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Chow polynomials of totally nonnegative matrices and posets

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Abstract. Huh-Stevens and Ferroni-Schröter independently conjectured that Hilbert-Poincaré series of Chow rings of geometric lattices have only real zeros, and Ferroni, Matherne and the second author extended this conjecture to Chow polynomials of Cohen-Macaulay posets. In this paper we address the above conjectures by providing new defining relations and properties of Chow functions of posets and matrices. These are used to prove that Chow polynomials of totally nonnegative matrices have only real zeros, which proves the above conjectures for projective and affine geometries, face lattices of cubical polytopes, partition lattices and Dowling lattices, perfect matroid designs, and lattices of flats of paving matroids.

We also study Chow polynomials of Toeplitz matrices in greater detail, and show how these are related to the combinatorics of binomial and Sheffer posets, as well as to a family of generalized Eulerian polynomials that have been studied by e.g. Stanley, Brenti, Stembridge and Shareshian-Wachs.

Keywords: Totally nonnegative matrix, real-rooted polynomial, Pólya frequency sequence, Chow polynomial, Chow ring, geometric lattice, binomial poset, Sheffer poset

1 Introduction

In their influential work [17], Feichtner and Yuzvinsky introduced the Chow ring of an atomistic lattice L, and gave a presentation for it via relations in terms of chains in the lattice. This ring is graded and finite-dimensional, allowing one to study its Hilbert-Poincaré series as a polynomial invariant of the lattice. When L is a geometric lattice, then the Chow ring satisfies the Kähler package [2], which implies that its Hilbert series is always palindromic and unimodal. Moreover, for specific classes of geometric lattices, these Hilbert series coincide with well-known families of polynomials; for example for boolean algebras one recovers the Eulerian polynomials, and for truncations of boolean algebras one recovers the derangement polynomials [22]. For these reasons, the Hilbert

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series of Chow rings of geometric lattices (or matroids) have garnered importance on their own and are now known as the *Chow polynomials* of matroids.

Ferroni-Schröter [20, Conjecture 8.18] and Huh-Stevens [29, Conjectures 4.1.3 and 4.3.3] independently conjectured that Chow polynomials of matroids have only real zeros. This was recently verified in the uniform case in [11, Theorem 1.1]. Ferroni, Matherne and the second author extended the definition of Chow polynomials to arbitrary bounded posets using the framework of Kazhdan-Lusztig-Stanley theory [19]. These are palindromic polynomials that are conjectured to have only real zeros for all Cohen-Macaulay posets and Bruhat intervals (with the appropriate P-kernels [19, Conjectures 4.26 and 6.14]). In this larger context, Hoster and Stump [23, Theorem 1.1] proved that Chow polynomials of boolean complexes with nonnegative h-vectors are real-rooted.

In this paper we develop a systematic approach to the study of zeros of Chow polynomials for a class of posets called TN-*posets*, which were introduced in [8, 9]. This class includes (rank-selected subposets of) affine and projective geometries, perfect matroid designs, dual partition lattices and dual Dowling lattices. To do so, we define and study Chow polynomials of *totally nonnegative matrices* (TN-matrices), i.e., matrices whose minors are all nonnegative. Our main result is Theorem 3.8, which states that the Chow polynomial of any lower triangular TN-matrix with all diagonal entries equal to one is real-rooted. A direct consequence of this is Theorem 4.2, which says that the Chow polynomial of any (dual) TN-poset is real-rooted. We also use our methods to prove that the Chow polynomial of any paving matroid is real-rooted, Corollary 5.2.

We mention that Coron, Ferroni and Li [12] independently proved that the Chow polynomials of so called UMEL-shellable posets are real-rooted. While there seems to be great intersection between this class and the one of TN posets, a clear relation between the two is at the moment only conjectured, see [9, Conjecture 4.3] and [12, Section 8].

For proofs of the results presented in this extended abstract, we refer to [10].

