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Dimension of the GL_N skein module of the mapping torus

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Abstract. We calculate the dimensions of the GL_N skein module of the mapping torus M_γ, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We collect the results into a generating function over N.

Keywords: skein algebra, skein module, mapping torus

1 Introduction

The G-skein module of a closed, oriented 3-manifold M, denoted by $\mathrm{Sk}_G(M)$ is a vector space formed as the span of framed labelled graphs in M, modulo local *skein relations*. The labels, and the skein relations, are modelled on the category $\mathrm{Rep}_q(G)$ of finite dimensional modules of the quantum group associated to the reductive algebraic group G. It was conjectured by Witten, and then proved in [4], that the skein module is finite dimensional for generic values of the quantum parameter q.

To date, few skein module dimensions are known and it is an important question to compute more examples. The case $G = \mathrm{GL}_1 = \mathbb{C}^\times$ is known by [11]. In the case $G = \mathrm{SL}_2$, the product manifolds $M = \Sigma_g \times S^1$, where Σ_g is a genus g surface, and the mapping tori $M_\gamma = T^2 \times_\gamma S^1$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ are known thanks to [3] and [8], respectively. Moving beyond $\mathbb{C}^\times$ and SL_2, the only known dimensions are for $G = \mathrm{SL}_N, \mathrm{GL}_N$ and for the 3-torus $M = T^3$ [5, 6].

In this paper we determine the dimension of the GL_N-skein module of the mapping torus $M_\gamma = T^2 \times_\gamma S^1$, for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and for all $N \geq 1$. We collect the dimensions over all N together into a single generating function, which we call the *skein partition function*.

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Definition 1.1. The **skein partition function** of a closed oriented 3-manifold M is the generating function,

$$\mathcal{Z}_M(t) = 1 + \sum_{N=1}^{\infty} \dim \mathrm{Sk}_{\mathrm{GL}_N}(M) \cdot t^N. \tag{1.1}$$

Our main result is Theorem 3.7, where we provide a closed product formula for $\mathcal{Z}_{M_\gamma}(t)$ of the form $\prod_{k=1}^{\infty}(1-t^k)^{-c_k(\gamma)}$, where the $c_k(\gamma)$ are non-negative integers depending on γ and are given in Figure 1.

This extended abstract summarizes the results in [1], and more complete proofs and details can be found there.

2 Preliminaries

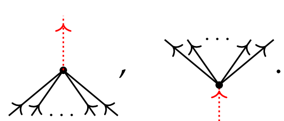
2.1 GL_N skein theory

Throughout the paper, $\mathcal{K}$ denotes the field $\mathbb{C}(q)$. Let us denote the quantum integers and quantum factorials by

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}}, \qquad [N]! = [N] \cdot [N-1] \cdots [1].$$

Skein modules were originally introduced by [10], [13]. Skein theory can be developed for arbitrary ribbon categories. One can recover the following definitions, when considering the ribbon category $\mathrm{Rep}_q(\mathrm{GL}_N)$. For a discussion of the general theory, we refer the interested reader to [14]. First, we introduce the notion of a GL_N-skein.

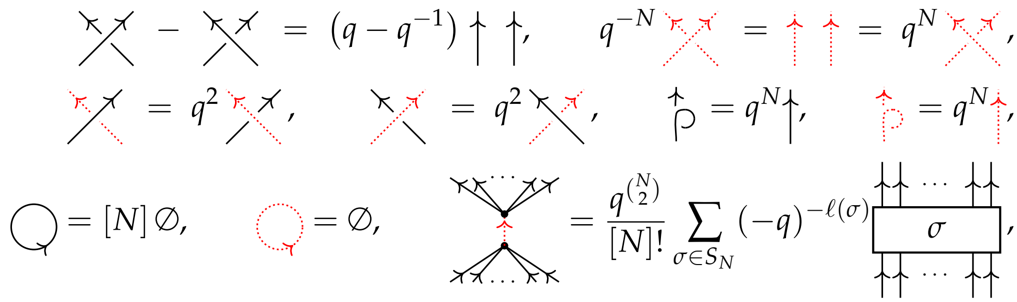
Definition 2.1. Let M be an oriented 3-manifold. We define a **basic** GL_N**-skein** in M to be an oriented, embedded ribbon graph, where the edges are either solid (black) or dashed (red) and where each vertex is one of the following $(N+1)$-valent vertices



When we draw GL_N-skeins on the page without indicating a framing, we assume blackboard framing; that is, the framing vector is orthogonal to the "board." If you prefer to picture the skeins as ribbons, the framing vector points along the width of the ribbon.

Proposition/Definition 2.2. *Let M be an oriented, compact 3-manifold. The* GL_N***-skein module*** $\mathrm{Sk}_{\mathrm{GL}_N}(M)$*, is defined here as the $\mathcal{K}$-module generated by the isotopy classes of basic* GL_N-

skeins embedded in M, modulo the following skein relations



(2.1)

applied in embedded cylinders $D^2 \times I \hookrightarrow M$.

