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Asymptotics of higher Lie characters (extended abstract)

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Abstract. Higher Lie characters form a distinguished family of symmetric group characters, which appear in many areas of algebra and combinatorics. An old open problem of Thrall is to decompose them into irreducibles. We propose a novel asymptotic approach to this problem, showing that many families of higher Lie characters tend to be proportional to the regular character. In particular, a random higher Lie character tends, in probability, to be proportional to the regular character.

Keywords: Asymptotic representation theory, symmetric group, higher Lie characters, conjugacy representation, Thrall's problem

1 Introduction

Higher Lie characters form a distinguished family of symmetric group characters. Their study can be traced back to Schur and Thrall. The important special case of Lie characters was thoroughly studied by Klyachko, Kraśkiewicz and Weyman, and Garsia, who pointed out their significant role in combinatorics. A new era in the study of higher Lie characters began when Gessel and Reutenauer [9] presented them by means of quasisymmetric functions, leading to surprising applications in various areas of algebra and combinatorics: Lyndon words, root enumeration, construction of Gelfand models, permutation statistics, and card shuffling. Strong connections were found to the Whitney homology of the partition lattice [20], the cohomology of configuration spaces [12], and Varchenko–Gelfand and Orlik–Solomon algebras [4].

For a partition $\lambda \vdash n$, denote by χ^λ and by ψ^λ the irreducible character and the higher Lie character (see definition in Subsection 2.1) indexed by λ, respectively.

Problem 1.1. (Thrall [23]) *Find a combinatorial interpretation of the inner products* $c^\lambda_\nu := \langle \psi^\lambda, \chi^\nu \rangle$.

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This (more than eighty years old) open problem of Thrall has attracted significant attention. Solutions are known for distinguished families of partitions. The partitions $\lambda = (2^k 1^{n-2k})$ yield $c_\nu^\lambda \in \{0,1\}$, an important multiplicity-free property that was applied to construct Gelfand models [13]. The case $\lambda = (n)$ was solved by Kraśkiewicz and Weyman [14], using the major index statistic; see Theorem 6.1 below. The best general result so far is Schocker's expansion [19, Theorem 3.1], which however involves signs and rational coefficients. Thrall's problem was approached from different viewpoints in the last decade; see, e.g., [12, 16, 3, 21, 5] and references therein.

In this paper we propose a novel asymptotic approach to Thrall's problem. Instead of focusing on the multiplicities $\langle \psi^\lambda, \chi^\nu \rangle$ for fixed finite partitions λ and ν, we study their asymptotic behavior as both partitions grow. It turns out that, for many families of partitions λ, the higher Lie character ψ^λ tends to be proportional to the regular character.

2 Preliminaries

For two functions $f(n)$ and $g(n)$, we write $f \sim g$ if $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots) = (1^{m_1} 2^{m_2} \cdots) \vdash n$, λ_1 and λ'_1 are the lengths of the first row and the first column, respectively, of the corresponding Young diagram (also denoted λ); χ^λ is the irreducible character of $\mathcal{S}_n$ corresponding to λ; $f^\lambda := \dim \chi^\lambda$; C_λ is the conjugacy class in $\mathcal{S}_n$ consisting of all the permutations with cycle type λ; and $z_\lambda := \prod_i i^{m_i} m_i!$, so that $|C_\lambda| = n!/z_\lambda$. By e_λ, h_λ, p_λ, and s_λ we denote the elementary, complete homogeneous, power sum, and Schur symmetric functions, respectively.

2.1 Higher Lie characters

In this subsection we give the definition of higher Lie characters in two languages — the language of representation theory, and the language of symmetric functions.