2 The incidence algebra and Chow functions

In this section we recall the construction of Chow functions and Chow polynomials of partially ordered sets (posets). For undefined poset terminology, we refer to [28]. Recall that an interval of a poset is a subposet of P of the form $[x, y] = \{z \in P : x \leq z \leq y\}$, where $x, y \in P$. All posets P considered in this paper are *locally finite*, i.e., each interval of P is finite. We also say that a poset is *bounded* if it has unique least and largest elements, which we denote by $\hat{0}$ and $\hat{1}$, respectively. Given a poset P, a *weak rank function* is a function $\rho : P \times P \to \mathbb{N}$ such that

- $\rho(x, y) = \rho_{x,y} > 0$ if and only if $x < y$, and
- $\rho_{x,y} = \rho_{x,z} + \rho_{z,y}$, for all $x \leq z \leq y$ in P.

A *weakly ranked* poset consists of a pair (P, ρ), where ρ is a weak rank function for P. By slight abuse of notation, we say that P is a weakly ranked poset, when this does not create confusion. If the poset has a least element $\hat{0}$, we write $\rho(x) := \rho_{\hat{0},x}$ for $x \in P$ and call this the *(weak) rank* of x in P. If $\rho_{x,y} = 1$ whenever y covers x, then we say that P is *graded* (or *ranked*). Moreover if P is bounded, then we say that P has (weak) rank $\rho(\hat{1})$.

The *incidence algebra*, $I(P)$, of a poset P is the free $\mathbb{R}[t]$-module spanned by the intervals of P. More explicitly, an element f associates to every interval $[x, y]$ of P an element in $\mathbb{R}[t]$ which we denote by $f_{x,y}$. The product (*convolution*) is defined by

$$(fg)_{x,y} = \sum_{x \le z \le y} f_{x,z} \cdot g_{z,y}.$$

For a polynomial $f \in \mathbb{R}[t]$ of degree at most n, we define $\mathcal{I}_n(f)(t) = t^n f(t^{-1})$ and say that f is *palindromic with center of symmetry* $n/2$, if $\mathcal{I}_n(f) = f$. Let $I_\rho(P)$ be the subalgebra of $I(P)$ consisting of functions f such that $\deg f_{x,y} \le \rho_{x,y}$ for all $x, y \in P$, and define $\mathcal{I} : I_\rho(P) \to I_\rho(P)$ by $\mathcal{I}(f)_{x,y} = \mathcal{I}_{\rho_{x,y}}(f_{x,y})$. Hence, $\mathcal{I}$ is an involution and an automorphism, i.e., $\mathcal{I}^2 = \delta$ and $\mathcal{I}(fg) = \mathcal{I}(f)\mathcal{I}(g)$ for all $f, g \in I_\rho(P)$.

Given a weakly ranked poset P, an element κ in $I_\rho(P)$ is called a *P-kernel* if $\kappa_{x,x} = 1$ for each $x \in P$, and $\kappa^{-1} = \mathcal{I}(\kappa)$. Equivalently, $\kappa \in I_\rho(P)$ is a P-kernel if and only if $\kappa = g^{-1}\mathcal{I}(g)$ for some $g \in I(P)$ such that

$$g_{x,x} = 1 \text{ for each } x \in P, \quad \text{and} \quad \deg g_{x,y} < \rho_{x,y}/2 \text{ for every } x < y \text{ in } P, \tag{2.1}$$

see [7, Theorem 6.2], [25, Proposition 2.5]. The function g is called the *left Kazhdan-Lusztig-Stanley function* associated to κ. Following [19, Definition 3.2], we define the *reduced P-kernel* to be the function $\overline{\kappa} = (\kappa - t\delta)/(t-1)$ in $I_\rho(P)$ and the *Chow function* associated to κ (or g) as the element $(-\overline{\kappa})^{-1}$ in $I_\rho(P)$.