The crossing relations come from R-matrices, the next relations from the ribbon structure, and the constants that the closed loops yield from computing quantum dimension. The last relation, in which $\ell(\sigma)$ denotes Coxeter length, captures the projection from the N^{th} exterior product of the defining representation to the determinant representation.

Remark 2.3. The GL_N-skein module defined here can easily be shown to be equivalent to: (a) the special case $\mathcal{A} = \mathrm{Rep}_q(\mathrm{GL}_N)$ of the general definition as in [14], and (b) the evident GL_N-modification of SL_N-webs from [12], and SL_N-spiders from [2].

Definition 2.4. Let Σ be an oriented, compact surface. The $\mathbf{GL}_N$**-skein category of** Σ, denoted $\mathrm{SkCat}_{\mathrm{GL}_N}(\Sigma)$ is the $\mathcal{K}$-linear category with

- objects: finite unions of oriented, framed points on Σ, colored by either "black solid" or "red dashed". This data is also called a labelling of Σ.
- morphisms space: the $\mathcal{K}$-vector space spanned by isotopy classes of GL_N-skeins in $M = \Sigma \times I$ compatible with the labelling of the source and target objects, modulo the skein relations (2.1) applied in embedded cylinders $D^2 \times I \hookrightarrow \Sigma \times I$.

The composition is defined by stacking in the I-direction.

Definition 2.5. The endomorphism algebra $\mathrm{End}_{\mathrm{SkCat}_{\mathrm{GL}_N}(\Sigma)}(\varnothing)$ for the empty labeling $\varnothing$, is called the $\mathbf{GL}_N$**-skein algebra of** Σ and is denoted by $\mathrm{SkAlg}_{\mathrm{GL}_N}(\Sigma)$.

2.2 Symmetric group conventions

For a permutation $w \in S_N$, we write $\lambda = 1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \ldots N^{\mathrm{r}_N}$ to denote its cycle type, that is w is the product of $\mathrm{r}_k = \mathrm{r}_k(w)$ disjoint cycles of length k. When w or λ is fixed, we merely write r_k to lighten notation. We write $[w]$ or $[1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \ldots N^{\mathrm{r}_N}]$ to denote the conjugacy class with representative w, and we denote by $\mathrm{cl}(S_N)$ the set of conjugacy classes of S_N. When we need to choose a specific representative w of the class $[1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \ldots N^{\mathrm{r}_N}]$, we will take for simplicity the *natural* one. That is, we write w as a product of $r = \sum_j r_j(w)$ disjoint cycles by writing the numbers $12 \cdots N$ and inserting r pairs of parenthesis in the obvious way.

See Example 2.6 below. In (2.1), $\ell(w)$ denotes the Coxeter length of w. For w of cycle type k^{r}, its centraliser is isomorphic to the wreath product $Z_{S_{k\mathrm{r}}}(w) \cong C_k \wr S_{\mathrm{r}}$. For a general element w of cycle type $1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \ldots N^{\mathrm{r}_N}$, its centraliser satisfies $Z_{S_N}(w) \cong \prod_{k=1}^{N} C_k \wr S_{\mathrm{r}_k}$.

Example 2.6. The conjugacy class $[1^3 3^2]$ in S_9 has $r_1 = 3, r_3 = 2$ and $r_j = 0$ otherwise. We have not written the parts j^0 when $r_j = 0$. This conjugacy class has natural representative given by $w = (1)(2)(3)(4,5,6)(7,8,9)$. The centraliser of w in S_9 is isomorphic to $S_3 \times (C_3 \wr S_2)$, generated by $\{\underbrace{(1,2),(2,3)}_{S_3}, \underbrace{(4,5,6)}_{C_3}, \underbrace{(7,8,9)}_{C_3}, \underbrace{(4,7)(5,8)(6,9)}_{S_2}\}$. In the notation of Theorem 3.3 below, $w_1 = (1)(2)(3)$ and $w_3 = (4,5,6)(7,8,9)$.

2.3 Mapping torus and $\mathrm{SL}_2(\mathbb{Z})$

A standard reference for this section is [7, Section 2.2].

Definition 2.7. The **mapping torus** of the diffeomorphism γ of the torus T^2 is defined as

$$M_\gamma := T^2 \times I \big/ (x,0) \sim (\gamma(x),1) \, .$$

Note that up to homeomorphism, M_γ depends only on the representative in the mapping class group $\mathrm{SL}_2(\mathbb{Z})$, and that moreover $M_\gamma, M_{\gamma'}$, for $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ are orientation-preserving diffeomorphic if and only if γ and γ' are conjugate in $\mathrm{SL}_2(\mathbb{Z})$.