For a partition $\lambda = (i^{m_i}) \vdash n$, let Z_λ be the centralizer in $\mathcal{S}_n$ of a permutation of cycle type λ. Then, as a group,

$$Z_\lambda \simeq \prod_{i=1}^{n} \mathcal{S}_{m_i}[\mathbb{Z}_i],$$

where $\mathbb{Z}_i$ is the cyclic subgroup of $\mathcal{S}_n$ generated by a cycle of length i, and $\mathcal{S}_{m_i}[\mathbb{Z}_i]$ is the wreath product of $\mathbb{Z}_i$ by $\mathcal{S}_{m_i}$, also denoted $\mathbb{Z}_i \wr S_{m_i}$. For each i, let ζ_i be a primitive irreducible character of $\mathbb{Z}_i$, and let $\mathrm{id}_{m_i}[\zeta_i]$ be the wreath product of ζ_i by the trivial character id_{m_i} of $\mathcal{S}_{m_i}$. Define the higher Lie character ψ^λ as the induced character

$$\psi^\lambda := \mathrm{Ind}_{Z_\lambda}^{\mathcal{S}_n} \left(\mathrm{id}_{m_1}[\zeta_1] \otimes \mathrm{id}_{m_2}[\zeta_2] \otimes \cdots \otimes \mathrm{id}_{m_n}[\zeta_n] \right). \tag{2.1}$$

In particular, the character $\psi^{(n)}$, corresponding to the one-row diagram $\lambda = (n)$, is the much-studied ordinary Lie character, corresponding to the action of $\mathcal{S}_n$ on the multilinear component of the free Lie algebra with n generators.

The following combinatorial approach to higher Lie characters was established by Gessel and Reutenauer. For each subset $D \subseteq [n-1] := \{1, \ldots, n-1\}$, define the *fundamental quasisymmetric function*

$$F_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For any permutation $\pi \in \mathcal{S}_n$, define the *descent set*

$$\mathrm{Des}(\pi) := \{i \in [n-1] \,:\, \pi(i) > \pi(i+1)\}.$$

Given any subset $A \subseteq \mathcal{S}_n$, define the corresponding quasisymmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} F_{n,\mathrm{Des}(\pi)}.$$

For a partition $\lambda \vdash n$, let L_λ be the image of the higher Lie character ψ^λ under the Frobenius characteristic map; the symmetric functions L_λ are sometimes called *Lyndon symmetric functions*. Let C_λ be the conjugacy class in $\mathcal{S}_n$ consisting of all permutations of cycle type λ. Gessel and Reutenauer [9] showed that $L_\lambda = \mathcal{Q}(C_\lambda)$. Thrall's problem (Problem 1.1) can, thus, be recast as follows: Find a combinatorial interpretation for the coefficients $c_\nu^\lambda := \langle \mathcal{Q}(C_\lambda), s_\nu \rangle$.

2.2 The Plancherel measure

The (infinite-dimensional) *Plancherel measure* is a distinguished measure on the space of infinite Young tableaux, which plays an important role in the Vershik–Kerov asymptotic representation theory [24]. In particular, it corresponds to the character of the regular representation of the infinite symmetric group.

This measure is defined as follows (see [24]). Denote by $\mathcal{T}_n$ the space of all sequences $\nu = (\nu_1, \ldots, \nu_n)$ of Young diagrams $\nu_i \vdash i$, where $\nu_i \subset \nu_{i+1}$; this is the same as the space of all standard Young tableaux with n cells. Consider the probability measure on $\mathcal{T}_n$ defined by

$$M_n(\nu) = \frac{f^{\nu_n}}{n!} \qquad (\forall\, \nu \in \mathcal{T}_n).$$

We have the natural projections $p_n : \mathcal{T}_n \to \mathcal{T}_{n-1}$, forgetting the last element of a sequence. Then $\mathcal{T} := \varprojlim \mathcal{T}_n$ is the space of infinite sequences $\nu = (\nu_1, \nu_2, \ldots)$ of Young diagrams $\nu_n \vdash n$, where $\nu_n \subset \nu_{n+1}$. It is easy to check that the pushforward of M_n by p_n coincides with M_{n-1}.

Definition 2.1. The measure $M := \varprojlim M_n$ on the space $\mathcal{T}$ of infinite sequences of Young diagrams is called the (infinite-dimensional) *Plancherel measure*. We say that a property holds for a *Plancherel-typical* sequence if it holds with probability 1 with respect to Plancherel measure.

The finite-dimensional marginals of M coincide with the finite Plancherel measures: for every $\alpha \vdash n$,

$$M\left(\{\nu = (\nu_1, \nu_2, \ldots) \in \mathcal{T} \, : \, \nu_n = \alpha\}\right) = \frac{(f^\alpha)^2}{n!}.$$

Remark 2.2. The famous Vershik–Kerov limit shape theorem [24] implies that, for a Plancherel-typical sequence of Young diagrams ν, the first row and the first column of ν_n grow like $2\sqrt{n}$ as $n \to \infty$.