For every $n \in \mathbb{N}$, define a family of operators $\mathcal{S}_n$ on polynomials of degree at most n in $\mathbb{R}[t]$ by $\mathcal{S}_n(f) = (\mathcal{I}_n(f) - f)/(t-1)$. The following theorem provides an alternative definition of the Chow function.

Theorem 2.1. *Let $g \in I(P)$ be an element that satisfies (2.1). Then there exist unique functions H and d in $I_\rho(P)$ for which*

(i) $d_{x,x} = 1$ *for each* $x \in P$,

(ii) $\mathcal{I}(H) = tH + (1-t)\delta$,

(iii) $\mathcal{I}(d) = d$,

(iv) $H = dg$.

Moreover, H is the Chow function associated to g, and the polynomials $d_{x,y}$ satisfy the recursion

$$d_{x,y} = t\mathcal{S}_{\rho_{x,y}-1}\left(\sum_{x \le z < y} d_{x,z} g_{z,y}\right), \quad x < y. \tag{2.2}$$

We call the unique function d achieved by Theorem 2.1 the *Chow-derangement function* associated to g. If the poset is bounded, we call $H_P = H_{\hat{0},\hat{1}}$ the *Chow polynomial* of P (with respect to g) and $d_P = d_{\hat{0},\hat{1}}$ the *Chow-derangement polynomial* of P.

In this paper we will be mostly concerned with a special class of functions g, where $g_{x,y} \in \mathbb{R}$ for all $x \leq y$ and $g_{x,x} = 1$ for each $x \in P$. We call such functions *scalar*. An example of a scalar function is ζ, the *zeta function* of P, which is defined by $\zeta_{x,y} = 1$ for all $x \leq y$ in P.

Example 2.2. When $g = \zeta$, the corresponding P-kernel is the *characteristic function* χ, which associates to each interval its characteristic polynomial. Hence, the associated Chow function H was given the name of *characteristic Chow function* in [19]. When P is a geometric lattice, the polynomial H_P coincides with the Hilbert series of the Chow ring of P [18, Theorem 1.4]. The polynomials H_P are γ-positive for Cohen-Macaulay posets [19, Theorem 1.4], and are conjectured to be real-rooted for every Cohen-Macaulay poset [19, Conjecture 1.5].

Example 2.3. When P is a boolean algebra of rank r and $g = \zeta$, then all intervals of the same rank in P are isomorphic, from which it follows that $H_{x,y} = H_{\rho(y)-\rho(x)}$ and $d_{x,y} = d_{\rho(y)-\rho(x)}$ for some polynomials $\{H_n\}_{n=0}^r$ and $\{d_n\}_{n=0}^r$. Theorem 2.1 then says that these two families of polynomials are the unique ones that satisfy $H_0 = d_0 = 1$ and, for $1 \leq n \leq r$, $H_n = \sum_{k=0}^n \binom{n}{k} d_k$, $\mathcal{I}_n(d_n) = d_n$ and $\mathcal{I}_n(H_n) = tH_n$. These properties are known to hold for the *derangement polynomials* $d_n = \sum_{\sigma \in \mathfrak{D}_n} t^{\mathrm{exc}(\sigma)}$, which count *derangements* (fixed point free permutations) in the symmetric group $\mathfrak{S}_n$ by the number of *excedances* $\mathrm{exc}(\sigma) = |\{i : \sigma(i) > i\}|$, and the *Eulerian polynomials* $A_n = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$, see [6, Corollary 1]. Hence we deduce that $d_{x,y}$ and $H_{x,y}$ are the traditional derangement polynomials and Eulerian polynomials, respectively.

Remark 2.4. Given a geometric lattice, one may also define its *augmented* Chow ring. In analogy with that, one defines an *augmented Chow polynomial* G_P for any poset P so that, when P is a geometric lattice, G_P computes the Hilbert series of the corresponding augmented Chow ring. By [19, Corollary 4.6] G_P is itself a Chow polynomial associated to the poset $\mathrm{aug}(P)$, which is built by adding a new least element to P. Our results then naturally extend to the augmented case, see [10].

