G}_{3412}$. ♢

Example 2.5. Let $w = 2143$. The BPDs of w are listed below, along with the co-BPDs of w.



The co-BPDs are all reduced. Thus, applying Theorem 2.2 gives $\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{3142} - \mathfrak{S}_{2341} + \mathfrak{S}_{3241}$. By Theorem 2.3, $\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{3142} + \mathfrak{G}_{2341}$. In this expansion, we do not have a term for the fourth co-BPD because its corresponding BPD is not reduced. ♢

3 Changing bases with pipe dreams

The statements of Theorem 2.2 and Theorem 2.3 closely parallel the change of basis theorems of Lenart and Lascoux. These earlier results were originally stated using binary triangular arrays. Here, we provide a reformulation using pipe dreams and co-pipe dreams. Fix $n \in \mathbb{Z}_+$. A ***pipe dream*** of size n is a tiling of the $n \times n$ grid using the tiles

such that every tile on the antidiagonal is a , every tile strictly below the antidiagonal is a , and every tile strictly above the antidiagonal is either a  or a . Each such tiling gives rise to a network of n pipes. A ***co-pipe dream*** is an upside down pipe dream. To a pipe dream $\mathcal{P}$, we associate a co-pipe dream $\check{\mathcal{P}}$ as follows. First, remove each tile on the main antidiagonal. Then, change each bump  to a cross  and each cross  to an upside down bump . Keeping tiles in order within their columns, move them downward so that the crosses and bumps are bottom-justified in the $n \times n$ grid. Finally, place a downward bump  in every cell along the diagonal. See Example 3.3 and Example 3.4 for examples of this map. Similarly to BPDs, if $\mathcal{P}$ is a pipe dream or co-pipe dream, there is a natural way to associate a permutation $\delta(\mathcal{P})$ by reading along pipes. If $w \in S_n$, we denote the set of pipe dreams of w by $\overline{\mathsf{Pipes}}(w)$, and write $\mathsf{Pipes}(w)$ for the subset of reduced pipe dreams.

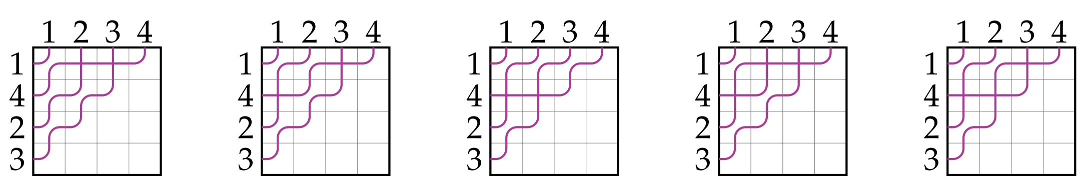
The next result, due to Lenart, gives a formula for expanding a Grothendieck polynomial in the basis of Schubert polynomials. Here, we have reformulated Lenart's result in terms of pipe dreams and co-pipe dreams.

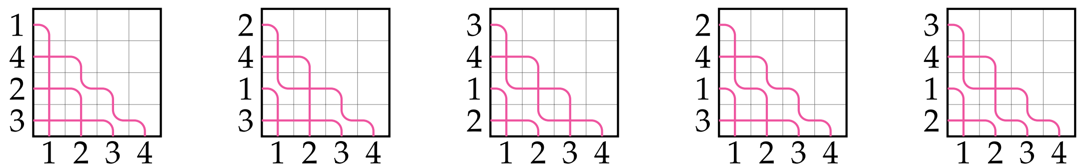
Theorem 3.1 ([29])**.** *Let $w \in S_n$. Then* $\mathfrak{G}_w = \sum_{\substack{\mathcal{P} \in \overline{\mathsf{Pipes}}(w) \\ \check{\mathcal{P}} \text{ is reduced}}} (-1)^{\ell(\delta(\check{\mathcal{P}}))-\ell(w)} \mathfrak{S}_{\delta(\check{\mathcal{P}})}.$

Lascoux gave the following formula for expressing a Schubert polynomial as a positive sum of Grothendieck polynomials.