3 Asymptotic proportionality of characters

There are various ways to measure how asymptotically close two character sequences $\rho = (\rho_n)$ and $\rho' = (\rho'_n)$ are. Our choice is to compare multiplicities of irreducible characters, $\langle \rho_n, \chi^{\nu_n} \rangle$ and $\langle \rho'_n, \chi^{\nu_n} \rangle$, where the sequence of diagrams $\nu = (\nu_n)$ is typical with respect to the Plancherel measure; see Subsection 2.2. The choice of Plancherel-typical "test characters" χ^ν is based on the following consideration. The Plancherel probability of a diagram ν is proportional to the dimension of the isotypic component of the regular representation Reg_n corresponding to χ^ν, which is $(\dim \chi^\nu)^2$. Thus, from a representation-theoretic point of view, it is natural to assume that a "randomly chosen" irreducible character χ^ν is distributed according to the Plancherel measure. We therefore introduce the following definition.

Definition 3.1. Two character sequences $\rho = (\rho_n)$ and $\rho' = (\rho'_n)$, where ρ_n and ρ'_n are characters of $\mathcal{S}_n$, are said to be *asymptotically proportional* if

$$\frac{\langle \rho_n, \chi^{\nu_n} \rangle}{\dim \rho_n} \sim \frac{\langle \rho'_n, \chi^{\nu_n} \rangle}{\dim \rho'_n} \quad \text{as} \quad n \to \infty$$

for a Plancherel-typical sequence of Young diagrams $\nu = (\nu_1, \nu_2, \ldots)$.

In particular, if $\rho'_n = \mathrm{Reg}_n$, the regular character of $\mathcal{S}_n$, then $\dim \mathrm{Reg}_n = n!$ and $\langle \mathrm{Reg}_n, \chi^{\nu_n} \rangle = f^{\nu_n}$.

Definition 3.2. A sequence of characters $\rho = (\rho_n)$ *tends to be regular* if it satisfies

$$\frac{\langle \rho_n, \chi^{\nu_n} \rangle}{\dim \rho_n} \sim \frac{f^{\nu_n}}{n!} \quad \text{as} \quad n \to \infty \tag{3.1}$$

for Plancherel-typical $\nu = (\nu_n)$.

We also consider random sequences of characters, with the following definition.

Definition 3.3. A random sequence of characters $\rho = (\rho_n)$, with ρ_n a (random) character of $\mathcal{S}_n$, *tends in probability to be regular* if the following holds: if

$$\langle \rho_n, \chi^{\nu_n} \rangle = \frac{\dim \rho_n}{n!} \cdot f^{\nu_n} \cdot (1 + R_n),$$

then, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \operatorname{Prob}\left(|R_n| > \varepsilon\right) = 0$$

for Plancherel-typical $\nu = (\nu_1, \nu_2, \ldots)$.

The results obtained in this paper hold under a weaker assumption on the test sequence ν, namely for balanced sequences in the sense of the following definition; but see Remark 3.5 below.

Definition 3.4. [7] A sequence of diagrams $\nu = (\nu_1, \nu_2, \ldots)$ with $\nu_n \vdash n$ is said to be *balanced* if the lengths of the first row and the first column of ν_n are both $O(\sqrt{n})$.

By Remark 2.2, a Plancherel-typical sequence of diagrams is balanced, so if (3.1) holds for balanced sequences ν, then ρ tends to be regular.

Remark 3.5. The condition of being balanced is very powerful, but technical. We believe that the right notion of asymptotic proportionality is the one with Plancherel-typical test functions. This condition is most natural from the point of view of the representation theory of symmetric groups, and plays a very important role in it. An illustration is Regev's result (Theorem 4.1 below), whose proof heavily relied on ν being Plancherel-typical.

