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The chromatic symmetric function of graphs glued at a single vertex

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Abstract. We describe how the chromatic symmetric function of two graphs glued at a single vertex can be expressed as a matrix multiplication using certain information of the two individual graphs. We then prove new e-positivity results by using a connection between forest triples, defined by the first author, and Hikita's probabilities associated to standard Young tableaux. Specifically, we prove that gluing a sequence of unit interval graphs and cycles results in an e-positive graph. We also prove e-positivity for a graph obtained by gluing the first and last vertices of such a sequence. This generalizes e-positivity of cycle-chord graphs and supports Ellzey's conjectured e-positivity for proper circular arc digraphs.

Keywords: chromatic symmetric function, e-positivity, Stanley–Stembridge conjecture

1 Introduction

The Stanley–Stembridge conjecture [15, 16] is a monumental problem in algebraic combinatorics. It states that the chromatic symmetric function $X_G(\boldsymbol{x})$ of a $(\mathbf{3}+\mathbf{1})$-free graph G is e-positive. By a reduction due to Guay-Paquet [7], it suffices to prove e-positivity whenever G is a unit interval graph. Shareshian and Wachs [13] refined this conjecture by introducing the chromatic quasisymmetric function $X_G(\boldsymbol{x};q)$, showing that it is symmetric whenever G is a unit interval graph, and conjecturing that in this case, its e-coefficients are positive polynomials in q. Hikita [10] proved the Stanley–Stembridge conjecture by giving a combinatorial formula for $X_G(\boldsymbol{x};q)$ in terms of probabilities associated to standard Young tableaux, whose summands are all nonnegative when $q=1$. The Shareshian–Wachs conjecture is still open and has seen keen interest because of its connection to Hessenberg varieties [1, 3, 4, 8, 9, 12] and Hecke algebras [5, 14]. Ellzey [6] generalized further by defining a chromatic quasisymmetric function $X_{\vec{G}}(\boldsymbol{x};q)$ for any directed graph $\vec{G}$, showing symmetry whenever $\vec{G}$ is a proper circular arc digraph, and conjecturing e-positivity in this case.

In this extended abstract, we consider the operation of gluing graphs G and H together at a single vertex. In Section 3, we show that the resulting chromatic symmetric function can be expressed as a matrix multiplication using certain information about G

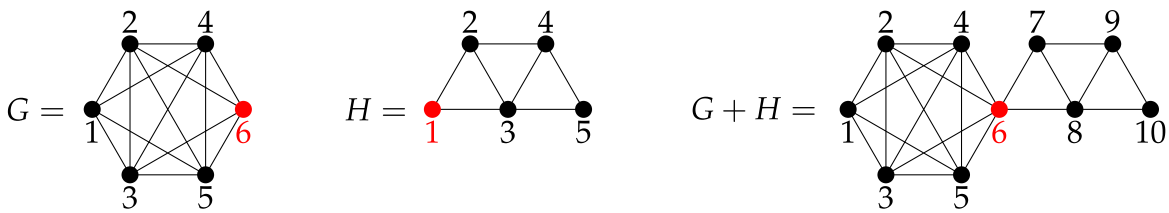


Figure 1: Graphs G and H glued at a single vertex.

and H. We also show a surprising connection between these matrices and Hikita's probabilities on standard Young tableaux (Theorem 3.10). From this connection and Hikita's remarkable breakthrough, we prove e-positivity of graphs obtained from gluing a sequence of unit interval graphs and cycles (Corollary 3.12). In Section 4, we show that if G° is obtained by gluing two vertices of a graph G, then $X_{G^\circ}(\boldsymbol{x})$ is equal to the trace of the matrix corresponding to G (Theorem 4.2). In particular, we prove e-positivity for graphs obtained by gluing the first and last vertices of a unit interval graph (Corollary 4.3). These graphs are proper circular arc graphs, lending support to Ellzey's conjecture. This also generalizes e-positivity of cycle-chord graphs (Corollary 4.6).