Theorem 3.2 ([25])**.** *Let $w \in S_n$. Then* $\mathfrak{S}_w = \sum_{\mathcal{P} \in \mathsf{Pipes}(w)} \mathfrak{G}_{\delta(\check{\mathcal{P}})}.$

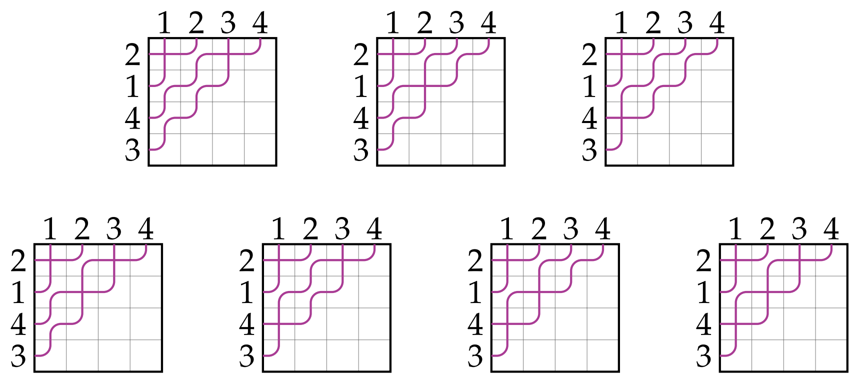
Example 3.3. Let $w = 1423$. The corresponding pipe dreams and their associated co-pipe dreams are shown below.



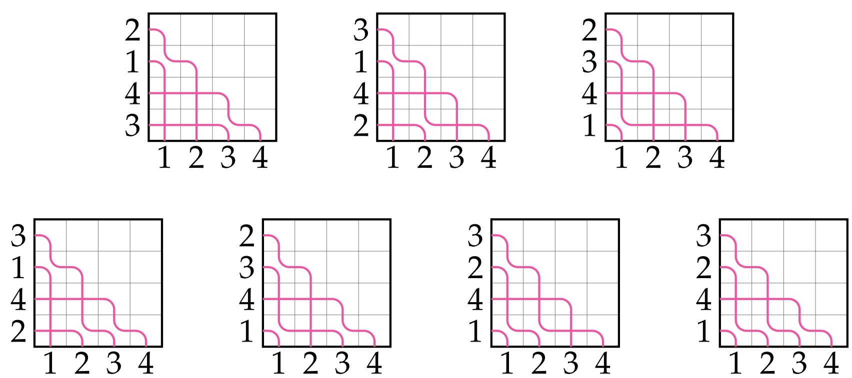


Of the co-pipe dreams, only the first and fourth are reduced. Lenart's formula says that $\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$. Of the pipe dreams, the first three are reduced. Thus, Lascoux's formula tells us that $\mathfrak{S}_{1423} = \mathfrak{G}_{1423} + \mathfrak{G}_{2413} + \mathfrak{G}_{3412}$. Both of these equalities agree with the expansions in terms of BPDs from Example 2.4. ♢

Example 3.4. Let $w = 2143$. The pipe dreams of w are pictured below.



Here are the associated co-pipe dreams.



Of the co-pipe dreams, the first, fourth, fifth, and seventh are reduced. This tells us that $\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{3142} - \mathfrak{S}_{2341} + \mathfrak{S}_{3241}$. Of the pipe dreams, the first three are reduced. Thus, $\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{3142} + \mathfrak{G}_{2341}$. These expansions agree with Example 2.5. ♢

In the full version of this abstract, we present new proofs of Theorem 3.1 and Theorem 3.2. Our approach relies on the combinatorial co-transition recurrence of [19]. One advantage of these new proofs is that they lead to a method for constructing the pipe dreams for w that have reduced co-pipe dreams recursively in terms of certain chains in Bruhat order. This chain theoretic construction is similar in spirit to the climbing chains of [2] and [31], which were used to give formulas for Schubert and Grothendieck polynomials, respectively. Though we still sum over a subset of these chains, there are typically fewer chains to check than there are pipe dreams for w.