4 Existing asymptotic results

It was shown by Thrall that the multiplicity-free sum of all higher Lie characters of $\mathcal{S}_n$ is the regular character Reg_n. Coarse versions of Thrall's problem (see [16]) aim to describe the decomposition into irreducibles of other (large) sums of higher Lie characters. For example, the *derangement character* $\mathcal{D}_n$ of the symmetric group $\mathcal{S}_n$ is the sum of higher Lie characters over all partitions with no parts of size 1. Thus, the dimension of $\mathcal{D}_n$ is equal to the number d_n of derangements (fixed-point-free permutations) in $\mathcal{S}_n$. Following a combinatorial interpretation by Désarménien and Wachs [6], Regev [15] presented the following remarkable asymptotic result.

Theorem 4.1. (Regev [15]) *For Plancherel-typical* ν,

$$\langle \mathcal{D}_n, \chi^\nu \rangle \sim \frac{f^\nu}{e} \quad \textit{as} \quad n \to \infty.$$

Note that $\lim_{n\to\infty} d_n/n! = 1/e$, so that Regev's result can be written as

$$\frac{\langle \mathcal{D}_n, \chi^\nu \rangle}{\dim \mathcal{D}_n} \sim \frac{f^\nu}{n!} \quad \text{as} \quad n \to \infty$$

for Plancherel-typical ν. In other words, $\mathcal{D}_n$ tends to be regular in the sense of Definition 3.2.

The action of $\mathcal{S}_n$ on itself by conjugation is closely related to higher Lie characters, as explained in Section 7 below; denote by $\mathcal{C}_n$ the corresponding character. The asymptotics of $\mathcal{C}_n$ was first studied in [1], where it was shown that it tends, in a certain sense, to the regular character. A well known problem (see, e.g., [11]) is to decompose $\mathcal{C}_n$ into irreducible characters. The following result follows from [18, Theorem 2.1].

Theorem 4.2. *For Plancherel-typical ν,*

$$\frac{\langle \mathcal{C}_n, \chi^\nu \rangle}{\dim \mathcal{C}_n} \sim \frac{f^\nu}{n!} \quad \textit{as} \quad n \to \infty.$$

Thus, $\mathcal{C}_n$ also tends to be regular in the sense of Definition 3.2.

5 Main results

In this paper we consider Thrall's original problem (rather than its coarse versions) in an asymptotic setting; namely, we study the multiplicities of irreducibles in single higher Lie characters (rather than sums of characters), indexed by diagrams of growing sizes.

Our first main result deals with sequences of higher Lie characters indexed by rectangular diagrams. Note that Thrall's problem may be reduced, via the Littlewood–Richardson rule, to the special case of a rectangular diagram; see, e.g., [16] and references therein.

Theorem 5.1. *Any sequence of higher Lie characters, indexed by rectangular diagrams $\lambda = (m^k) \vdash n$ with $m \to \infty$ and $k = o(m)$, tends to be regular. Furthermore,*

$$\langle L_{(m^k)}, s_\nu \rangle \sim \frac{f^\nu}{m^k k!} \quad \textit{as} \quad m \to \infty$$

for balanced (and hence for Plancherel-typical) $\nu \vdash mk$.

For the validity of the conclusion beyond the stated assumptions see Section 8 below.

The second main result of this paper deals with the asymptotics of a random higher Lie character. As mentioned above, the sum of higher Lie characters ψ^λ over all partitions $\lambda \vdash n$ is the regular character Reg_n. Also, $\dim \psi^\lambda = |C_\lambda|$, where C_λ is the conjugacy class consisting of all permutations in $\mathcal{S}_n$ with cycle type λ. Thus, exactly the same logic as in the choice of Plancherel measure for irreducible characters χ^ν suggests that a random higher Lie character ψ^λ ($\lambda \vdash n$) be chosen with probability $|C_\lambda|/n!$.

Theorem 5.2. *A random higher Lie character ψ^{λ_n}, where $\lambda_n \vdash n \to \infty$, tends in probability to be regular.*

For stronger quantitative versions, see [2, Theorems 7.2 and 7.3].

Remark 5.3. Theorems 5.1 and 5.2, as well as all other results in this paper, hold also when higher Lie characters are replaced by any other characters induced from linear characters of centralizers, in particular for the characters of the conjugacy action of $\mathcal{S}_n$ on its conjugacy classes; see Theorem 7.1 and Corollary 7.2 below.