The group $\mathrm{SL}_2(\mathbb{Z})$ can be generated by the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. With respect to these generators, $\mathrm{SL}_2(\mathbb{Z})$ has corresponding presentation

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = \mathrm{Id}, (ST)^3 = S^2 \rangle.$$

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we say that γ is **hyperbolic** if $|\operatorname{tr}(\gamma)| > 2$. If γ is not hyperbolic then it is conjugate in $\mathrm{SL}_2(\mathbb{Z})$ to one of the following: $\pm \mathrm{Id}, \pm TS, \pm (TS)^{-1}, \pm S, \pm T^m$ for $m \in \mathbb{Z}$. See Figure 1 for an overview of our results for all possible conjugacy classes of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Note that above we have written Id for Id_2 and from now on, if not specified, $\mathrm{Id} = \mathrm{Id}_2$.

2.4 Skein module of the mapping torus via Hochschild homology

Because $\mathrm{SkAlg}_{\mathrm{GL}_N}(T^2) = \mathrm{End}_{\mathrm{SkCat}_{\mathrm{GL}_N}(T^2)}(\varnothing)$, the skein algebra (considered as category with one object) embeds into the skein category $\mathrm{SkCat}_{\mathrm{GL}_N}(T^2)$. In [6], it was shown that this inclusion induces a Morita equivalence. We have isomorphisms,

$$\mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma) \cong \mathrm{HH}_0^\gamma(\mathrm{SkAlg}_{\mathrm{GL}_N}(T^2)), \qquad \mathrm{SkAlg}_{\mathrm{GL}_N}(T^2) \cong (A_N)^{S_N}, \tag{2.2}$$

where the first follows from the Morita invariance of Hochschild homology, and the second was proved in [5]. A_N denotes the algebra, also known as the quantum torus, defined as follows.

Let $\bullet$ denote the dot product on $\mathbb{Z}^{\oplus N}$. Let ω be the nondegenerate skew pairing on $(\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$ given by

$$\omega(\mathbf{u}, \mathbf{u}') = (j_1, \dots, j_N) \bullet (\ell'_1, \dots, \ell'_N) - (\ell_1, \dots, \ell_N) \bullet (j'_1, \dots, j'_N),$$

for $\mathbf{u} = (j_1, \ell_1 \dots, j_N, \ell_N)$, $\mathbf{u}' = (j'_1, \ell'_1 \dots, j'_N, \ell'_N)$ in $(\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$.

Definition 2.8. Let $A_N = \mathcal{K}_\omega[(\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}]$ denote the twisted group algebra, whose underlying vector space is the $\mathcal{K}$-span of $m_{\mathbf{u}}$, for $\mathbf{u} \in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$, and with multiplication

$$m_{\mathbf{u}} m_{\mathbf{v}} = q^{\frac{1}{2}\omega(\mathbf{u},\mathbf{v})} m_{\mathbf{u}+\mathbf{v}}.$$

The group S_N acts diagonally on $m_{\mathbf{u}} \in A_N$ by permuting the j_i's and ℓ_i's simultaneously in $\mathbf{u} = (j_1, \ell_1 \dots, j_N, \ell_N)$, i.e., $w(m_{\mathbf{u}}) = m_{w(\mathbf{u})}$ for $w \in S_N$. Note that ω is S_N-invariant. For $w \in S_N$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ one can define their conjoint action on A_N by multiplication by the tensor product $w \otimes \gamma$ on $\mathbb{Z}^{\oplus N} \otimes_{\mathbb{Z}} \mathbb{Z}^{\oplus 2} \simeq (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$.

Because S_N acts on A_N, we can form the smash product algebra $A_N \# S_N$. When N is clear from context, to lighten notation we merely write A for A_N.

In [6], and then in [8] for the twisted version, the authors employ the isomorphism

$$\mathrm{HH}_0^\gamma(A^{S_N}) \cong \mathrm{HH}_0^\gamma(A \# S_N) \cong \bigoplus_{[w] \in \mathrm{cl}(S_N)} \mathrm{HH}_0(A, A_{w\gamma})_{Z_{S_N}(w)}, \tag{2.3}$$

where the right hand side is defined as follows. Given $w \in S_N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the action of $w \otimes \gamma$ allows us to define the A-bimodule $A_{w\gamma}$ having the same underlying space A, on which A acts on the left in the usual way, and on the right *twisted* by $w \otimes \gamma$. The zero$^{\text{th}}$ Hochschild homology of A with coefficients in the twisted bimodule $A_{w\gamma}$ is then given by

$$\mathrm{HH}_0(A, A_{w\gamma}) = A_{w\gamma} \Big/ N_{w,\gamma}, \tag{2.4}$$

where we quotient by the *subspace of twisted commutators*

$$N_{w,\gamma} := \mathcal{K}\{m_{\mathbf{u}} m_{\mathbf{v}} - m_{\mathbf{v}} m_{w \otimes \gamma(\mathbf{u})} \mid \mathbf{u}, \mathbf{v} \in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}\}. \tag{2.5}$$

The sum in (2.3) is over conjugacy classes and the subscript $Z_{S_N}(w)$ indicates that we take coinvariants with respect to the centraliser. Implicit in the statement is the independence of choice of representative of $[w]$.