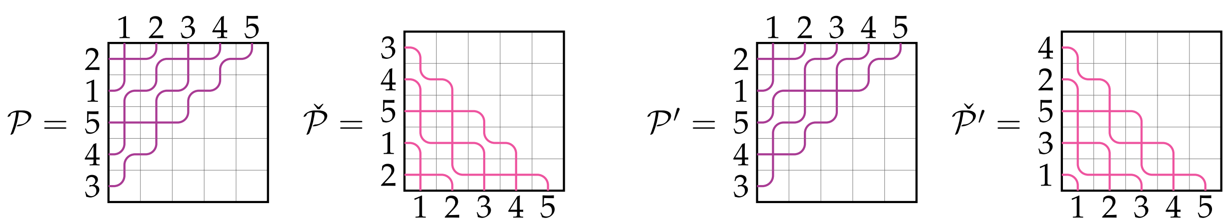
4 Co-objects and the canonical bijection

Our new results on BPDs follow from transferring properties from pipe dreams to BPDs. We use the *column-weight preserving canonical bijection* between these objects, which was given by Gao and Huang [11]. For pipe dreams, column-weight tracks how many ⊞ tiles appear in each column, and for BPDs, the column-weight is determined by the number of □ tiles in each column. Our key observation is the following:

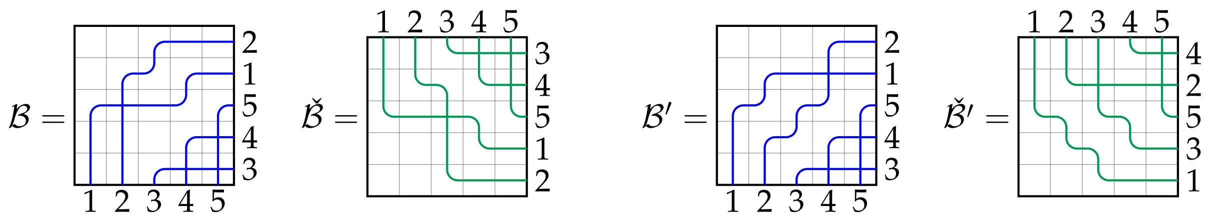
Theorem 4.1. *Suppose $\mathcal{P} \in \mathsf{Pipes}(w)$ maps to $\mathcal{B} \in \mathsf{BPD}(w)$ under the column-weight preserving canonical bijection of Gao–Huang. Then $\delta(\check{\mathcal{P}}) = \delta(\check{\mathcal{B}})$.*

This statement provides evidence that, despite pipe dreams and BPDs exhibiting different properties, there is underlying structure governing aspects of their behavior in parallel. We prove Theorem 4.1 by analyzing the co-transition recurrence on both pipe dreams and BPDs, focusing on how co-transition modifies the associated co-objects.

Example 4.2. Let $w = 21543$. There are two reduced pipe dreams for w that have the column-weight $\alpha = (2,1,1,0,0)$. We list them, and their associated co-pipe dreams below.



Below are the reduced BPDs of w with column-weight α, and their associated co-BPDs.



By applying Theorem 4.1, we conclude that under the canonical bijection, $\mathcal{P}$ maps to $\mathcal{B}$ and $\mathcal{P}'$ maps to $\mathcal{B}'$. ♢

Remark 4.3. *Huang, Shimozono, and Yu [15] generalized [11] to give a weight preserving bijection between marked bumpless pipe dreams and pipe dreams. It is likely that considering co-objects would give further insight into this K-theoretic bijection, but we do not pursue this direction here.*

5 Expansion of back stable Grothendieck polynomials into back stable Schubert polynomials

One advantage of the new BPD-based expansion of Grothendieck polynomials into Schubert polynomials is an application to back stable Schubert calculus. In this section, we state the first combinatorial formula for expanding back stable Grothendieck polynomials into back stable Schubert polynomials (see Theorem 5.1). We refer the reader to [22, 23] for background.