3 Chow polynomial of totally nonnegative matrices

Let $R = (r_{n,k})_{n,k=0}^N$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix with all diagonal entries equal to one. Consider the chain $[0, N] = \{0 < 1 < \cdots < N\}$ graded by the rank function $\rho(n) = n$. Then the matrix R corresponds to an element $g = g[R]$ in the incidence algebra of $[0, N]$ defined by

$$g_{k,n} = r_{n,k}, \quad \text{for all } 0 \leq k \leq n \leq N.$$

Clearly g satisfies (2.1) and therefore we can define the corresponding Chow and Chow-derangement polynomials. The following corollary provides the defining relations for these polynomials.

Corollary 3.1. *Let $R = (r_{n,k})_{n,k=0}^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix with all diagonal entries equal to one. There are unique polynomials $\{d_n\}_{n=0}^N$ and $\{H_n\}_{n=0}^N$ for which*

(i) $d_0 = 1$,

(ii) $\mathcal{I}_n(H_n) = tH_n$,

(iii) $\mathcal{I}_n(d_n) = d_n$,

(iv) $H_n = \sum_{k=0}^n r_{n,k} d_k$,

for each $1 \le n \le N$.

Moreover $d_n = d_{0,n}[R]$ and $H_n = H_{0,n}[R]$, $n \le N$, are the Chow-derangement polynomials and the Chow polynomials associated to $g[R]$, respectively, and

$$d_n = t\mathcal{S}_{n-1}\left(\sum_{k=0}^{n-1} r_{n,k} d_k\right), \quad 1 \le n \le N. \tag{3.1}$$

The main result of this section is Theorem 3.8, which says that the Chow polynomial of any lower triangular and totally nonnegative matrix with all diagonal entries equal to one is real-rooted. We start by collecting some notation and results from [8]. Recall that a matrix with real entries is *totally nonnegative* (or TN) if all of its minors are nonnegative.

Definition 3.2. Let $R = (r_{n,k})_{n,k=0}^N$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix with real entries whose diagonal entries are all equal to one, and let $R_n = \sum_{k=0}^n r_{n,k} t^k$ be the generating polynomial of the nth row. The matrix R is called *resolvable* if there is an array $(\lambda_{n,k})_{0 \le k \le n < N}$ of nonnegative numbers, and an array of monic polynomials $(R_{n,k})_{0 \le k \le n \le N}$ in $\mathbb{R}[t]$ for which

- $R_{n,0} = R_n$ and $R_{n,n} = t^n$ for each $0 \le n \le N$,
- t^k divides $R_{n,k}$ for all $0 \le k \le n \le N$, and
- if $0 \le k \le n < N$, then $R_{n+1,k} = R_{n+1,k+1} + \lambda_{n,k} R_{n,k}$.

If the matrix R is resolvable, then we say that the polynomials $(R_{n,k})_{0 \le k \le n \le N}$ *resolve* R.

Example 3.3. The identity matrix is resolvable, with $R_{n,k} = t^n$ and $\lambda_{n,k} = 0$ for all $0 \le k \le n$. Pascal's triangle $(\binom{n}{k})_{n,k=0}^\infty$ is resolvable, with $R_{n,k} = t^k(1+t)^{n-k}$ and $\lambda_{n,k} = 1$ for all $0 \le k \le n$. For more examples we refer to [8, 9].

The next theorem, proved in [8, Theorem 2.6], characterizes resolvability in terms of totally nonnegative matrices.

Theorem 3.4. *Let R be a lower triangular matrix with all diagonal entries equal to one. Then R is resolvable if and only if R is* TN.