Notation 2.9. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $w \in S_N$, and $\mathbf{v} \in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$, we perform the same rescaling of generators of $\mathrm{HH}_0(A, A_{w\gamma})$ as in [6]:

$$\widetilde{m}_{\mathbf{v}} := q^{\frac{1}{2}\omega((\mathrm{Id}_{2N} - w\otimes\gamma)^{-1}\mathbf{v},\mathbf{v})} m_{\mathbf{v}},$$

which is well defined even when $\mathrm{Id}_{2N} - w \otimes \gamma$ is not invertible because we quotient by $N_{w,\gamma}$, see [6] for a proof, or its modification in [8].

After rescaling, the twisted commutator relations (2.5) are proportional to $\widetilde{m}_{\mathbf{u}} - \widetilde{m}_{\mathbf{u}+(\mathrm{Id}_{2N} - w\otimes\gamma)\mathbf{v}}$, hence as vector spaces we have the isomorphism

$$\mathrm{HH}_0(A, A_{w\gamma}) \cong \mathcal{K}[\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}}], \tag{2.6}$$

first introduced in [6] and then extended in [8]. This allows us to focus on the exponent vectors $\in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$ and use tools from linear algebra and finitely generated abelian groups. Observe that any $\sigma \in Z_{S_N}(w)$ commutes with $\mathrm{Id}_{2N} - w \otimes \gamma$, so the quotient $\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}}$ carries an action of $Z_{S_N}(w)$. Further, since the skew pairing ω is S_N-invariant we have that for all $\sigma \in Z_{S_N}(w)$, $\sigma(\widetilde{m}_{\mathbf{u}}) = \widetilde{m}_{\sigma(\mathbf{u})}$ in $\mathrm{HH}_0(A, A_{\gamma w})$. This allows us to reduce the computation of coinvariants to one of orbit-counting.

3 Dimension of the skein module

In this section, we compute the dimension of $\mathrm{HH}_0(A, A_{w\gamma})_{Z_{S_N}(w)}$ for $w \in S_N$. In order to do so, we will first use equation (2.6) and compute the image of $\mathrm{Id}_{2N} - w \otimes \gamma$ in $(\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}$, and consequently its cokernel. Then, we will study the action of $Z_{S_N}(w)$ on $\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}}$, starting with the case that $w = (1 \cdots k)$ is a single k-cycle. In this case, it reduces to understanding $\mathrm{coker}(\mathrm{Id}_2 - \gamma^k)_{\mathrm{tors}}$ with γ acting by multiplication. Finally, by counting the number of orbits of the action by $Z_{S_N}(w)$ for any w, we will give an explicit formula for the dimension of $\mathrm{HH}_0(A, A_{w\gamma})_{Z_{S_N}(w)}$.

The first step is to compute the image of $\mathrm{Id}_{2N} - w \otimes \gamma$ via column reduction. We find we need only compute the image of the 2×2 matrices $\mathrm{Id} - \gamma^k$ for appropriate k.

Lemma 3.1. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $w \in S_N$ have cycle type $1^{\mathrm{r}_1}2^{\mathrm{r}_2}\dots N^{\mathrm{r}_N}$. We have an isomorphism of abelian groups,*

$$\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma) \cong \bigoplus_{k=1}^{N} (\mathrm{coker}(\mathrm{Id} - \gamma^k))^{\oplus \mathrm{r}_k}.$$

Consequently combining equation (2.6) with taking torsion and linearizing, we have:

$$\mathrm{HH}_0(A, A_{w\gamma}) \cong \mathcal{K}[\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}}] \cong \mathcal{K}[\bigoplus_{k=1}^{N} [(\mathrm{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}})^{\oplus \mathrm{r}_k}].$$

Proof. By the independence of conjugacy class representative, we may choose the natural representative w as in Section 2.2. Then the matrix $\mathrm{Id}_{2N} - w \otimes \gamma$ is a diagonal block matrix,

$$\mathrm{Id}_{2N} - w \otimes \gamma = \mathrm{diag}(Q_{1,\mathrm{r}_1}, Q_{2,\mathrm{r}_2}, \ldots, Q_{n,\mathrm{r}_n}),$$

consisting of blocks Q_{k,r_k} defined by

$$Q_{k,\mathrm{r}_k} := \mathrm{diag}(\underbrace{P_k, P_k, \ldots, P_k}_{\mathrm{r}_k \ \text{times}}), \quad P_k := \mathrm{Id}_{2k} - (1 \cdots k) \otimes \gamma, \quad 1 \le k \le n, \tag{3.1}$$

and $P_1 = \mathrm{Id} - \gamma$. Hence, we have isomorphisms

$$\mathrm{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}} \cong \bigoplus_{k=1}^{N} \mathrm{coker}(Q_{k,\mathrm{r}_k})_{\mathrm{tors}} \cong \bigoplus_{k=1}^{N} \mathrm{coker}(P_k)_{\mathrm{tors}}^{\oplus \mathrm{r}_k}. \tag{3.2}$$