2 Background

All graphs in this paper will have vertex set $[n] = \{1, \ldots, n\}$ for some n. For graphs $G = ([n], E)$ and $H = ([n'], E')$, we define the *sum*

$$G + H = ([n + n' - 1], \{E \cup \{\{i + n - 1, j + n - 1\} : \{i, j\} \in E'\}).$$

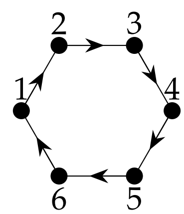
Informally, we glue vertex n of G to vertex 1 of H, as shown in Figure 1.

2.1 Chromatic symmetric functions and unit interval graphs

A *proper colouring* of $G = ([n], E)$ is a function $\kappa : [n] \to \{1, 2, 3, \ldots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $\{u, v\} \in E$. The *chromatic symmetric function* of G is [15, Definition 2.1]

$$X_G(\boldsymbol{x}) = \sum_{\kappa \text{ proper colouring}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

We say that $X_G(\boldsymbol{x})$ is *e-positive* if $X_G(\boldsymbol{x})$ has nonnegative coefficients when expanded in the basis of *elementary symmetric functions* e_λ, defined by $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where $e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$. We say that G is a *natural unit interval graph (or NUIG)* if there are unit intervals $I_1 = [a_1, a_1 + 1], \ldots, I_n = [a_n, a_n + 1] \subset \mathbb{R}$ with $a_1 < \cdots < a_n$ and $\{i, j\} \in E(G)$ if and only if $I_i \cap I_j \neq \emptyset$. Hikita proved the famous Stanley–Stembridge conjecture.



$$X_{\vec{C}_6}(\boldsymbol{x};q) = 2q^3 e_{222} + 4q^2[3]_q e_{42} + 2q^2[3]_q e_{42} + 3q^2[2]_q[2]_q e_{33} + 6q[5]_q e_6$$
$$\alpha = \quad 222 \quad 42 \quad 24 \quad 33 \quad 6$$

Figure 2: The directed cycle $\vec{C}_6$ and the chromatic quasisymmetric function $X_{\vec{C}_6}(\boldsymbol{x};q)$.

Theorem 2.1. [15, Conjecture 5.5], [10, Theorem 3] If G is an NUIG, then $X_G(\boldsymbol{x})$ is e-positive.

An *ascent* of a proper colouring κ of digraph $\vec{G} = ([n], \vec{E})$ is an edge $\{u, v\} \in \vec{E}$ where $u \to v$ and $\kappa(u) < \kappa(v)$. The *chromatic quasisymmetric function* of $\vec{G}$ is [2, Definition 8]

$$X_{\vec{G}}(\boldsymbol{x};q) = \sum_{\kappa \text{ proper colouring}} q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

Note that when $q = 1$, we get $X_{\vec{G}}(\boldsymbol{x};1) = X_G(\boldsymbol{x})$. For NUIGs, we implicitly direct every edge from the smaller to the larger vertex. Shareshian and Wachs showed that if G is an NUIG, then $X_G(\boldsymbol{x};q)$ is in fact symmetric [13, Theorem 4.5]. They conjectured that in this case, $X_G(\boldsymbol{x};q)$ is *e-positive*, meaning that its e-coefficients, which are polynomials in the variable q, have positive coefficients. This refined conjecture is still open.

Conjecture 2.2. [13, Conjecture 5.1] If G is an NUIG, then $X_G(\boldsymbol{x};q)$ is e-positive.

Example 2.3. The complete graph K_n is an NUIG and has $X_{K_n}(\boldsymbol{x};q) = [n]_q! e_n$, where

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q \text{ and } [k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}.$$

Example 2.4. The path P_n is an NUIG and has [13, Section 5]

$$X_{P_n}(\boldsymbol{x};q) = \sum_{\alpha \vDash n} [\alpha_1]_q([\alpha_2]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\operatorname{sort}(\alpha)},$$

where $\operatorname{sort}(\alpha)$ is the partition obtained by sorting the parts of α in decreasing order. Thus $X_{P_n}(\boldsymbol{x};q)$ is e-positive.

Example 2.5. The directed cycle $\vec{C}_n$ has [6, Equation 6.6]

$$X_{\vec{C}_n}(\boldsymbol{x};q) = \sum_{\alpha \vDash n} \alpha_1([\alpha_1]_q - 1)([\alpha_2]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\operatorname{sort}(\alpha)}, \tag{2.1}$$

and thus $X_{\vec{C}_n}(\boldsymbol{x};q)$ is e-positive. The example of $\vec{C}_6$ is shown in Figure 2.