3.1 Interlacing zeros and $\mathcal{I}_n$-interlacing sequences of polynomials

Recall that a polynomial $p \in \mathbb{R}[t]$ is called *real-rooted* if all of its zeros are real. Suppose p and q are two real-rooted polynomials in $\mathbb{R}_{\geq 0}[t]$ with zeros $\cdots \leq \beta_2 \leq \beta_1$ and $\cdots \leq \alpha_2 \leq \alpha_1$, respectively. The zeros *interlace* if

$$\cdots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1.$$

We then write $p \prec q$. For technical reasons we consider the identically zero polynomial to be real-rooted and write $0 \prec p$ and $p \prec 0$ for any other real-rooted polynomial p. A sequence of polynomials $f_1, \ldots, f_m$ in $\mathbb{R}_{\geq 0}[t]$ is called *interlacing* if $f_i \prec f_j$ for all $i < j$.

The following notion will be fundamental in the proofs of our main results.

Definition 3.5. We say that a sequence $f_1, f_2, \ldots, f_m$ of polynomials in $\mathbb{R}_{\geq 0}[t]$ of degree at most n is $\mathcal{I}_n$*-interlacing* if

$$f_1, f_2, \ldots, f_m, \mathcal{I}_n(f_m), \mathcal{I}_n(f_{m-1}), \ldots, \mathcal{I}_n(f_1)$$

is an interlacing sequence of polynomials.

3.2 Chow-deranged maps

Let $R = (r_{n,k})_{n,k=0}^{N}$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix, with entries in $\mathbb{R}$, and with all diagonal entries equal to one. Let further $\mathbb{R}_N[t]$ be the linear space of all polynomials in $\mathbb{R}[t]$ of degree at most N. We define the $\mathbb{R}$-linear map

$$\mathcal{D} = \mathcal{D}_R : \mathbb{R}_N[t] \to \mathbb{R}_N[t], \qquad \mathcal{D}(t^n) = d_n$$

where d_n is the Chow-derangement polynomial associated to R. This map is called the *Chow-deranged map*. Notice that (iv) in Corollary 3.1 translates to $H_n = \mathcal{D}(R_n)$.

Remark 3.6. The deranged map was first studied by the first author and Solus [5], who proved Theorem 3.7 below for the case when $R = (\binom{n}{k})_{n,k=0}^{\infty}$.

Suppose R is resolvable, and define $d_{n,k} = \mathcal{D}(R_{n,k})$, $0 \leq k \leq n \leq N$.

Theorem 3.7. *Let R be a resolvable matrix, and $0 \leq n \leq N$. Then*

$$d_{n,0}, d_{n,1}, \ldots, d_{n,n}$$

is an $\mathcal{I}_n$-interlacing sequence. Moreover if $f = \sum_{k=0}^{n} h_k R_{n,k}$, where $h_k \geq 0$ for each k, then $\mathcal{D}(f)$ is real-rooted and $d_{n,0} \prec \mathcal{D}(f) \prec d_{n,n}$.

The first important consequence of Theorem 3.7 establishes the real-rootedness of the Chow polynomials and Chow-derangement polynomials of any lower triangular TN-matrix with all diagonal entries equal to one.

Theorem 3.8. *Let $R = (r_{n,k})_{n,k=0}^N$ be a lower triangular* TN*-matrix with all diagonal entries equal to one. Then the Chow polynomials $\{H_n\}_{n=0}^N$ and the Chow-derangement polynomials $\{d_n\}_{n=0}^N$ are real rooted.*

Moreover $H_n \prec d_n$, $H_n \prec H_{n+1}$ and $d_n \prec d_{n+1}$ for each n.