Write $S_{\mathbb{Z}}$ for the set of permutations of $\mathbb{Z}$ that fix all but finitely many elements. The set $S_{\mathbb{Z}}$ is a group under composition of functions and is generated by the set of simple reflections $\{s_i = (i\, i+1) : i \in \mathbb{Z}\}$. Let R denote the set of formal power series in the variables $\{x_i : i \in \mathbb{Z}\}$ that have *bounded degree* and *bounded support*. An element $f \in R$ is ***back symmetric*** if there exists some integer c so that $s_i \cdot f = f$ for all $i < c$. We write $\overleftarrow{R} \subsetneq R$ for the subset of ***back symmetric formal power series***. Lam, Lee, and Shimozono showed that $\overleftarrow{R} = \Lambda \otimes \mathbb{Q}[\ldots, x_{-1}, x_0, x_1, \ldots]$, where Λ denotes the ring of symmetric functions in the variables $\{\ldots, x_{-2}, x_{-1}, x_0\}$ [22, Proposition 3.1].

Let $w \in S_{\mathbb{Z}}$. Write $\mathsf{red}(w)$ for the set of reduced words for w, and let $\mathsf{hecke}(w)$ denote the set of *Hecke words* for w. By [22, Theorem 3.2], the ***back stable Schubert polynomial*** has the monomial expansion

$$\overleftarrow{\mathfrak{S}}_w = \sum_{(a_1, a_2, \ldots, a_{\ell(w)}) \in \mathsf{red}(w)} \sum_{\substack{b_1 \le b_2 \le \cdots \le b_{\ell(w)} \\ a_i \le a_{i+1} \Rightarrow b_i < b_{i+1} \\ b_i \le a_i}} x_{b_1} x_{b_2} \cdots x_{b_{\ell(w)}}. \tag{5.1}$$

We take this expansion as a definition here, although $\overleftarrow{\mathfrak{S}}_w$ is equivalently the limit of certain shifts of Schubert polynomials. Lam, Lee, and Shimozono showed that back stable Schubert polynomials form a $\mathbb{Q}$-linear basis for $\overleftarrow{R}$ [22, Theorem 3.5].

Similarly, by [23, Proposition 4.4], the ***back stable Grothendieck polynomial*** has the monomial expansion

$$\overleftarrow{\mathfrak{G}}_w = \sum_{(a_1, a_2, \ldots, a_L) \in \mathsf{hecke}(w)} (-1)^{L - \ell(w)} \sum_{\substack{b_1 \le b_2 \le \cdots \le b_L \\ a_i \le a_{i+1} \Rightarrow b_i < b_{i+1} \\ b_i \le a_i}} x_{b_1} x_{b_2} \cdots x_{b_L}. \tag{5.2}$$

Again, $\overleftarrow{\mathfrak{G}}_w$ is the limit of certain shifts of Grothendieck polynomials. Note that for $w \neq \mathrm{id}$, $\overleftarrow{\mathfrak{G}}_w$ is not an element of R, since it is not of bounded degree. However, the degree d component of $\overleftarrow{\mathfrak{G}}_w$ is a back symmetric formal power series, and it expands as a finite sum of back stable Schubert polynomials. See [23, Section 4.1] for a description of the back stable ring in which $\overleftarrow{\mathfrak{G}}_w$ resides.

We now recall the back stable version of BPDs from [22, 23]. An ***$S_\mathbb{Z}$ BPD*** is a tiling of $\mathbb{Z} \times \mathbb{Z}$ with the six tiles from Equation (2.1), forming a network of pipes labeled by $\mathbb{Z}$ such that pipes do not start or end within the grid, for each column c, there exists a row r_c such that the tile at position (r, c) is a for all $r > r_c$, for each row r, there exists a column c_r such that the tile at position (r, c) is a for all $c > c_r$, and there exist integers $p \leq q$ such that for all $i \notin [p, q]$, the tile at position (i, i) is a , and the pipe that passes through this tile is a hook. An $S_\mathbb{Z}$ BPD is ***reduced*** if each pair of pipes crosses at most one time.