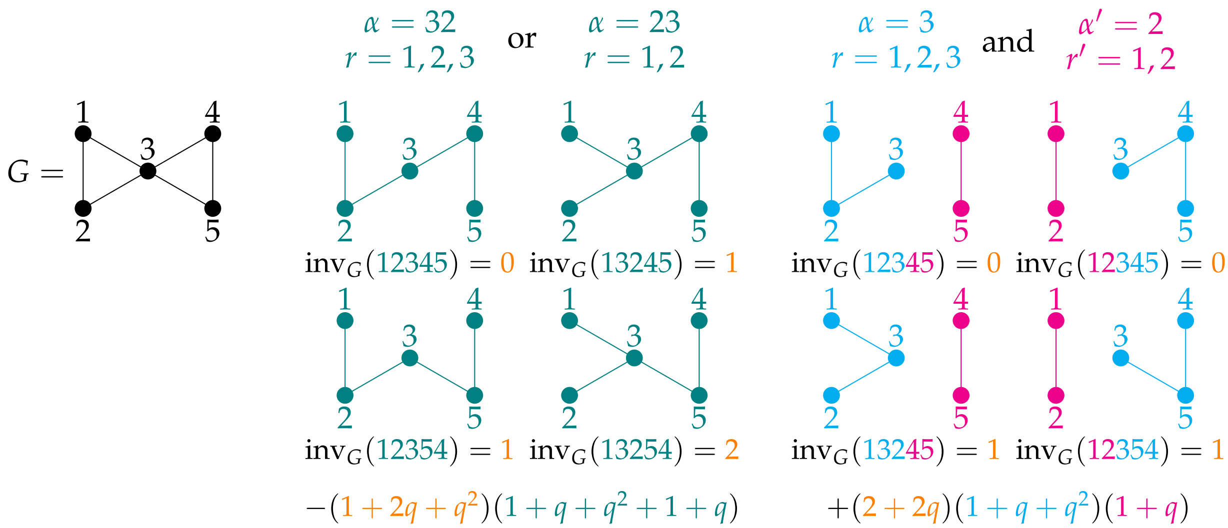


Figure 3: The bowtie graph G and the forest triples of G of type 32.

2.2 Forest triples

The first author proved a signed combinatorial formula for the e-expansion of $X_G(\boldsymbol{x})$. In the case that G is an NUIG, a similar formula is given for $X_G(\boldsymbol{x}; q)$. Let $G = ([n], E)$ be a graph and fix a total ordering $\lessdot$ on E.

Definition 2.6. A *no broken circuit (NBC) tree* of G is a subtree T whose edges do not contain a subset of the form $B = C \setminus \{\max(C)\}$, where C is a set of edges that form a cycle and $\max(C)$ is the largest edge of C under $\lessdot$.

Definition 2.7. A *tree triple of* G is an object $\mathcal{T} = (T, \alpha, r)$ consisting of the following data.

- T is an NBC tree of G.
- α is a composition of size $|T|$.
- r is an integer with $1 \le r \le \alpha_1$, the first part of α.

A *forest triple of* G is a sequence $\mathcal{F} = (\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1), \ldots, \mathcal{T}_m = (T_m, \alpha^{(m)}, r_m))$ of tree triples such that each vertex of G is in exactly one tree . Let $\mathrm{FT}(G)$ denote the set of forest triples of G. The *type* of $\mathcal{F}$ is the partition

$$\mathrm{type}(\mathcal{F}) = \mathrm{sort}(\alpha^{(1)} \cdots \alpha^{(m)}),$$

given by concatenating the compositions and sorting to make a partition. The *sign* of $\mathcal{F}$ is the integer

$$\mathrm{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^{m}(\ell(\alpha^{(i)})-1)} = (-1)^{\ell(\mathrm{type}(\mathcal{F}))-m}.$$

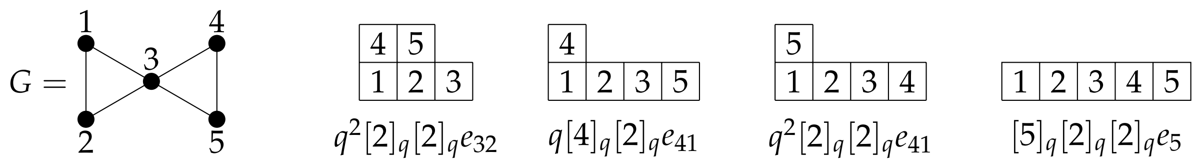


Figure 4: The bowtie graph G and the standard Young tableaux in $\mathrm{SYT}(G)$.

