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Blossoming bijection for bipartite maps: a new approach via orientations and applications to the Ising model

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Abstract. We develop a new bijective framework for the enumeration of bipartite planar maps with control on the degree distributions of black and white vertices. We introduce a family of orientations on bipartite maps that connects the bijection of Bousquet-Mélou and Schaeffer to the general scheme of Albenque and Poulalhon. This enables us to generalize Bousquet-Mélou and Schaeffer's bijection to several families of bipartite maps. As an application, we also derive a rational Lagrangian parametrization with positive integer coefficients for the generating series of quartic maps equipped with an Ising model.

Résumé. Nous développons un nouveau cadre bijectif pour l'énumération des cartes planaires biparties avec contrôle des distributions des degrés des sommets noirs et blancs. Nous introduisons une famille d'orientations sur les cartes bipartites qui relie la bijection de Bousquet-Mélou et Schaeffer au schéma général d'Albenque et Poulalhon. Cela nous permet de généraliser la bijection de Bousquet-Mélou et Schaeffer à plusieurs familles de cartes bipartites. Une paramétrisation rationnelle lagrangienne avec des coefficients entiers positifs pour la série génératrice des cartes quartiques équipées d'un modèle d'Ising en est déduite.

Keywords: Bijections, bipartite maps, orientations, blossoming trees.

1 Introduction

In this article, we address the problem of enumerating bijectively bipartite planar maps with a canonical black-white proper coloring, while controlling the distributions of vertex degrees of black and white vertices independently. This problem generalizes the enumeration of classical maps with control on vertex degrees.

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The enumeration of bipartite planar maps originated in the physics literature, where they appeared in connection with the so-called "2-matrix model" studied by Itzykson and Zuber [17]. The motivation behind their study stems from their connection with the Ising model on planar maps, solved by Boulatov and Kazakov via matrix integrals [8, 18]. The first bijective enumeration of bipartite maps was obtained by Bousquet-Mélou and Schaeffer in [10]. To this end, they built an explicit bijective correspondence between bipartite maps and a family of decorated bicolored trees. This allowed them to rederive fully rigorously the results of Kazakov [18]. Shortly after, another bijective proof of this result, relying on a different family of decorated trees – called mobiles – was developed by Bouttier, Di Francesco and Guitter in [12].

More recently, unified bijective frameworks have been developed to provide generic methods for constructing bijections between maps and tree-like families. Notable examples include the schemes of Bernardi and Fusy [7], as well as the scheme of Albenque and Poulalhon [4]. Both approaches extend a construction introduced by Bernardi [5] for maps equipped with a canonical orientation of their edges.

The starting point of the present work is to introduce a new family of orientations on bipartite maps that allows us to interpret Bousquet-Mélou and Schaeffer's bijection as an instance of the Albenque-Poulalhon bijective scheme. This family of orientations generalizes the classical Eulerian and quasi-Eulerian orientations.

To describe more precisely the main contribution of this paper, some terminology is needed (precise definitions will be given later in Section 2). A *blossoming tree* is a tree where vertices can carry opening and closing stems, which are half-edges oriented either from or towards their incident vertex. The *charge* of a blossoming tree is defined as the difference between the number of its closing and opening stems. The *closure* of a blossoming tree is the planar maps obtained by matching opening and closing stems cyclically around the tree. In [10], a family of trees with some charge constraints was defined and shown to be in bijection with bipartite planar maps through the closure operation. In this paper, we generalize this result in two directions.

First, we relax the *balancedness assumption* of [10], in which only trees whose root lies in the outer-face of their closure map are considered. We instead obtain a bijection between not necessarily balanced blossoming trees and planar bipartite maps with a marked face, see Theorem 3.2. The main consequence is of enumerative nature, since removing the balancedness assumption simplifies considerably the enumeration of the family of trees obtained.