We may assign a permutation to an $S_\mathbb{Z}$ BPD $\mathcal{B}$ by labeling each pipe with the index of the column in which it is eventually vertical, and propagating labels across crossings using standard local rules for northeast planar histories. We obtain an associated permutation $w \in S_\mathbb{Z}$ by setting $w(i) = j$ if the pipe labeled j is eventually horizontal in row i. We write $\delta(\mathcal{B}) = w$.

We now introduce a back stable version of co-BPDs. An ***$S_\mathbb{Z}$ co-BPD*** is a tiling of the $\mathbb{Z} \times \mathbb{Z}$ grid with the six tiles pictured below

forming a network of pipes labeled by $\mathbb{Z}$ so that: pipes do not start or end within the grid, for each column c, there exists a row r_c such that the tile at position (r, c) is a for all $r < r_c$, for each row r, there exists a column c_r such that the tile at position (r, c) is a for all $c > c_r$, and there exist integers $p \leq q$ such that for all $i \notin [p, q]$, the tile at position (i, i) is a , and the pipe that passes through this tile is a hook. An $S_\mathbb{Z}$ co-BPD is ***reduced*** if each pair of pipes crosses at most one time.

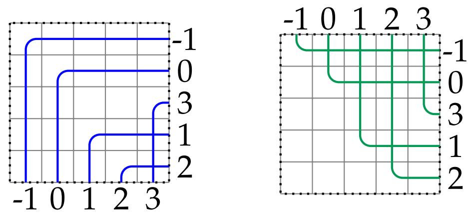
Note that in contrast to $S_\mathbb{Z}$ BPDs, each $S_\mathbb{Z}$ co-BPD contains infinitely many crossings. Nevertheless, we can still read off a permutation $w \in S_\mathbb{Z}$ from an $S_\mathbb{Z}$ co-BPD $\mathcal{B}$ by following the pipes in a slightly more subtle way.

Making the same tile-by-tile replacements as we did on finite BPDs induces a bijection from $S_\mathbb{Z}$ BPDs to $S_\mathbb{Z}$ co-BPDs. Given $\mathcal{B} \in \overleftarrow{\mathsf{BPD}}(w)$, we write $\check{\mathcal{B}}$ for the image of $\mathcal{B}$ under this map. In contrast to the situation with finite BPDs, we no longer have a well-defined map from $S_\mathbb{Z}$ BPDs to $S_\mathbb{Z}$ co-BPDs by reflection.

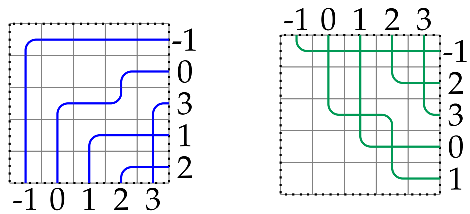
We now state the main theorem of this section.

Theorem 5.1. *Given $w \in S_\mathbb{Z}$, we have* $\overleftarrow{\mathfrak{G}}_w = \sum_{\substack{\mathcal{B} \in \overleftarrow{\mathsf{BPD}}(w) \\ \check{\mathcal{B}} \text{ is reduced}}} (-1)^{\ell(\delta(\check{\mathcal{B}})) - \ell(w)} \overleftarrow{\mathfrak{S}}_{\delta(\check{\mathcal{B}})}.$

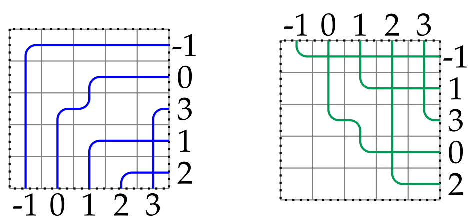
Example 5.2. Let $w = s_2 s_1 \in S_{\mathbb{Z}}$. The Rothe $S_{\mathbb{Z}}$ BPD of w (the unique $S_{\mathbb{Z}}$ BPD for w with no ╯ tiles) and its associated $S_{\mathbb{Z}}$ co-BPD, both restricted to $[-1,3] \times [-1,3]$, are shown below.