Now we can use forest triples to calculate the chromatic symmetric function.

Theorem 2.8. [17, Theorem 5.10] The chromatic symmetric function of G satisfies

$$X_G(\boldsymbol{x}) = \sum_{\mathcal{F} \in \mathrm{FT}(G)} \mathrm{sign}(\mathcal{F}) e_{\mathrm{type}(\mathcal{F})}.$$

For an NUIG G, we associate the integer $\mathrm{weight}(\mathcal{F}) = \mathrm{inv}_G(\sigma) + \sum_{i=1}^{m}(r_i - 1)$ to each forest triple, where σ defined in [17, Definition 3.3], which results in the following formula for $X_G(\boldsymbol{x}; q)$.

Theorem 2.9. [17, Theorem 3.4] The chromatic quasisymmetric function of an NUIG G satisfies

$$X_G(\boldsymbol{x}; q) = \sum_{\mathcal{F} \in \mathrm{FT}(G)} \mathrm{sign}(\mathcal{F}) q^{\mathrm{weight}(\mathcal{F})} e_{\mathrm{type}(\mathcal{F})}.$$

Example 2.10. Enumerating over the forest triples of type 32 for the bowtie graph $G = K_3 + K_3$ shown in Figure 3, we see that the coefficient of e_{32} in $X_G(\boldsymbol{x}; q)$ is

$$-(1+2q+q^2)(1+q+q^2+1+q) + (2+2q)(1+q+q^2)(1+q) = q^2(1+q)^2 = q^2[2]_q[2]_q.$$

2.3 Standard Young tableaux

Hikita proved a formula for the e-expansion of $X_G(\boldsymbol{x}; q)$ in terms of probabilities associated to standard Young tableaux.

Hikita's paper associates a rational function in q, $c_T(G)$, to every tableau, where $c_T(G) \geq 0$ when $q = 1$ [10, Definition 1-2]. He then proves the following equation.

Theorem 2.11. [10, Theorem 3] If G is an NUIG, then we have

$$X_G(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SYT}(G)} c_T(G) e_{\mathrm{shape}(T)}. \tag{2.2}$$

In particular, if we evaluate at $q = 1$, we have that $X_G(\boldsymbol{x})$ is e-positive.

Example 2.12. Figure 4 shows all tableaux associated to the bowtie graph with a nonzero coefficient c_T, where using Equation (2.2) results in $X_G(\boldsymbol{x}; q)$.

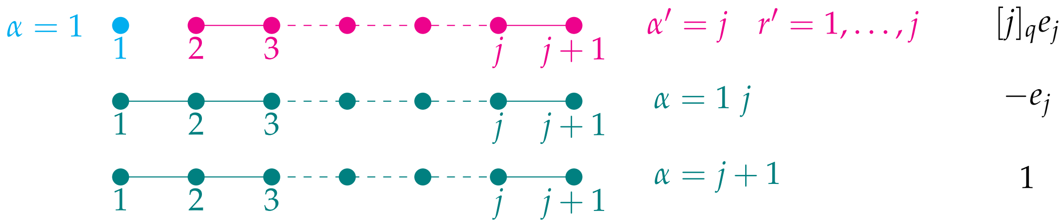


Figure 5: The forest triples used to calculate $F_{P_2}(q)$.

3 Forest triple matrix

In this section, we use forest triples to define a matrix that will allow us to calculate the chromatic symmetric function $X_{G+H}(\boldsymbol{x})$ in terms of information about G and H.