4 Chow polynomials of TN-posets

We shall now see that the Chow polynomials of certain matrices are actually characteristic Chow polynomials of posets, by following the construction in [8, Section 5]. We say that a weakly ranked poset P with a least element $\hat{0}$ is *weak-rank uniform* if for any x and y in P with $\rho(x) = \rho(y)$,

$$|\{z \leq x : \rho(z) = k\}| = |\{z \leq y : \rho(z) = k\}|, \quad \text{for each } 0 \leq k \leq \rho(x). \tag{4.1}$$

If $\rho(x) = n$, we define $r_{n,k} = r_{n,k}(P) = |\{z \leq x : \rho(z) = k\}|$, and write $R = R(P) = (r_{n,k})_{n,k \geq 0}$. Notice that $r_{n,0} = r_{n,n} = 1$ for each n. If P is graded and satisfies (4.1), then we say that P is *rank-uniform*.

Proposition 4.1. *Let P be a weak-rank uniform poset, and let $R = (r_{n,k}(P))_{n,k=0}^N$ be the corresponding matrix, where $N \in \mathbb{N} \cup \{\infty\}$. If $n \leq N$, then $H_n[R]$ is equal to the characteristic Chow polynomial $H_{[\hat{0},x]}$, where x is any element of rank n.*

In this paper, a weakly rank-uniform poset P is called a TN-*poset* if the matrix $R(P)$ is TN. Below is a list of examples of TN-posets, for proofs see [8, 9].

- **a**. Boolean cell complex with nonnegative h-vectors. For example Cohen-Macaulay simplicial complexes and face lattices of simplicial polytopes.
- **b**. Cubical complexes with nonnegative cubical h-vectors [1]. These include face lattices of cubical polytopes, i.e., polytopes for which all faces are hypercubes.
- **c**. q-poset with nonnegative h-vectors [4]. For example shellable q-complexes [21].
- **d**. Perfect matroid designs [13], i.e., rank uniform geometric lattices. These include (truncations of) projective geometries and affine geometries.
- **e**. Dual of Dowling lattices [15]. In particular, the dual of partition lattices.

Recall that if P is a weakly ranked poset of rank r and $S \subseteq [r-1]$, then

$$P_S = \{x \in P : \rho(x) \in S \cup \{0, r\}\}$$

is the *rank selected subposet* induced by S. Since submatrices of TN-matrices are TN it follows that the class of TN-posets is closed under rank selection. Hence any rank-selected poset of any poset listed above is TN.

Theorem 4.2. *Suppose P is a* TN*-poset, or the dual of a* TN*-poset, of rank n. Then the characteristic Chow polynomial H_P is real-rooted.*

If we apply Theorem 4.2 to **a** above, then we recover [23, Theorem 1.1]. Theorem 4.2 for the case **e** above implies in particular that Chow polynomials of Dowling lattices are real-rooted.

5 Chow polynomials of lattices of flats of paving matroids

Let P be a graded, rank uniform and bounded poset of rank n and let $0 \leq d < n$. Consider a subset $\mathcal{H} \subseteq P$ of pairwise incomparable elements such that $d < \rho(x) < n$ for each $x \in \mathcal{H}$. Let $P(d, \mathcal{H})$ be the weakly ranked poset of rank n obtained by adjoining $\hat{1}$ and $\mathcal{H}$ to the set $\{x \in P \mid \rho(x) \leq d\}$ and by declaring all elements in $\mathcal{H}$ to be of rank $d+1$ and $\rho(\hat{1}) = d+2$. Notice also that if

$$\text{for each } x \text{ of rank } \rho(x) \leq d \text{ there exists a } y \in \mathcal{H} \text{ such that } x \leq y, \tag{5.1}$$

then $P(d, \mathcal{H})$ is graded again. If P is a boolean algebra on n elements and $\mathcal{H}$ satisfies (5.1), then $P(d, \mathcal{H})$ is a paving geometric lattice of rank $d+2$.