Second, we give an interpretation of the closure of trees with *non-zero charge* in Theorem 3.7. In that case some stems remain unmatched, and by connecting them to an additional vertex we obtain a rooted bipartite map with an additional marked vertex and some connectivity constraints. This interpretation of trees with non-zero charge gives a combinatorial explanation of the rational parametrization for hypermaps given in [15, Chapter 8]. Recently, another combinatorial interpretation of this parametrization was given by Albenque and Bouttier [1], using the theory of slices. This allows them to give a combinatorial proof of several enumerative formulas for hypermaps, using as fundamental building blocks some particular subfamilies of hypermaps on the cylinder, called *trumpets* and *cornets*. These families correspond exactly (by duality) to the families we obtain as the closure of trees, which confirms the important role of trumpets and cornets in the context of the topological recursion, see Eynard [15].

Finally, let us mention an important byproduct of our bijection for maps decorated with an Ising model. As already mentioned, it is classical that the generating series of bipartite maps and of maps decorated with an Ising model are connected through a change of variables, see [10]. In particular, this connection allowed Bousquet Mélou and Schaeffer to derive a rational parametrization for the generating series of quartic maps with an Ising model in [10].

Our new bijection allows us to derive an equivalent rational and Lagrangian parametrization of this generating function that has a combinatorial interpretation, see Theorem 5.1. A consequence of this combinatorial interpretation is that the parameter series has non-negative integer coefficients. Both this property and the Lagrangian form of the parametrization allow us for a detailed analysis of the generating series of quartic maps with an Ising model. In turn, this allows to study probabilistic properties of these maps as was done for example in [2, 3, 13, 14, 23]. Tokka uses the parametrization of Theorem 5.1 to study random quartic maps with an Ising model in the presence of an external magnetic field in [22].

Let us also mention that our bijection for bipartite maps through orientation could be an important tool to address the problem of enumerating bipartite maps with prescribed vertex degrees in higher genus, building upon the work of Lepoutre [19]. This could pave the way for exploring the Ising model on planar maps in higher genus using a bijective approach. Note that the recent work by Bouquet-Mélou, Carrance and Louf [9] studies cubic maps of arbitrary genus equipped with an Ising model with a different method. They provide inequalities for the coefficients of the generating function using equations of the KP hierarchy.

2 Preliminaries and background

2.1 Planar maps and families of maps

A *planar map* is a proper embedding of a connected planar graph on the two dimensional sphere $\mathbb{S}^2$, considered up to orientation-preserving homeomorphisms. Note that loops and multiple edges are allowed.

Edges and *vertices* of a map are the natural counterparts of edges and vertices of the underlying graph. The *faces* of a map m are the connected components of the complement of the embedded graph. The sets of its vertices, edges and faces are respectively

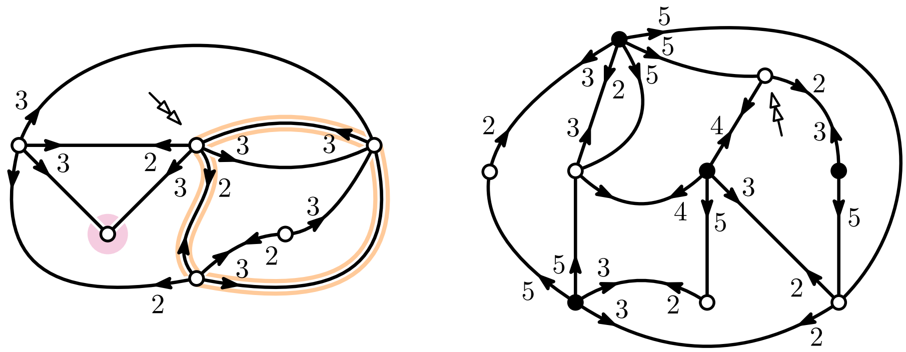

Figure 1: A map with a non-minimal (see the orange-highlighted cycle) and non-coaccessible (see the pink-highlighted vertex) 3-fractional orientation (left), and a bipartite map with its minimal α_4-orientation (right). By convention, integers are assigned to the half-edge on their right.

denoted by $\mathrm{V}(\mathrm{m})$, $\mathrm{E}(\mathrm{m})$ and $\mathrm{F}(\mathrm{m})$.