6 Proof strategy and additional results

6.1 Rectangular shapes

To prove Theorem 5.1, we first prove the relatively easy case of one-row shapes. We quote two results (Theorems 6.1 and 6.2) which, combined, essentially yield the asymptotic behavior of the ordinary Lie characters $\psi^{(n)}$, described in Theorem 6.3.

Denote by $\mathrm{SYT}(\lambda/\mu)$ the set of standard Young tableaux of skew shape λ/μ, and let $f^{\lambda/\mu} := |\mathrm{SYT}(\lambda/\mu)|$. For $T \in \mathrm{SYT}(\lambda)$, define the *descent set*

$$\mathrm{Des}(T) := \{i \,:\, i+1 \text{ is in a lower row of } T \text{ than } i\}$$

and the *major index*

$$\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i.$$

Theorem 6.1. (Kraśkiewicz–Weyman [14], see also [8, Theorem 8.4]) *For every partition $\nu \vdash n$, the multiplicity $\langle \psi^{(n)}, \chi^{\nu} \rangle$ is equal to the cardinality of the set*

$$\{T \in \mathrm{SYT}(\nu) \,:\, \mathrm{maj}(T) \equiv 1 \pmod{n}\}.$$

Theorem 6.2. (Swanson [22, Theorem 1.8]) *For every partition $\nu \vdash n \geq 1$ and all r,*

$$\left| \frac{|\{T \in \mathrm{SYT}(\nu) \,:\, \mathrm{maj}(T) \equiv r \pmod{n}\}|}{f^{\nu}} - \frac{1}{n} \right| \leq \frac{2n^{3/2}}{\sqrt{f^{\nu}}}.$$

Combining Theorems 6.1 and 6.2 with estimates on f^{ν}, we prove the following.

Theorem 6.3 (One-row shapes)**.** *The sequence of ordinary Lie characters $\psi^{(n)}$ tends to be regular. Furthermore, as $n \to \infty$,*

$$\langle \mathrm{Lie}_n, s_{\nu} \rangle \sim \frac{f^{\nu}}{n}$$

for balanced (and hence for Plancherel-typical) $\nu \vdash n$. In fact,

$$\langle \mathrm{Lie}_n, s_{\nu} \rangle = \frac{f^{\nu}}{n}(1 + r_{\nu})$$

with the following two bounds on the error term r_{ν}.

(a) *For every $s > 0$ there exist constants $N > 0$ and $c > 0$ (both depending only on s) such that, for every $n \geq N$, if $\nu \vdash n$ with $\nu_1, \nu_1' \leq n - c$ then*

$$|r_\nu| \leq n^{-s}. \tag{6.1}$$

(b) *For balanced $\nu \vdash n$, there exists a constant $C > 0$ (depending only on the constant in $O(\sqrt{n})$ in Definition 3.4) such that*

$$|r_\nu| \leq \left(\frac{C}{n}\right)^{n/4}. \tag{6.2}$$

To obtain the main result on rectangular shapes, Theorem 5.1, we combine Theorem 6.3 with symmetric function computations. We apply necessary conditions from [10] for the positivity of plethystic coefficients, and upper bounds from [17] on character ratios.

6.2 Shapes with distinct row lengths

The proof of Theorem 5.2 is outlined in the next three subsections. We first analyze diagrams with distinct row lengths, and show that under mild conditions they tend to be regular (Theorem 6.4 below). Another important tool is the Gluing Lemma (Lemma 6.5 below), which allows us to glue relatively small diagrams to large diagrams and preserve the tending-to-regularity property. An immediate consequence of the Gluing Lemma is a sufficient condition for higher Lie characters of hook shapes to tend to be regular (Corollary 6.6). The sketch of the proof is completed in Subsection 6.4.