Theorem 5.1. *Let P be a graded* TN*-poset of rank n and $P' = P(d, \mathcal{H})$ be as above. Then the polynomials $d_{P'}$ and $H_{P'}$ are real-rooted.*

Corollary 5.2. *If L is the lattice of flats of a paving matroid, then the polynomials d_L and H_L are real-rooted.*

6 Toeplitz matrices

In this section we will focus on the special case when R is a Toeplitz matrix. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers, where $a_0 = 1$. Associate to $\{a_n\}_{n=0}^{\infty}$ the lower triangular *Toeplitz matrix* $R = (a_{n-k})_{n,k=0}^{\infty}$, where $a_m = 0$ if $m < 0$.

Consider the formal power series $f = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{R}[[z]]$, and define formal power series in $\mathbb{R}[t][[z]]$ by

$$D(z,t) = \sum_{n=0}^{\infty} d_n(t) z^n, \quad H(z,t) = \sum_{n=0}^{\infty} H_n(t) z^n,$$

where $d_n(t), H_n(t)$, are the Chow-derangement polynomials and Chow polynomials associated to R, respectively.

Theorem 6.1. *Let* $f = \sum_{n=0}^{\infty} a_n z^n$ *be a formal power series in* $\mathbb{R}[[z]]$, *where* $a_0 = 1$. *Then*

$$D(z,t) = \sum_{n=0}^{\infty} d_n(t) z^n = \frac{1-t}{f(tz) - tf(z)} = \frac{1}{1 - \sum_{n \geq 2} a_n (t + t^2 + \cdots + t^{n-1}) z^n},$$
$$H(z,t) = \sum_{n=0}^{\infty} H_n(t) z^n = \frac{(1-t) f(z)}{f(tz) - tf(z)}.$$

A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers is a *Pólya frequency sequence* if the Toeplitz matrix $R = (a_{n-k})_{n,k=0}^{\infty}$ is TN. Pólya frequency sequences were characterized by Aissen, Schoenberg, Whitney and Edrei [3] as follows.

Theorem 6.2. *A sequence* $\{a_n\}_{n=0}^{\infty}$ *of real numbers is a Pólya frequency sequence if and only if its generating function is of the form*

$$f = \sum_{n=0}^{\infty} a_n z^n = C z^N e^{\gamma z} \prod_{i=1}^{\infty} \frac{1 + \alpha_i z}{1 - \beta_i z}, \tag{6.1}$$

where $C, \gamma, \alpha_i, \beta_i$ *are nonnegative real numbers,* $N \in \mathbb{N}$, *and* $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty$.

Applying Theorem 3.8 to Toeplitz matrices produces families of real-rooted polynomials associated to any series f of the form (6.1). In particular if $f(z) = e^z$ this implies the real-rootedness of Eulerian and derangement polynomials.

Theorem 6.3. *Suppose* f *is a power series as in* (6.1). *Then the polynomials* $d_n(t), H_n(t)$, $n \geq 0$, *defined via* f *by the identities in Theorem 6.1 are all real-rooted.*

6.1 Binomial and Sheffer posets

Recall [14, 27, 28] that a *binomial poset* is a locally finite poset P for which there exists an infinite chain in P, each interval of P is graded, and there exists a function $B : \mathbb{N} \to \mathbb{N}$, called the *factorial function* of P, such that the number of maximal chains in any interval $[x, y]$ in P is equal to $B(\rho(x, y))$.

Hence a binomial poset P is rank uniform with

$$r_{n,k}(P) = \frac{B(n)}{B(k) \cdot B(n-k)}, \quad 0 \leq k \leq n, \tag{6.2}$$

since each maximal chain in $[\hat{0}, x]$, $\rho(x) = n$, passes through a unique element of rank k.