Each edge e of m can be split into two *half-edges* h_1 and h_2, so we write $e = \{h_1, h_2\}$. The set of half-edges of m is denoted by $\mathrm{H}(\mathrm{m})$. For $v \in \mathrm{V}(\mathrm{m})$ and $h \in \mathrm{H}(\mathrm{m})$, we write $h \sim v$ if h is incident to v. The *degree* of a vertex or a face is defined as the number of its incident half-edges. A *corner* is a pair of consecutive half-edges around a vertex.

To avoid dealing with symmetries, planar maps will always be *rooted*, meaning that one of their corners – called the *root corner* – is distinguished (and indicated by a double arrow on figures). The vertex and the face incident to this corner are called the *root vertex* and the *root face*, respectively. In this article, all maps are rooted.

A *plane map* is a map with an additional marked face – called the *outer face*. We emphasize that the outer (infinite) face of a plane map is not necessarily its root face, contrary to what is often done in the literature.

A map m is called *bipartite* if the set of its vertices can be partitioned into two disjoint subsets, $\mathrm{V}_\bullet$ and $\mathrm{V}_\circ$, such that every edge connects a vertex of $\mathrm{V}_\bullet$ to a vertex of $\mathrm{V}_\circ$.

We denote by $\mathcal{M}$ the set of rooted bipartite planar maps, and for any $d \geq 0$, we denote by $\mathcal{M}^{(d)}$ its subset composed of the maps with maximal vertex degree d. Similarly, we denote by $\bar{\mathcal{M}}$ the set of rooted bipartite *plane* maps, and for any $d \geq 0$, we denote by $\bar{\mathcal{M}}^{(d)}$ its subset restricted to the maps with maximal vertex degree d .

2.2 Orientations, fractional orientations and α-orientations

An *orientation* of a planar map m is a mapping $\mathcal{O}$ from $\mathrm{H}(\mathrm{m})$ to $\mathbb{Z}_{\geq 0}$. The orientation is viewed as the value of an outgoing flow through the corresponding half-edge, see Figure 1.

For any oriented map $(\mathrm{m}, \mathcal{O})$, a directed edge $\overrightarrow{e} = (h_1, h_2)$ of m is said to be *forward* if $\mathcal{O}(h_1) > 0$, and to be *saturated* if $\mathcal{O}(h_2) = 0$.

Next, for $u, v \in \mathrm{V}(\mathrm{m})$, a directed path $\mathrm{p} := (\overrightarrow{e_1}, \ldots, \overrightarrow{e_p})$ from u to v in m is said to be *forward* if for any $i \in \{0, \ldots, p-1\}$, the directed edge $\overrightarrow{e_i}$ is forward. The vertex v is said to be *accessible* from u, if there exists a forward path from u to v. Moreover, an orientation $\mathcal{O}$ is said to be *root-coaccessible* (or simply *coaccessible*), if its root vertex is accessible from every vertex of m, see Figure 1.

In a plane map m endowed with an orientation, a *clockwise* (respectively *counterclockwise*) cycle is a forward cycle such that the marked face of m lies on its left (respectively on its right). We call *minimal* any orientation without counterclockwise cycles.

Fix an integer $k > 0$ and m a planar map. An orientation of m is said to be *k-fractional* if for every edge $e = \{h_1, h_2\} \in \mathrm{E}(\mathrm{m})$, we have $\mathcal{O}(h_1) + \mathcal{O}(h_2) = k$, see Figure 1. From now on, all the orientations considered are *fractional*, meaning that they are k-fractional for some $k \in \mathbb{Z}_{>0}$.