Definition 3.1. Let $G = ([n], E)$ be a graph and consider the subset of $\mathrm{FT}(G + P_j)$

$$\mathrm{FT}^{(i)}(G + P_j) = \{\mathcal{F} \in \mathrm{FT}(G + P_j) : \ \alpha_1 = i, \ r = 1, \ \{n, n+1, \dots, n+j-1\} \subseteq T', \ \alpha'_\ell \geq j\},$$

where $\mathcal{T} = (T, \alpha, r)$ is the tree triple of $\mathcal{F}$ with $1 \in T$ and $\mathcal{T}' = (T', \alpha', r')$ is the tree triple of $\mathcal{F}$ with $n \in T'$. Note that we could have $\mathcal{T} = \mathcal{T}'$.

Definition 3.2. We now define the infinite matrix F_G by

$$(F_G)_{i,j} = \frac{1}{e_i} \sum_{\mathcal{F} \in \mathrm{FT}^{(i)}(G+P_j)} \mathrm{sign}(\mathcal{F}) e_{\mathrm{type}(\mathcal{F})}.$$

If G is an NUIG, we also define the infinite matrix $F_G(q)$ by

$$(F_G(q))_{i,j} = \frac{1}{e_i} \sum_{\mathcal{F} \in \mathrm{FT}^{(i)}(G+P_j)} \mathrm{sign}(\mathcal{F}) q^{\mathrm{weight}(\mathcal{F})} e_{\mathrm{type}(\mathcal{F})}.$$

We first calculate some examples.

Example 3.3. For the graph P_1 with one vertex, $F_{P_1}(q)$ is the infinite identity matrix I.

Example 3.4. For the two-vertex path P_2, we have

$$(F_{P_2}(q))_{i,j} = \begin{cases} q[j-1]_q e_j & \text{if } i = 1, \\ 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases} \qquad F_{P_2}(q) = \begin{bmatrix} 0 & qe_2 & q[2]_q e_3 & q[3]_q e_4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

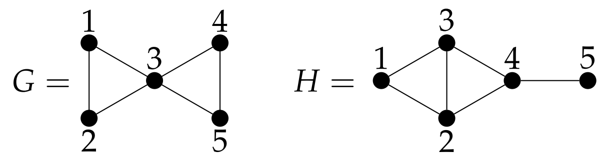


Figure 6: The bowtie graph G and the kite graph H.

Proof. We are considering forest triples in $\mathrm{FT}^{(i)}(P_2 + P_j)$. This is empty unless $i = 1$ or $i = j + 1$. These forest triples are of the form described in Figure 5, where the top two correspond to $i = 1$ and the bottom to $i = j + 1$, resulting in the given matrix. □

Although F_G is an infinite matrix, we show in [19, Proposition 3.6] that every column contains finitely many nonzero entries, allowing us to define matrix multiplication.

We can recover $X_G(\boldsymbol{x}; q)$ from $F_G(q)$ as follows.

$$X_G(\boldsymbol{x}; q) = \sum_i [i]_q (F_G(q))_{i,1} e_i. \tag{3.1}$$

In particular, if $F_G(q)$ has all e-positive entries, then $X_G(\boldsymbol{x}; q)$ is e-positive.

We now show that the operation of gluing graphs corresponds to multiplying matrices. Informally, the idea is that a forest triple of $G + H$ can be split into a forest triple of $G + P_j$ and a forest triple of H.

Proposition 3.5. [19, Proposition 3.10] For graphs G and H, we have

$$F_{G+H} = F_G F_H.$$

Moreover, if G and H are NUIGs, then $F_{G+H}(q) = F_G(q) F_H(q)$.

We can use this to calculate $F_{P_n}(q)$, because $F_{P_n}(q) = (F_{P_2}(q))^{n-1}$.

Proposition 3.6. The forest triple matrix of the path P_n is given by

$$(F_{P_n}(q))_{i,j} = \sum_{\alpha \vDash n+j-i-1,\ \alpha_\ell \geq j} ([\alpha_1]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\mathrm{sort}(\alpha)}. \tag{3.2}$$

Remark 3.7. Proposition 3.3 and Proposition 3.5 tell us that the forest triple matrix is a *monoid homomorphism* on graphs under the gluing operation.