We note that, for $1 < k \le N$,

$$P_k = \begin{pmatrix} \mathrm{Id} & & & & -\gamma \\ -\gamma & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \mathrm{Id} & \\ & & & -\gamma & \mathrm{Id} \end{pmatrix} \underset{\mathrm{col}}{\cong} \begin{pmatrix} \mathrm{Id} & & & & \\ -\gamma & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \mathrm{Id} & \\ & & & -\gamma & \mathrm{Id} - \gamma^k \end{pmatrix} \tag{3.3}$$

where the equivalence is obtained by elementary column operations. This yields isomorphisms $\mathrm{coker}(P_k) \cong \mathrm{coker}(\mathrm{Id} - \gamma^k)$ and $\mathrm{coker}(P_k)_{\mathrm{tors}} \cong \mathrm{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}$, with $P_k \in \mathrm{End}((\mathbb{Z} \oplus \mathbb{Z})^{\oplus k})$, $\mathrm{Id} - \gamma^k \in \mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$. □

Notation 3.2. We define $c_k(\gamma)$ to be the number of orbits of $Z_{S_k}((1 \cdots k))$ acting on $\mathrm{coker}(\mathrm{Id}_{2k} - (1 \cdots k) \otimes \gamma)_{\mathrm{tors}}$. When γ is fixed, to lighten notation we will often write c_k for $c_k(\gamma)$.

Theorem 3.3. *Let $w \in S_N$ have cycle type $1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \ldots N^{\mathrm{r}_N}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then*

$$\dim(\mathrm{HH}_0(A, A_{w\gamma})_{Z_{S_N}(w)}) = \prod_{k=1}^{N} \binom{c_k(\gamma) + \mathrm{r}_k - 1}{\mathrm{r}_k}.$$

Proof. Since the space of coinvariants is defined by

$$\mathrm{HH}_0(A, A_{w\gamma})_{Z_{S_N}(w)} = \mathrm{HH}_0(A, A_{w\gamma}) \Big/ \mathcal{K}\{\widetilde{m}_{\mathbf{v}} - \widetilde{m}_{\sigma(\mathbf{v})} \mid \sigma \in Z_{S_N}(w), \mathbf{v} \in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus N}\}$$

its dimension is the number of orbits of the action of $Z_{S_N}(w)$ on $\operatorname{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}}$. Let w be the natural representative of $[1^{\mathrm{r}_1} 2^{\mathrm{r}_2} \dots N^{\mathrm{r}_N}]$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be fixed. Let w_k denote the product of all the disjoint cycles of w of length k, viewed as element of $S_{k\,\mathrm{r}_k}$; thus w_k has cycle type k^{r_k}. The centraliser of w in S_n satisfies $Z_{S_N}(w) \cong \prod_{k=1}^{N} Z_{S_{k\,\mathrm{r}_k}}(w_k)$. Note that the centraliser $Z_{S_N}(w)$ only permutes entries from the cycles of w that have the same lengths. Therefore by (3.2) the set of orbits, denoted $\operatorname{coker}(\mathrm{Id}_{2N} - w \otimes \gamma)_{\mathrm{tors}} \big/ Z_{S_N}(w)$, is in bijection with the Cartesian product of orbits $\prod_{k=1}^{N} \left(\operatorname{coker}(Q_{k,\mathrm{r}_k})_{\mathrm{tors}} \big/ Z_{S_{k\,\mathrm{r}_k}}(w_k) \right)$. Recall that in this notation $c_k = \#(\operatorname{coker}(P_k)_{\mathrm{tors}} / C_k)$. Thus, after a standard stars and bars counting argument, one finds that the total number of orbits is equal to $\binom{c_k + \mathrm{r}_k - 1}{\mathrm{r}_k}$. □

As a result of Theorem 3.3, it suffices to study the action of $Z_{S_k}((1 \cdots k)) = \langle (1 \cdots k) \rangle \cong C_k$ on $\operatorname{coker}(P_k)_{\mathrm{tors}}$. We carry out this analysis below.

We have reduced computing $\mathrm{HH}_0(A, A_{w\gamma})$ to a computation involving 2×2 matrices. To compute the invariants with respect to $Z_{S_N}(w)$ we need to see how this action interacts with the reduction. This is achieved in Lemma 3.4. Taking $w = (1 \cdots k)$, we regard $\mathcal{K}[\operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}]$ as a module for $Z_{S_k}(w)$ via the surjective group homomorphism given by $w \mapsto \bar{\gamma}$ where we have written $\bar{\gamma}$ for its coset in $\langle \gamma \rangle / \langle \gamma^k \rangle$ and where γ, and hence $\bar{\gamma}$, acts on $\operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}$ by multiplication.

Lemma 3.4. *For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $w = (1 \cdots k) \in S_k$, the natural action of $\langle \gamma \rangle$ on* $\operatorname{coker}(\mathrm{Id} - \gamma^k)$ *yields that the isomorphism*

$$\mathrm{HH}_0(A, A_{w\gamma}) \cong \mathcal{K}[\operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}],$$

of Lemma (3.1) intertwines the natural $Z_{S_k}(w)$ action on the left hand side with the $Z_{S_k}(w)$ action on the right hand side as constructed above.