Let $\mathcal{O}$ be an orientation of m. Then, for $v \in \mathrm{V}(\mathrm{m})$, the *outdegree* and the *indegree* of v for $\mathcal{O}$ are defined by:

$$\deg_{\mathrm{out}}(v) := \sum_{h \sim v} \mathcal{O}(h) \qquad \text{and} \qquad \deg_{\mathrm{in}}(v) := \sum_{\substack{h_1 \sim v \\ \{h_1, h_2\} \in \mathrm{E}(\mathrm{m})}} \mathcal{O}(h_2).$$

Fix a function $\alpha : \mathrm{V}(\mathrm{m}) \to \mathbb{Z}_{\geq 0}$. An *$\alpha$-orientation* $\mathcal{O}$ of m is a fractional orientation such that for every vertex $v \in \mathrm{V}(\mathrm{m})$, we have: $\deg_{\mathrm{out}}(v) = \alpha(v)$. If m admits an α-orientation, then the function α said to be *feasible* on m.

2.3 Blossoming maps and bijective scheme

A *blossoming map* is a plane map in which each *outer corner* can carry some half-edges. These half-edges are called *closing stems* (for ingoing half-edges) and *opening stems* (for outgoing half-edges), also commonly known in the literature as buds and leaves (see [10, 11]). A *blossoming tree* is a blossoming map with only one face.

An orientation $\mathcal{O}$ of a blossoming map is defined as an orientation for maps, subject to the additional condition that $\mathcal{O}(h) = 0$ for any closing stem h. Moreover, if $\mathcal{O}$ is k-fractional, we also require that $\mathcal{O}(h) = k$ for any opening stem h.

The *charge* of a blossoming map is defined as the difference between the number of its closing stems and the number of its opening stems. Furthermore, for a blossoming tree t, we define the *charge* of any $v \in \mathrm{V}(\mathrm{t})$ as the charge of the subtree of t rooted at v.

Following [4, 5, 11, 20, 21], given a blossoming map m, we define its *closure* as follows. Consider the cyclic sequence of opening and closing stems, obtained by turning clockwise around the border of its marked face. This induces a partial matching between

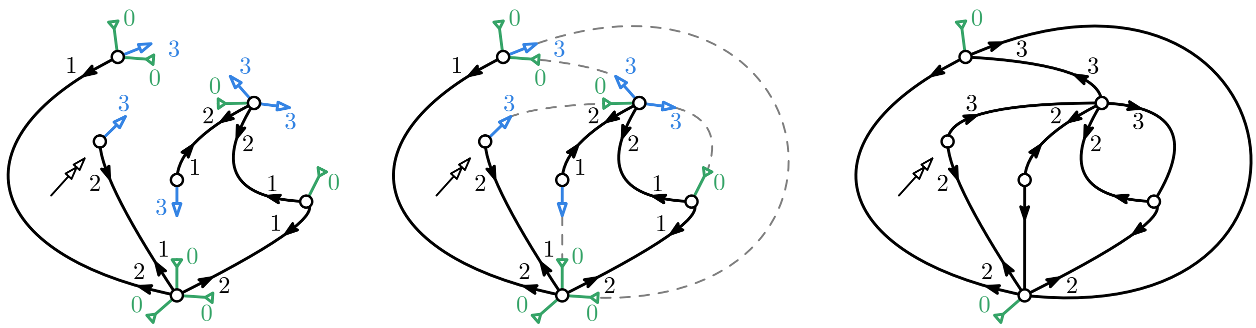

Figure 2: A blossoming tree of charge 2 with a 3-fractional orientation (left). The same tree with the matching of stems in dashed lines (middle). Its closure, which is a blossoming map with two unmatched closing stems (right).

opening and closing stems, as in a parenthesis word. Note that some opening or closing stems remain unmatched if the charge of the whole tree is strictly negative or strictly positive, respectively. For any matched pair made of an opening and a closing stem, we merge them together into a new edge such that the outer face lies on its left when oriented from the opening stem to the closing stem, see Figure 2. It is easy to prove that the resulting plane blossoming map does not depend on the order of local closures. Additionally, the vertex degree distribution is preserved through the closure operation.