Remark 3.8. Note that $X_G(\boldsymbol{x})$ and $X_H(\boldsymbol{x})$ are not enough to calculate $X_{G+H}(\boldsymbol{x})$. For example, the bowtie graph G and the kite graph H in Figure 6 satisfy $X_G(\boldsymbol{x}) = X_H(\boldsymbol{x})$, but $X_{G+H}(\boldsymbol{x}) \neq X_{H+H}(\boldsymbol{x})$.

$$(F_G(q))_{i,1} = \Bigg[\underset{i=1}{\overbrace{q^2[2]_q[2]_q \cdot e_4}^{\begin{array}{c}\begin{array}{|c|c|c|c|}\cline{1-1} 5 & \multicolumn{3}{c}{} \\ \hline 1 & 2 & 3 & 4 \\ \hline\end{array} \\ q^2[2]_q[2]_q \cdot \frac{e_{41}}{[1]_q \cdot e_1}\end{array}}},\ \underset{i=2}{\overbrace{q^2[2]_q \cdot e_3}^{\begin{array}{c}\begin{array}{|c|c|c|}\cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline\end{array} \\ q^2[2]_q[2]_q \cdot \frac{e_{32}}{[2]_q \cdot e_2}\end{array}}},\ \underset{i=3}{0},\ \underset{i=4}{\overbrace{q[2]_q \cdot e_1}^{\begin{array}{c}\begin{array}{|c|c|c|c|}\cline{1-1} 4 & \multicolumn{3}{c}{} \\ \hline 1 & 2 & 3 & 5 \\ \hline\end{array} \\ q[4]_q[2]_q \cdot \frac{e_{41}}{[4]_q \cdot e_4}\end{array}}},\ \underset{i=5}{\overbrace{[2]_q[2]_q}^{\begin{array}{c}\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array} \\ [5]_q[2]_q[2]_q \cdot \frac{e_5}{[5]_q \cdot e_5}\end{array}}},\ 0, \ldots\Bigg]^T$$

Figure 7: The tableau corresponding to the first column of the bowtie graph G.

In previous work, the authors found sign-reversing involutions on forest triples. This allows us to prove e-positivity of forest triple matrices.

Proposition 3.9. [17, Theorem 4.10], [18, Theorem 6.5] The matrices $F_{K_n}(q)$ and F_{C_n} have all e-positive entries.

We also proved a connection between our forest triple matrix and Hikita's tableaux.

Theorem 3.10. [19, Theorem 4.9] If G is an NUIG, then

$$(F_G(q))_{i,j} = \frac{1}{[i]_q[j-1]_q! \cdot e_i} \sum_{T \in \mathrm{SYT}^{(i)}(G+K_j)} c_T(G+K_j) e_{\mathrm{shape}(T)},$$

where $\mathrm{SYT}^{(i)}(G+K_j)$ is the set of tableaux where the largest entry $(n+j-1)$ is in column i. In particular, when $q=1$, all entries of F_G are e-positive.

Example 3.11. Figure 7 shows the tableau associated with the first column of $F_G(q)$ for the bowtie $G = K_3 + K_3$.

Combining this with Proposition 3.5 results in a new e-positivity result.

Corollary 3.12. If G is a sum of NUIGs and cycles, then $X_G(\boldsymbol{x})$ is e-positive.

We can also use the forest triple matrix to gain new insight on Hikita's standard Young tableaux. This was also found independently in [11, Corollary 1.10].

Corollary 3.13. Let G be an NUIG. Then for every i and every partition λ, the sum

$$\sum_{T \in \mathrm{SYT}^{(i)}(G):\ \mathrm{shape}(T)=\lambda} c_T(G)$$

is a polynomial and is divisible by $[i]_q$.