Proof. Starting with Notation 2.9, for $\mathbf{v} \in \mathbb{Z} \oplus \mathbb{Z}$, we let $\widetilde{m}_i^{\mathbf{v}} := \widetilde{m}_{\mathbf{v}_i}$, with $\mathbf{v}_i \in (\mathbb{Z} \oplus \mathbb{Z})^{\oplus k}$ having all components being 0 except in the i^{th} component (that is the (2i-1)$^{\mathrm{st}}$ and 2i$^{\mathrm{th}}$ coordinates), where it is equal to $\mathbf{v}$, i.e., $\mathbf{v}_i = (0, 0, \dots, 0, 0, j, \ell, 0, 0, \dots, 0, 0)$ if $\mathbf{v} = (j, \ell)$.

From the reduction (3.3), we know that $\mathrm{HH}_0(A, A_{w\gamma})$ is spanned by monomials $\widetilde{m}_k^{\mathbf{v}}$, with k fixed as above, and $\mathbf{v}$ ranging over $\mathbb{Z} \oplus \mathbb{Z}$, quotiented by the subspace $\mathcal{K}\{\widetilde{m}_k^{\mathbf{v}} - \widetilde{m}_k^{\gamma^k \mathbf{v}} \mid \mathbf{v} \in \mathbb{Z} \oplus \mathbb{Z}\}$.

Now for $(1 \cdots k)^{k-r} \in Z_{S_k}((1 \cdots k)) = \langle (1 \cdots k) \rangle \subset S_k$, with $0 \le r < k$, we obtain

$$(1 \cdots k)^{k-r}(\widetilde{m}_k^{\mathbf{v}}) = \widetilde{m}_{k-r}^{\mathbf{v}}.$$

Since we already had, from the cokernel relations (2.5), that $\widetilde{m}_{k-r}^{\mathbf{v}} = \widetilde{m}_{k-r+1}^{\gamma \mathbf{v}} = \cdots = \widetilde{m}_k^{\gamma^r \mathbf{v}}$, we now have that the action of the centraliser agrees with

$$(1 \cdots k)^{k-r}(\widetilde{m}_k^{\mathbf{v}}) = \widetilde{m}_k^{\gamma^r \mathbf{v}}, \quad 0 \le r < k. \tag{3.4}$$

□

Lemma 3.5. *If* $\det(\mathrm{Id} - \gamma^k) \neq 0$*, then for* $1 \leq p < k$*, the number of fixed points of* γ^p *on* $\operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}$ *is*

$$\# \operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}^{\gamma^p} = |\det(\mathrm{Id} - \gamma^d)|.$$

with $d = \gcd(k, p)$ *or* $d = \gcd(k, p, m)$ *in the case* γ *has finite order* m*.*

Proof. We consider γ of infinite order, respectively finite order m. Let $\bar{\gamma}^p$ denote the elements of the finite cyclic group $\langle \gamma \rangle / \langle \gamma^k \rangle = \langle \bar{\gamma} \rangle$. Note the order of $\bar{\gamma}$ is k, respectively $m / \gcd(k, m)$.

We have an equality of the cyclic subgroups $\langle \bar{\gamma}^p \rangle = \langle \bar{\gamma}^d \rangle$ exactly when $\gcd(k, p) = \gcd(k, d)$, (respectively $\gcd(k, p, m) = \gcd(k, d, m)$); and note that any fixed point of $\bar{\gamma}^p$ will also be a fixed point of $\bar{\gamma}^{rp}$ for $r \in \mathbb{Z}$. Thus we may assume that $p = d$, which divides k (respectively divides m and k).

When $\det(\mathrm{Id} - \gamma^k) \neq 0$, $\operatorname{coker}(\mathrm{Id} - \gamma^k) = \operatorname{coker}(\mathrm{Id} - \gamma^k)_{\mathrm{tors}}$ and $\mathrm{Id} - \gamma^k$ and $\mathrm{Id} - \gamma^d$ are both endomorphisms of $\mathbb{Z}^2$ with finite cokernel, i.e. whose images are of finite index. Let $L = \mathrm{Im}(\mathrm{Id} - \gamma^d) \subseteq \mathbb{Z}^2$ and $M = \mathrm{Im}(\mathrm{Id} - \gamma^k) \subseteq \mathbb{Z}^2$. Because d divides k we have an inclusion of lattices $M \subseteq L \subseteq \mathbb{Z}^2$. Observe $\operatorname{coker}(\mathrm{Id} - \gamma^k) = \mathbb{Z}^2 / M$. We compute the index of these sublattices

$$[\mathbb{Z}^2 : L] = |\det(\mathrm{Id} - \gamma^d)|, \qquad [\mathbb{Z}^2 : M] = |\det(\mathrm{Id} - \gamma^k)| = [\mathbb{Z}^2 : L][L : M].$$