The closure of a blossoming map with charge 0 is a plane map (i.e., with no remaining unmatched stems). In contrast, the closure of a blossoming map with charge $k > 0$ (respectively $k < 0$) yields a blossoming map with k unmatched closing (respectively. opening) stems.

If a blossoming map m is equipped with an orientation $\mathcal{O}$, then its closure naturally inherits this orientation – still denoted by $\mathcal{O}$, by a slight abuse of notation. Since $\mathcal{O}(h) = 0$ for any closing stem, the edges created during the closure are all saturated.

Note that both coaccessibility and minimality are preserved under the closure operation. Coaccessibility follows immediately, while minimality holds because the closure process is performed in the clockwise direction and so does not create any counterclockwise cycles.

The inverse of the closure operation is *a priori* not well-defined. Indeed, a map can be obtained as the closure of any of its spanning trees. The following result by Albenque and Poulalhon, which generalizes earlier results of Poulalhon and Schaeffer [20] and Bernardi [5], gives a framework in which a canonical inverse can be defined:

Theorem 2.1 ([4, Corollary 2.4]). *Let* m *be a rooted plane map with a minimal coaccessible orientation* $\mathcal{O}$*. Then, there exists a unique blossoming tree* $\mathrm{t_m}$ *of charge* 0*, equipped with an coaccessible orientation, whose closure yields* $(\mathrm{m}, \mathcal{O})$*.*

The statement also holds if m *is a blossoming map with k unmatched closing stems (respec-*

tively, opening stems). In that case, $\mathrm{t_m}$ *is a blossoming tree of charge* k *(respectively,* $-k$*).*

3 New family of orientations and blossoming bijections for bipartite maps

3.1 Definitions and first properties of α_d-orientations

In all this section, $\mathrm{m} \in \mathcal{M}$ is a rooted bipartite planar map (endowed with one of its two proper colorings). For $d \geq 1$, let $\alpha_d : \mathrm{V}(\mathrm{m}) \to \mathbb{Z}_{\geq 0}$ be the function defined by:

$$\alpha_d(v) := \begin{cases} d \deg(v) & \text{if } v \text{ is black,} \\ \deg(v) & \text{if } v \text{ is white.} \end{cases} \tag{3.1}$$

The function α_d is feasible on m as a $(d+1)$-fractional orientation and each α_d-orientation on m is coaccessible. As a direct consequence of [6, Lemma 2] (see also [16]) we have:

Property 3.1. *For any* $d \geq 1$*, every bipartite plane map* m *admits a unique minimal coaccessible* α_d*-orientation. Moreover, if* m *has vertex degree at most* d*, then for any* α_d*-orientation* $\mathcal{O}$*, every saturated edge is oriented from its black endpoint to its white endpoint.*

3.2 Recovering the BMS bijection: the case of plane maps

In this section, we apply Theorem 2.1 to bipartite maps endowed with their minimal α_d-orientation, to obtain a new bijection between bipartite maps and the family of *well-charged trees* introduced in [10], which we now define.

A *well-charged tree* is a rooted bipartite blossoming tree satisfying the following conditions:

- black vertices have no incident closing stems, and every non-root black vertex has charge at most one,
- white vertices have no incident opening stems, and every non-root white vertex has non-negative charge.

For any $k \in \mathbb{Z}$ and $d \geq 0$, we denote by $\mathcal{T}_k$ the set of well-charged trees of total charge k, and by $\mathcal{T}_k^{(d)}$ the subset consisting of trees whose maximal vertex degree is d.