Theorem 6.4. *Let* P *be a binomial poset with factorial function* B, *and let*

$$b(z) = \sum_{n=0}^{\infty} \frac{z^n}{B(n)}.$$

The generating functions for the various Chow-polynomials of P have the following expressions.

$$\sum_{n=0}^{\infty} d_n(t)\frac{z^n}{B(n)} = \frac{1-t}{b(tz)-tb(z)}, \qquad \sum_{n=0}^{\infty} H_n(t)\frac{z^n}{B(n)} = \frac{(1-t)b(z)}{b(tz)-tb(z)}.$$

Example 6.5. Let $\mathbb{F}_q$ be a field with q elements, and let $V(q)$ be the free $\mathbb{F}_q$-linear space over the set $\{e_1, e_2, \ldots\}$. The *projective geometry* $\mathbb{B}(q)$ is the lattice of all finite dimensional subspaces of $V(q)$. Then $\mathbb{B}(q)$ is binomial with factorial function given by

$$(\mathbf{n})! = 1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-1}).$$

By Theorem 6.4,

$$\sum_{n=0}^{\infty} H_n(t)\frac{z^n}{(\mathbf{n})!} = \frac{(1-t)e_q(z)}{e_q(tz)-te_q(z)},$$

where $e_q(z) = \sum_{n=0}^{\infty} z^n/(\mathbf{n})!$ is the *q-exponential function*.

For $\sigma \in \mathfrak{S}_n$, let $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{D}(\sigma)} i$. From the work of Shareshian and Wachs [26] it follows that

$$H_n(t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)},$$

the q-analog, $A_n(q,t)$, of the Eulerian polynomial studied in [26]. This was first proved by Hameister, Rao and Simpson [22].

Ehrenborg and Readdy generalized the notion of binomial posets in [16]. A poset P is called a *Sheffer poset* if there are two functions $B : \mathbb{N} \to \mathbb{N}$ and $C : \mathbb{N} \to \mathbb{N}$ such that P is locally finite with a least element $\hat{0}$, P contains an infinite chain, each interval in P is graded, the number of maximal chains in $[\hat{0}, x]$ is equal to $C(\rho(x))$, for each $x \in P$, and if $\hat{0} < x \le y$, then the number of maximal chains in $[x, y]$ is equal to $B(\rho(x,y))$.

The functions B and C are called the *factorial functions* associated to P. Hence Sheffer posets are rank uniform, and binomial posets are the Sheffer posets for which $B = C$.

Theorem 6.6. *Let P be a Sheffer poset with factorial functions B and C, and let*

$$b(z) = \sum_{n=0}^{\infty} \frac{z^n}{B(n)} \quad \text{and} \quad c(z) = \sum_{n=0}^{\infty} \frac{z^n}{C(n)}.$$

Then

$$\sum_{n=0}^{\infty} d_n(t)\frac{z^n}{C(n)} = 1 + \frac{1-t}{b(tz)-tb(z)} - \frac{c(tz)-tc(z)}{b(tz)-tb(z)},$$

$$\sum_{n=0}^{\infty} H_n(t)\frac{z^n}{C(n)} = \frac{(1-t)b(z)}{b(tz)-tb(z)} + \frac{c(z)\cdot b(tz) - c(tz)\cdot b(z)}{b(tz)-tb(z)}.$$

Example 6.7. The *affine geometry* $\mathbb{A}(q)$, ordered by inclusion, may be defined as $\mathbb{A}(q) = \{U \setminus H : U \in \mathbb{B}(q)\}$, where H is the subspace of $V(q)$ spanned by $\{e_2, e_3, \ldots\}$. The poset $\mathbb{A}(q)$ is Sheffer (see [24, Proposition 6.2.5]). The factorial functions of $\mathbb{A}(q)$ are $B(n) = (\mathbf{n})!$ and $C(n) = q^{n-1} \cdot (\mathbf{n-1})!$. Then

$$\sum_{n\geq 0} H_{n+1}(t)\frac{z^n}{q^n(\mathbf{n})!} = \frac{e_q(z/q)\cdot e_q(tz) - te_q(tz/q)\cdot e_q(z)}{e_q(tz) - te_q(z)}.$$

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