The First Isomorphism Theorem applied to multiplication by $\mathrm{Id} - \gamma^d$ shows the number of fixed points for γ^d, and in general for γ^p with $\gcd(k, p) = d$, (respectively $\gcd(k, p, m) = d$) is $|\det(\mathrm{Id} - \gamma^d)|$, and the Lemma follows. □

Theorem 3.6. *Suppose that* $\mathrm{tr}(\gamma) \neq \pm 2$*, and* $\gamma^k \neq \mathrm{Id}$*. Then the following formula holds:*

$$c_k(\gamma) = \frac{1}{k} \sum_{d|k} \phi(\frac{k}{d}) |\operatorname{tr}(\gamma^d) - 2|. \tag{3.5}$$

where ϕ *is the Euler's totient function.*

The remaining special cases occur when γ *is conjugate to a shear* T^m*, its negative* $-T^m$*, or when* γ *is of finite-order with* $\gamma^k = \mathrm{Id}$*. In those cases, we have the following expressions,*

$$c_k(\gamma) = \begin{cases} 1, & \text{if } \gamma^k = \mathrm{Id}, \\ |m|k, & \text{if } \gamma \text{ is conjugate to } T^m \text{ for } m \neq 0, \\ |m|\frac{k}{2} + 1, & \text{if } \gamma \text{ is conjugate to } -T^m \text{ for } m \neq 0 \text{ , and } k \text{ is even}, \\ 4, & \text{if } \gamma \text{ is conjugate to } -T^m \text{ and } k \text{ is odd}. \end{cases}$$

Proof. By Burnside's Lemma, the average number of fixed points of $\langle\gamma\rangle/\langle\gamma^k\rangle$ acting on $\operatorname{coker}(\mathrm{Id}-\gamma^k)_{\mathrm{tors}}$ is given by:

$$\frac{1}{k}\sum_{p=1}^{k}\#\operatorname{coker}(\mathrm{Id}-\gamma^k)_{\mathrm{tors}}^{\gamma^p}=\frac{1}{k}\sum_{d|k}\phi\left(\frac{k}{d}\right)|\det(\mathrm{Id}-\gamma^d)|.$$

The equation follows from Lemma 3.5 and the definition of ϕ. We note that the formula holds even in the case that γ has finite order m. In this case it still counts the average number of fixed points of a group of size $m/\gcd(k,m)$ repeated $k/\gcd(k,m)$ times then normalized appropriately. Note that for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, $\det(\mathrm{Id}-\gamma^d)=2-\operatorname{tr}(\gamma^d)$.

Note that if $|\operatorname{tr}(\gamma)|>2$, $k>0$ then $\det(\mathrm{Id}-\gamma^k)\neq 0$. When γ is of finite order dividing k, the matrix $\mathrm{Id}-\gamma^k$ is equal to the zero matrix. Hence, $\operatorname{coker}(\mathrm{Id}-\gamma^k)_{\mathrm{tors}}$ is trivial and there is only one orbit. We obtain

$$\dim\mathrm{HH}_0(A,A_{(1\cdots k)\gamma})=\dim\mathrm{HH}_0(A,A_{(1\cdots k)\gamma})_{Z_{S_k}((1\cdots k))}=1.$$

For the remaining cases it suffices to take γ equal to either T^m or $-T^m$. In the following, we use the notation from the proof of Lemma 3.4.

When $\gamma=T^m$, we have $\operatorname{coker}(\mathrm{Id}-\gamma^k)_{\mathrm{tors}}\cong\{\widetilde{m}_k^{(i,0)}\mid 0\le i<|m|k\}$ as sets. Each $\widetilde{m}_k^{(i,0)}$ is fixed by T, hence by equation (3.4) the action of the centraliser is trivial on $\mathrm{HH}_0(A,A_{(1\cdots k)\gamma})$. Thus

$$c_k(T^m)=\dim\mathrm{HH}_0(A,A_{(1\cdots k)T^m})_{Z_{S_k}((1\cdots k))}=\dim\mathrm{HH}_0(A,A_{(1\cdots k)T^m})=|m|k.$$

When $\gamma=-T^m$ and k is odd, we have $\det(\mathrm{Id}-\gamma^k)=4$, hence the formula (3.5) applies.

When k is even, we have $\operatorname{coker}(\mathrm{Id}-\gamma^k)_{\mathrm{tors}}\cong\{\widetilde{m}_k^{(i,0)}\mid 0\le i<|m|k\}$. However, the action by the centraliser $Z_{S_k}((1\cdots k))=\langle(12\cdots k)\rangle$ is non-trivial. Each $-T^m$ will send $\widetilde{m}_k^{(i,0)}$ to $\widetilde{m}_k^{(-i,0)}$, which is identified with $\widetilde{m}_k^{(|m|k-i,0)}$ modulo the relations (2.5). Hence, the centraliser has two singleton orbits $\{\widetilde{m}_k^{(0,0)}\}$ and $\{\widetilde{m}_k^{(|m|\frac{k}{2},0)}\}$ and $|m|\frac{k}{2}-1$ orbits of the form $\{\widetilde{m}_k^{(i,0)},\widetilde{m}_k^{(|m|k-i,0)}\}$. In conclusion, for k even, we obtain $c_k(-T^m)=|m|\frac{k}{2}+1$. □