The main result of this section is the following bijection, illustrated in Figure 3:

Theorem 3.2. *The closure operation is a one-to-one correspondence between* $\mathcal{T}_0$ *and* $\bar{\mathcal{M}}$*. Moreover, this correspondence preserves the vertex-degree distribution and the color of the root vertex.*

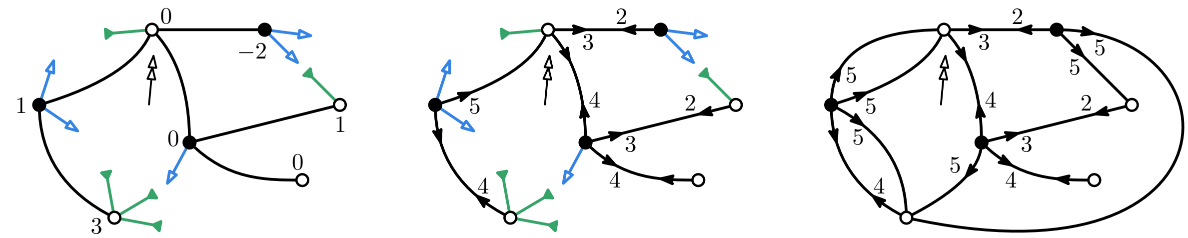

Figure 3: A well-charged tree of charge 0 with its charge function (left), and with its α_4-orientation (middle). Its closure, which is endowed with its unique minimal coaccessible α_4-orientation (right).

Remark 3.3. *This result is closely related to Bousquet-Mélou and Schaeffer bijection [10] for rooted bipartite planar maps. The main difference is that they deal with planar maps (with the additional technical assumption that the root vertex has degree* 2*) and balanced well-charged trees, while we deal with plane maps and non-necessarily balanced well-charged trees. While this complicates our bijective proofs, it simplifies greatly enumeration. As we will see, our setting also extends to non zero charged trees.*

The proof Theorem 3.2 is divided into two parts. First, we define a family of blossoming trees – referred to as α_d-trees – which arise from applying the general bijective framework of Theorem 2.1 to the family of bipartite maps equipped with their canonical α_d-orientations. Second, we prove that α_d-trees are in bijection with well-charged trees.

3.3 Extending the BMS bijection: the case of dual-trumpets and dual-cornets

Definition 3.4. Consider a rooted planar map m with an additional marked vertex τ. We write ρ for the root vertex of m. Then, a *cut* of (m, τ) is defined as a partition of $\mathrm{V}(\mathrm{m})$ into two sets $C = (R, S)$ such that $\rho \in R$ and $\tau \in S$. Its *cut-set* is the subset of edges that connect a vertex in R to a vertex in S, and its *weight* is the number of such edges.

Moreover, if m is bipartite, $C = (R, S)$ is said to be *black* (respectively *white*) if all vertices in S incident to edges of the cut-set are black (respectively white).

A black (respectively white) cut is said to be *minimal* if its weight is minimal among all the black (respectively white) cuts.

Note that the trivial partition $(\mathrm{V} \setminus \{\tau\}, \{\tau\})$ defines a cut – called the *trivial cut* – which has the same color as τ, and whose weight is equal to $\deg(\tau)$. This justifies the following definition:

Definition 3.5. A map with an additional marked vertex (m, τ) is said to be *tight* if the trivial partition is minimal, and is said to be *strictly tight* if it is the unique minimal cut

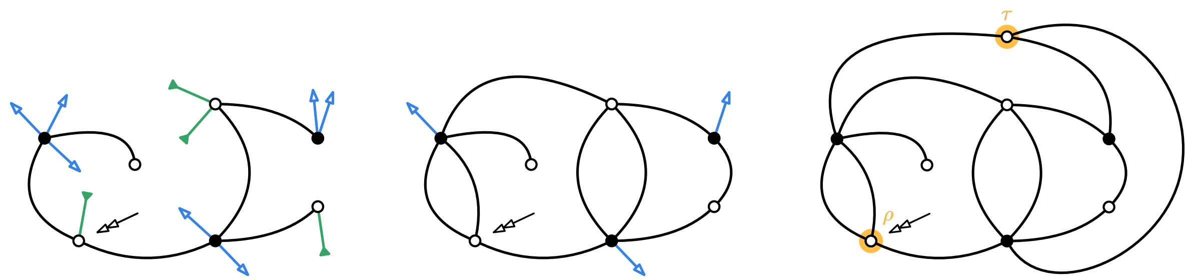

Figure 4: A well-charged tree of charge -3 (left), its closure (middle), and its complete closure (m, ρ, τ) which is a dual-cornet with $\deg(\tau) = 3$ (right).

of this color.