Theorem 6.4 (Distinct row lengths)**.** *Let $\mu = (m_1, \ldots, m_t) \vdash n$ with $m_1 > \cdots > m_t$. If $t \leq g(n)$ and $m_t \geq f(n)$ for some functions f, g such that $f(n) \to \infty$, $g(n) = O(f(n)/\ln n)$, and $g(n) = o(\sqrt{n})$ (as $n \to \infty$), then ψ^μ tends to be regular. Furthermore,*

$$\langle L_\mu, s_\nu \rangle \sim \frac{f^\nu}{z_\mu} \quad as \quad n \to \infty,$$

for balanced (and hence for Plancherel-typical) ν. Moreover, the error term is uniform over ν:

$$\langle L_\mu, s_\nu \rangle = \frac{f^\nu}{z_\mu}(1 + R_\nu) \quad where \quad |R_\nu| = o(f(n)^{-q})$$

for any fixed (arbitrarily large) q.

The proof of Theorem 6.4 applies a case analysis of the error term and bounds on the number of standard Young tableaux of a skew shape, including those from [7].

6.3 The Gluing Lemma

Lemma 6.5 (Gluing Lemma)**.** *Let $\lambda \vdash n$ be the disjoint union $\lambda = \mu \cup \tau$ of diagrams $\mu \vdash n-k$ and $\tau \vdash k$. Assume that $k = o(n^{1/4})$, the smallest part of μ is larger than the largest part of τ, and the character ψ^μ tends to be regular strongly and uniformly, in the sense that for any balanced $\alpha \vdash n-k$*

$$\langle L_\mu, s_\alpha \rangle = \frac{f^\alpha}{z_\mu}(1 + R_\alpha) \quad \textit{with} \quad |R_\alpha| < \omega(n-k) \quad \textit{as} \quad n \to \infty,$$

for some function $\omega(n) = o(1)$. Then ψ^λ tends to be regular. Furthermore,

$$\langle L_\lambda, s_\nu \rangle \sim \frac{f^\nu}{z_\mu z_\tau} \quad \textit{as} \quad n \to \infty,$$

for balanced (and hence for Plancherel-typical) $\nu \vdash n$. More precisely,

$$\langle L_\lambda, s_\nu \rangle = \frac{f^\nu}{z_\mu z_\tau}(1 + R_{\lambda,\nu}) \quad \textit{with} \quad |R_{\lambda,\nu}| \le O(\omega(n-k) + k^2 n^{-1/2}).$$

Corollary 6.6 (Hook shapes)**.** *Any sequence of higher Lie characters indexed by hook shapes $(n-k, 1^k)$ with $k = o(n^{1/4})$ tends to be regular.*

6.4 Random shapes

To prove Theorem 5.2 we also need the following lemma.

Lemma 6.7. *The probability for a random permutation, uniformly distributed on $\mathcal{S}_n$, to have two cycles of the same length m for some $m > M$ is at most $1/M$.*

The idea of the proof of Theorem 5.2 is as follows. The cycle type of a random permutation of $\mathcal{S}_n$ can be decomposed into two disjoint parts, with one part consisting of long rows (with length greater than $f(n)$, for some function f) and the other part consisting of short rows. By Lemma 6.7, all the rows in the first part are distinct, with high probability, so we can apply Theorem 6.4. On the other hand, the number of cycles in a random permutation is logarithmic (less than $g(n)$, for some function g). This implies that the size of the second part is relatively small, with high probability, so we can apply the Gluing Lemma 6.5. This completes the proof.

Remark 6.8. There is a trade-off between the sizes of the error term and the error probability, as in Definition 3.3. Choosing f and g appropriately can optimize one or the other. For details see Theorem 7.3 in [2].

7 Conjugation and other actions

Replacing, in the construction of higher Lie characters described in (2.1), the primitive irreducible character ζ_k of $\mathbb{Z}_k$ by any character of the form ζ_k^r for $r \in \mathbb{Z}$, we obtain the more general character

$$\psi_r^\lambda := \mathrm{Ind}_{Z_\lambda}^{\mathcal{S}_n} \left(\mathrm{id}_{m_1}[\zeta_1^r] \otimes \mathrm{id}_{m_2}[\zeta_2^r] \otimes \ldots \otimes \mathrm{id}_{m_n}[\zeta_n^r] \right),$$

where $\mathrm{id}_{m_i}[\zeta_i^r]$ is the wreath product of ζ_i^r by the trivial character id_{m_i} of $\mathcal{S}_{m_i}$.