Theorem 3.7. *Let $\gamma\in\mathrm{SL}_2(\mathbb{Z})$. The skein partition function $\mathcal{Z}_{M_\gamma}(t)$ of the mapping torus M_γ admits an Euler product expansion,*

$$\mathcal{Z}_{M_\gamma}(t)=\prod_{k=1}^{\infty}(1-t^k)^{-c_k(\gamma)},$$

where $c_k(\gamma)$ is a non-negative integer given in Theorem 3.6, and in particular is determined by $\operatorname{tr}(\gamma)$ *when* $\operatorname{tr}(\gamma)\neq 2$.

We omit the proof in the interest of space, noting it follows easily from Theorem 3.3.

Remark 3.8. We note that the trace $\mathrm{tr}(\gamma^d)$ can be expressed via Chebyshev polynomials as a polynomial of degree d in $\mathrm{tr}(\gamma)$. To illustrate this, we write out $c_k(\gamma)$ for some values of k, as a polynomial in $x := \mathrm{tr}(\gamma)$ in the case when $\mathrm{tr}(\gamma) > 2$. $c_1(x) = x - 2$, $c_2(x) = \frac{1}{2}(x^2 + x - 6)$, $c_3(x) = \frac{1}{3}(x^3 - x - 6)$, $c_4(x) = \frac{1}{4}(x^4 - 3x^2 + 2x - 8)$.

Example 3.9. The skein module dimensions for GL_N and the sequences c_k are given below, for some matrices γ.

γ	$c_k(\gamma)$ for $k = 1, 2, 3, \ldots$	$\dim \mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma)$ for $N = 1, 2, 3, \ldots$
$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$	2, 7, 18, 52, 146, 463, 1442, 4732, ...	2, 10, 36, 142, 520, 1980, 7344,...
T	1, 2, 3, 4, 5, 6, 7, 8, 9, ...	1, 3, 6, 13, 24, 48, 86, 160, 282, ...
$-T$	4, 2, 4, 3, 4, 4, 4, 5, 4, 6, 4, 7, 4, 8,...	4, 12, 32, 77, 172, 366, 744, 1460, ...
$-\mathrm{Id}$	4, 1, 4, 1, 4, 1, 4, 1, 4, ...	4, 11, 28, 63, 132, 264, 504, 928, ...

Remark 3.10. For $\gamma = T$, the dimension of the GL_N-skein module is the number of plane partitions of N. Its generating function was first presented by MacMahon in [9].

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$[\gamma]$	$\mathrm{tr}(\gamma)$	$\mathrm{ord}(\gamma)$	$\dim \mathrm{HH}_0(A, A_{(1\cdots k)\gamma})$	$c_k(\gamma)$
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $S^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	0	4	1 if $k \equiv 0 \mod 4$ 4 if $k \equiv 2 \mod 4$ 2 if $k \equiv \pm 1 \mod 4$	1 if $k \equiv 0 \mod 4$ 3 if $k \equiv 2 \mod 4$ 2 if $k \equiv \pm 1 \mod 4$
$TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ or $(TS)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	1	6	1 if $k \equiv 0, \pm 1 \mod 6$ 4 if $k \equiv 3 \mod 6$ 3 if $k \equiv \pm 2 \mod 6$	1 if $k \equiv 0, \pm 1 \mod 6$ 2 if $k \equiv \pm 2, 3 \mod 6$
$-TS = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ or $-(TS)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	-1	3	1 if $k \equiv 0 \mod 3$ 3 if $k \equiv \pm 1 \mod 3$	1 if $k \equiv 0 \mod 3$ 3 if $k \equiv \pm 1 \mod 3$
$\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	1	1	1
$-\mathrm{Id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	-2	2	1 if $k \equiv 0 \mod 2$ 4 if $k \equiv 1 \mod 2$	1 if $k \equiv 0 \mod 2$ 4 if $k \equiv 1 \mod 2$
$T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$	2	∞	$\lvert m \rvert k$	$\lvert m \rvert k$
$-T^m = \begin{pmatrix} -1 & -m \\ 0 & -1 \end{pmatrix}$	-2	∞	$\lvert m \rvert k$ if $k \equiv 0 \mod 2$ 4 if $k \equiv 1 \mod 2$	$\lvert m \rvert \frac{k}{2} + 1$ if $k \equiv 0 \mod 2$ 4 if $k \equiv 1 \mod 2$
hyperbolic	$\lvert \mathrm{tr}(\gamma) \rvert > 2$	∞	$\lvert \mathrm{tr}(\gamma^k) - 2 \rvert$	$\frac{1}{k} \sum_{d \mid k} \phi(\frac{k}{d}) \lvert \mathrm{tr}(\gamma^d) - 2 \rvert$

Figure 1: Overview of the results for all possible conjugacy classes of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

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