A *tight* pair (m, τ) such that τ is black is called a *dual-trumpet*, and a *strictly tight* pair (m, τ) such that τ is white is called a *dual-cornet*.

Remark 3.6. *By duality, rooted planar maps with an additional marked vertex correspond to rooted planar maps with an additional marked face, colloquially known as annular maps or maps on the cylinder.*

Annular maps satisfying a tightness condition were first considered in [6] to enumerate d-angulations of girth d. By duality, bipartite maps correspond to Eulerian maps. In that setting, annular maps with tightness conditions – called cornets and trumpets – were introduced and enumerated in [1, Section 3.3]. They correspond exactly to the dual of dual-cornets and dual-trumpets.

The closure $\Psi(\mathrm{t})$ of a well-charged tree t with non-zero charge is a blossoming map with unmatched closing stems if $c(\mathrm{t}) > 0$, and unmatched opening stems otherwise. The *complete closure* of t is the map obtained from $\Psi(\mathrm{t})$ as follows: add a new marked vertex τ in the outer face, and replace each unmatched stem by an edge connecting its extremity to τ, see Figure 4. To ensure that the complete closure is bipartite, we set that τ is black if $c(\mathrm{t}) > 0$ and white otherwise.

Following the proof of Theorem 3.2 combined with the max-flow min-cut theorem we can prove the following theorem, which is the main result of the section:

Theorem 3.7. *Let $k > 0$. The complete closure operation is a one-to-one correspondence between $\mathcal{T}_k$ and the set of dual-trumpets (m, τ) with $\deg(\tau) = k$.*

Similarly, the complete closure operation is a one-to-one correspondence between $\mathcal{T}_{-k}$, and the set of dual-cornets (m, τ) with $\deg(\tau) = k$.

Both these bijections preserve the vertex degree distribution and the color of the root vertex.

The emergence of tight and strictly tight maps may be surprising but comes from technical tweaks that are needed to generalize our arguments to this setting.

4 Enumerative consequences

Our new bijective framework allows us to establish parametrizations for the generating functions of the following families of maps. It is based on the relatively straightforward (recursive) enumeration of families of well-charged trees, which are in bijection with these maps.

1. Bipartite *plane* maps, as a direct consequence of Theorem 3.2. This is closely related to [10, Theorem 3] that established similar formulas for bipartite planar maps rooted at a vertex of degree 2.

2. Bipartite *planar* maps, by applying an extra formal integration step that corresponds to forgetting the marked face. However, the formulas obtained are not necessarily rational. Still, this is the case for quartic maps, and we recover a rational parametrization that is equivalent to the one given in [10].

3. The so-called dual-trumpets, dual-cornets, and the hypermaps with two monochromatic boundaries, as discussed in [1, Corollary 3.13, Corollary 4.2]. We obtain these results in the dual setting using Theorem 3.7, which also provides a simpler proof.

4. Quartic planar maps decorated with an Ising model, which is the object of the next section. A different parametrization was established in [10], but the one we derive has the additional property that it has non-negative integer coefficients. This is an important property for the probabilistic study of the Ising model on planar maps with external magnetization, as discussed in [22].

Note that these enumerations are performed with control on the vertex or face degrees.

5 Application to the Ising model

A *spin configuration* on a map m is a coloring in black and white of its vertices, that is a mapping $\sigma : \mathrm{V}(\mathrm{m}) \to \{\bullet, \circ\}$. An edge $e = \{v_1, v_2\}$ is *monochromatic* if $\sigma(v_1) = \sigma(v_2)$, and *frustrated* otherwise. The number of monochromatic edges of (m, σ) is denoted by $m(\mathrm{m}, \sigma)$.