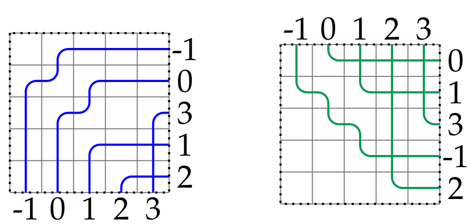
The following $S_{\mathbb{Z}}$ BPD for w has a nonreduced $S_{\mathbb{Z}}$ co-BPD.



Furthermore, any $S_{\mathbb{Z}}$ BPD obtained from it by a sequence of *droop moves* will still have a nonreduced $S_{\mathbb{Z}}$ co-BPD. One can check that the $S_{\mathbb{Z}}$ BPDs contributing to the expansion of $\overleftarrow{\mathfrak{S}}_w$ are the Rothe $S_{\mathbb{Z}}$ BPD for w, plus $S_{\mathbb{Z}}$ BPDs obtained from a sequence of droop moves applied to the $S_{\mathbb{Z}}$ BPD below, each of which leaves the pipe with label 0 fixed.



For instance, we first droop the pipe with label -1 and obtain the diagrams below.



Then we droop the pipe with label -2, then -3, and so on. From this and Theorem 5.1, we deduce that $\overleftarrow{\mathfrak{S}}_{s_2 s_1} = \overleftarrow{\mathfrak{S}}_{s_2 s_1} - \overleftarrow{\mathfrak{S}}_{s_0 s_2 s_1} + \overleftarrow{\mathfrak{S}}_{s_{-1} s_0 s_2 s_1} - \overleftarrow{\mathfrak{S}}_{s_{-2} s_{-1} s_0 s_2 s_1} + \cdots$. The reader is invited to compare the expansion of $\overleftarrow{\mathfrak{S}}_{s_2 s_1}$ into back stable Schubert polynomials directly with the monomial expansions in Equation (5.1) and Equation (5.2). ♢

6 Connections to the literature

In addition to the binary triangular array formulas of [29, 25] for changing bases between Schubert and Grothendieck polynomials, Lenart [30] also gave a tableau formula for the special case of symmetric Grothendieck polynomials and *Schur polynomials*. The *stable Grothendieck polynomials* of [8] arise as specializations of back stable Grothendieck polynomials. The results of Lenart [30] naturally extend to give expansions of stable Grothendieck polynomials indexed by partition shapes into *Schur functions*, and vice versa. Chan and Pflueger [6] generalized this result to provide analogous expansions for *skew stable Grothendieck polynomials* into *skew Schur functions*, and vice versa. Skew stable Grothendieck polynomials (and likewise skew Schur functions) are not linearly independent, thus the resulting expansions are not canonical. It would be interesting to compare the result of specializing Theorem 5.1 with the work of Chan–Pflueger, but we do not pursue this direction here.

The expansion of Grothendieck polynomials into the Schubert basis has been a useful tool for studying related algebraic and geometric questions. For instance, Monical, Tokcan, and Yong conjectured that Grothendieck polynomials have saturated Newton polytopes [34]. In the special case of symmetric Grothendieck polynomials, Escobar and Yong [7] proved the conjecture using Lenart's tableau change of basis formula. Although there has been additional progress [33, 5], the full conjecture remains open. Lenart's formula was also used to prove a combinatorial rule for computing the Castelnuovo–Mumford regularity of Grassmannian matrix Schubert varieties [35]. A number of open questions about the monomial supports of Grothendieck polynomials remain, see, for example, [32]. Additional understanding of the Grothendieck to Schubert expansion may be useful for studying these problems.

Acknowledgements

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