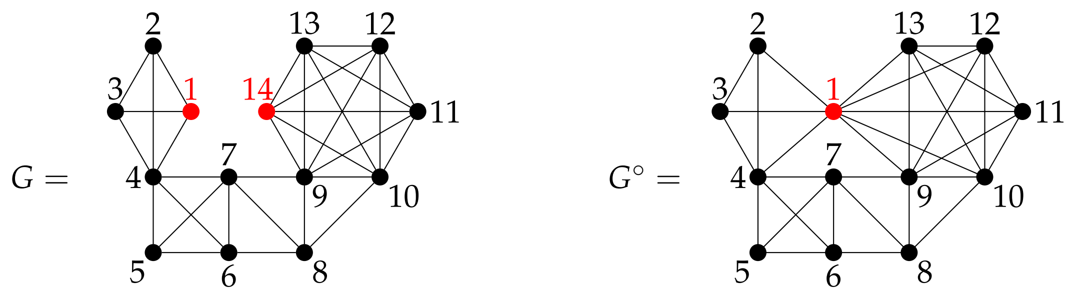


Figure 8: The graph G° obtained by gluing the first and last vertices of G.

4 Trace of the forest triple matrix

In this section, we show that the trace of F_G is equal to the chromatic symmetric function of G°, the graph obtained by gluing vertices 1 and n of G together (shown in Figure 8). From this we prove the e-positivity of many graphs including certain proper circular arc graphs and non-crossing cycle-chord graphs.

Definition 4.1. Let $G = ([n], E)$ be a graph with $n \geq 2$ vertices. We define the graph

$$G^\circ = ([n-1], \{\{i,j\} \in E : \ 1 \leq i < j \leq n-1\} \cup \{\{i,1\} : \ \{i,n\} \in E\}).$$

Now the chromatic symmetric function of G° is given as follows. The idea of the proof is to find a bijection that maps $\mathcal{F} \in \mathrm{FT}(G^\circ)$ to $\mathcal{F}' \in \mathrm{FT}^{(k)}(G + P_k)$ for some k.

Theorem 4.2. Let $G = ([n], E)$ be a graph with $n \geq 2$ vertices. Then we have

$$X_{G^\circ}(\boldsymbol{x}) = \mathrm{trace}(F_G) = \sum_{k=1}^{n-1} (F_G)_{k,k}.$$

From this, we can find new e-positivity results.

Corollary 4.3. If G is an NUIG, then $X_{G^\circ}(\boldsymbol{x})$ is e-positive.

Proof. This follows from Theorem 3.10 and Theorem 4.2. □

Remark 4.4. The graph G° arising from an NUIG G in this way will be a *proper circular arc graph*. Ellzey showed that these are symmetric and conjectured e-positivity. Thus, this result confirms a subset of this conjecture.

Corollary 4.5. If G is a sum of NUIGs and cycles, then $X_{G^\circ}(\boldsymbol{x})$ is e-positive.

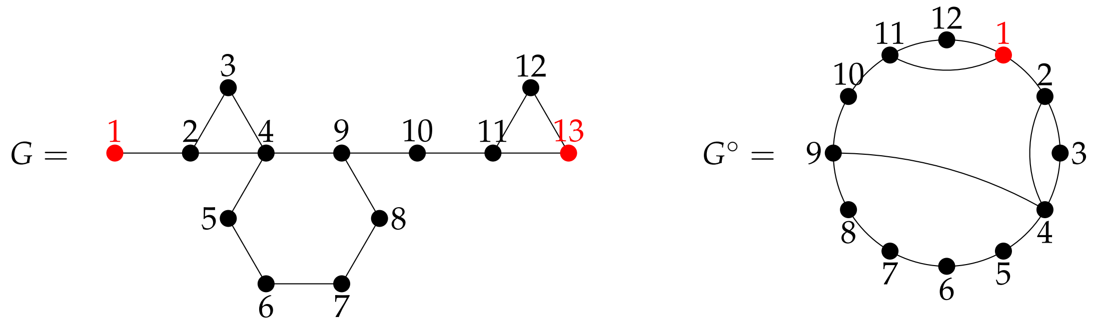


Figure 9: The graph $(P_2 + C_3 + P_1 + C_6 + P_3 + C_3)^\circ$ is a cycle C_{12} with three noncrossing chords.

Wang proved [20] e-positivity of *cycle-chord graphs*, which are graphs obtained by adding a single edge to a cycle. We can use Corollary 4.5 to generalize e-positivity to *noncrossing cycle-chord graphs*, which are obtained from a cycle by adding any number of edges, as long as they do not cross (as shown in Figure 9).

Corollary 4.6. Noncrossing cycle-chord graphs are e-positive.