Let $\mathcal{I}_\circ$ be the set of rooted quartic planar maps, meaning that all vertices have degree 4, endowed with a spin configuration, in which the root vertex is white. Its associated weighted generating function is the formal power series defined, in $\mathbb{Q}[x, y, \nu, u][\![t]\!]$, as follows:

$$I_\circ(x, y, t, \nu, u) := \sum_{(\mathrm{m},\sigma)\in\mathcal{I}_\circ} x^{|V_\circ(\mathrm{m}|} y^{|V_\bullet(\mathrm{m})|} t^{\#\mathrm{E}(\mathrm{m})} \nu^{m(\mathrm{m},\sigma)} u^{\#\mathrm{F}(\mathrm{m})}. \tag{5.1}$$

Note that the variable u is redundant due to Euler's formula, but its use simplifies the proof. As a consequence of Theorem 3.2 and Theorem 3.7, we obtain:

Theorem 5.1. *The generating function of white-rooted quartic planar maps endowed with a spin configuration is given by:*

$$I_\circ(x,y,t,\nu,u) = \frac{\mathrm{Pol}_{I_\circ}(x,y,t,\nu,u,Q)}{9\,(1-\nu^2)\,t^4\Big(1+3x\,(1-\nu^2)\,Q\Big)}, \tag{5.2}$$

where the series $Q \equiv Q(x,y,t,\nu,u)$ *is the unique formal power series in* $\mathbb{Q}(x,y,\nu,u)[\![t]\!]$ *with constant term 0 that satisfies the Lagrangian equation:*

$$t^2 = \frac{Q\left(1-3\nu^2\,(x+y)\,Q-3xy\,(1-\nu^2)\,(3\nu^2+7)\,Q^2+135x^2y^2\,(1-\nu^2)^3\,Q^4-243x^3y^3\,(1-\nu^2)^5\,Q^6\right)}{u\left(1-9xy\,(1-\nu^2)^2\,Q^2\right)^2}, \tag{5.3}$$

and where the polynomial $\mathrm{Pol}_{I_\circ}(x,y,t,\nu,u,q) \in \mathbb{Q}\,[x,y,t,\nu,u,q]$ *is defined as follows:*

$$\begin{aligned}
&\mathrm{Pol}_{I_\circ}(x,y,t,\nu,u,q)\\
&\quad := 405x^3y^2\left(1-\nu^2\right)^4 q^7+351x^2y^2\left(1-\nu^2\right)^3 q^6+27xy\left(1-\nu^2\right)^2\left(\nu^2 y-\left(5+12t^2u\left(1-\nu^2\right)^2 y\right)x\right)q^5\\
&\quad +3xy\left(1-\nu^2\right)\left(36t^2u\left(1-\nu^2\right)^2 x-3\nu^2-47\right)q^4+\left(\left(252t^2u\left(1-\nu^2\right)^2 y-6\nu^2-9\right)x-15\nu^2 y\right)q^3\\
&\quad +\left(\left(36t^2u\left(1-\nu^2\right)-108t^4u^2y\left(1-\nu^2\right)^3\right)x+5+9\nu^2t^2u\left(1-\nu^2\right)y\right)q^2-t^2u\left(27t^2u\left(1-\nu^2\right)^2 x-3\nu^2+8\right)q\\
&\quad +3t^4u^2\left(1-\nu^2\right).
\end{aligned}$$

Moreover, the series Q *has non-negative integer coefficients.*

The fact that the series Q has non-negative integer coefficients follows from it corresponds to the generating series of rooted planar maps endowed with a spin configuration where all vertices have degree 4 except the root vertex and an extra marked vertex, both of which have degree 1. This interpretation is a byproduct of Theorem 3.7 as this family of maps is related to that of well-charged trees of charge 1.

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