Theorem 7.1. *All the previous results in this paper, in particular Theorems 5.1 and 5.2, are valid for ψ_r^λ, for arbitrary r, instead of the higher Lie character $\psi^\lambda = \psi_1^\lambda$.*

In particular, the case $r = 0$ corresponds to the conjugacy characters of $\mathcal{S}_n$. Namely, consider the linear representation π_C of $\mathcal{S}_n$ on the group algebra $\mathbb{C}[\mathcal{S}_n]$, defined by the action of $\mathcal{S}_n$ on itself by conjugation: $\pi_C(g)(h) = ghg^{-1}$. Clearly, for each $\lambda \vdash n$, the subspace $H_\lambda \subset \mathbb{C}[\mathcal{S}_n]$ spanned by the permutations with cycle structure λ is invariant under π_C. The *conjugacy character* ϕ^λ of $\mathcal{S}_n$ is the character of the restriction of π_C to H_λ. It is not difficult to see that $\phi^\lambda = \psi_0^\lambda$.

Corollary 7.2. *All the previous results in this paper are valid for the conjugacy character ϕ^λ instead of the higher Lie character ψ^λ.*

Characters close to higher Lie characters appear in natural symmetric group actions on certain homology and cohomology groups. For instance, the character π_n of the action of $\mathcal{S}_n$ on the top component of the cohomology ring of the partition lattice Π_n, which is isomorphic to the cohomology ring of the Orlik–Solomon algebra of the braid hyperplane arrangement, is equal to $\varepsilon_n \psi^{(n)}$, where ε_n is the sign character of $\mathcal{S}_n$ [20].

Corollary 7.3. *The sequence of characters π_n tends to be regular.*

More generally, one can show that results in this paper are valid for the characters

$$\mathrm{Ind}_{\mathcal{S}_{m_1}[\mathcal{S}_1] \times \mathcal{S}_{m_2}[\mathcal{S}_2] \times \ldots}^{\mathcal{S}_n} \left(\mathrm{id}_{m_1}[\pi^{(1)}] \otimes \varepsilon_{m_2}[\pi^{(2)}] \otimes \ldots \otimes \mathrm{id}_{m_{2i-1}}[\pi^{(2i-1)}] \otimes \varepsilon_{m_{2i}}[\pi^{(2i)}] \otimes \ldots \right),$$

which appear in the description of the actions of $\mathcal{S}_n$ on the Whitney homology of the partition lattice and on the cohomology of the configuration space of points in $\mathbb{R}^d$ [12].

8 Final remarks and open problems

Here is a brief intuitive explanation of the "tending to be regular" phenomenon. The conjugation action of $\mathcal{S}_n$ on itself tends to be regular [1, 17]. Large conjugacy classes in $\mathcal{S}_n$ essentially resemble the whole group. For this "reason", $\mathcal{S}_n$-characters induced from linear characters of centralizers of large conjugacy classes tend to be regular. For small conjugacy classes the behavior is different, as demonstrated below.

In Theorem 5.1, regarding rectangular diagrams, it is assumed that the height of the rectangle is asymptotically smaller than its width. Indeed, there are families of rectangles which do not tend to be regular. Here is an important example.

Proposition 8.1. *The sequence of higher Lie characters indexed by rectangular diagrams $\lambda = (2^k)$ with constant width $m = 2$ and height $k \to \infty$ does not tend to be regular.*

In fact, each irreducible character appears with multiplicity 0 or 1.

Problem 8.2. *What is the threshold for the height k, as a function of the width m, for a sequence of higher Lie characters indexed by rectangular diagrams (m^k) to tend to be regular?*

An analogous problem can be posed for hook shapes.

Problem 8.3. *What is the threshold for the leg length k, as a function of the total size n, for a sequence of higher Lie characters indexed by hook shapes $(n-k, 1^k)$ to tend to be regular?*

Corollary 6.6 provides a lower bound for the threshold. Here is an upper bound.

Proposition 8.4. *For any $\varepsilon > 0$, any sequence of higher Lie characters indexed by hook shapes $(n-k, 1^k)$ with $k > (2+\varepsilon)\sqrt{n}$ for infinitely many values of n does not tend to be regular.*

In fact, Pieri's rule implies that only shapes with first row length at least k appear.

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