Proof. We have $G = H^\circ$, where H is a sum of paths and cycles. $\square$

We can use the formula for F_{P_n} to provide an alternative proof for the chromatic symmetric function of cycles.

Corollary 4.7. [6, Corollary 6.2] The chromatic symmetric function of the cycle C_n is

$$X_{C_n}(\boldsymbol{x}) = \sum_{\alpha \models n} \alpha_1(\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\mathrm{sort}(\alpha)}.$$

Proof. Using Proposition 3.6 and Theorem 4.2, we have

$$\begin{aligned} X_{C_n}(\boldsymbol{x}) = \mathrm{trace}(F_{P_{n+1}}) &= \sum_{k \geq 1} \left(\sum_{\alpha \models n,\ \alpha_\ell \geq k} (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\mathrm{sort}(\alpha)} \right) \\ &= \sum_{\alpha \models n} \alpha_\ell (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\mathrm{sort}(\alpha)} = \sum_{\alpha \models n} \alpha_1 (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\mathrm{sort}(\alpha)}. \end{aligned}$$

$\square$

Curiously, if we include the parameter q, we find that the trace of $F_{P_{n+1}}(q)$ is

$$\mathrm{trace}(F_{P_{n+1}}(q)) = \sum_{\alpha \models n} \alpha_1([\alpha_1]_q - 1) \cdots ([\alpha_l]_q - 1) e_{\mathrm{sort}(\alpha)} = X_{\vec{C}_n}(\boldsymbol{x}; q).$$

This suggests the following q-analogue of Theorem 4.2. Note that we need to carefully keep track of our multiple directed edges in $\vec{G}^\circ$.

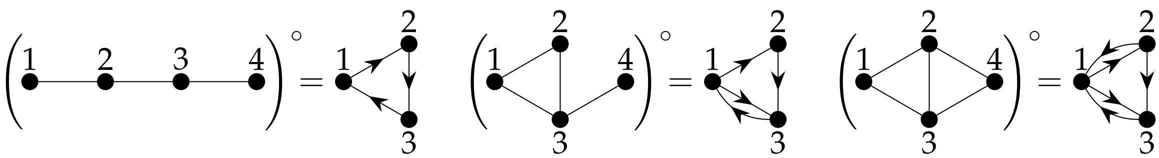


Figure 10: The directed multigraphs obtained from P_4, K_3+P_2, and K_4'.

Conjecture 4.8. Let $G = ([n], E)$ be an NUIG and let $\vec{G}^\circ$ be the directed multigraph obtained by directing the edges of G from the smaller vertex to the larger vertex, and then gluing vertices 1 and n. Then we have

$$X_{\vec{G}^\circ}(\boldsymbol{x}; q) = \mathrm{trace}(F_G(q)).$$

Example 4.9. Figure 10 shows this construction of $\vec{G}^\circ$ for the NUIGs $G = P_4$, $G = K_3 + P_2$, and $G = K_4'$. In each case, the underlying simple graph of $\vec{G}^\circ$ is K_3, and indeed

$$\mathrm{trace}(F_{P_4}) = \mathrm{trace}(F_{K_3+P_2}) = \mathrm{trace}(F_{K_4'}) = 6e_3 = X_{K_3}(\boldsymbol{x}).$$

When we calculate the q-analogues, multiple edges contribute to the number of ascents, so the $X_{\vec{G}^\circ}(\boldsymbol{x}; q)$ are no longer the same. It turns out that

$$\begin{aligned} X_{(\vec{P}_4)^\circ}(\boldsymbol{x}; q) &= 3q[2]_q e_3 = \mathrm{trace}(F_{P_4}(q)), \\ X_{(\overrightarrow{K_3+P_2})^\circ}(\boldsymbol{x}; q) &= (q^3 + 4q^2 + q)e_3 = \mathrm{trace}(F_{K_3+P_2}(q)), \text{ and} \\ X_{(\vec{K}_4')^\circ}(\boldsymbol{x}; q) &= 3q^2[2]_q e_3 = \mathrm{trace}(F_{K_4'}(q)), \end{aligned}$$

which is consistent with Conjecture 4.8. We have checked Conjecture 4.8 by computer for all NUIGs with at most 8 vertices.

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