Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #36, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

m is for meet-congruence

Nathan Reading* and Tanvi Thummar*

Department of Mathematics, North Carolina State University, USA

Abstract. Lattice congruences and quotients of the weak order on a finite Coxeter group are a rich source of combinatorial insight. Recently, Stump, Thomas, and Williams defined the m-eralized weak order associated to a finite Coxeter group, as a way of constructing m-eralized Cambrian lattices. The case $m = 1$ recovers the usual weak order and Cambrian lattices. Lattice congruences and quotients are less useful in the case $m > 1$, in a sense that we make precise. We show, by several explicit constructions, that meet-congruences and meet-quotients play a crucial role in the combinatorics of the m-weak order. Most notably, we show that each m-eralized Cambrian lattice is a meet-quotient of the m-eralized weak order and construct a meet-congruence that relates to an m-version of the descent map.

1 Introduction

The m-eralized weak order $\mathsf{Weak}^{(m)}(W)$ associated to a finite Coxeter group W (for $0 \leq m \leq \infty$) was defined by Stump, Thomas, and Williams [14]. Their motivations lie in Coxeter-Catalan combinatorics, or more correctly Coxeter-Fuss-Catalan combinatorics. More specifically, the goal was to define and study the m-eralized Cambrian (or m-Cambrian) lattices. These are sublattices of the m-weak order that generalize the Cambrian lattices [7, 9]. An m-Cambrian lattice depends on the choice of a Coxeter element c, and the elements of the m-Cambrian lattice are the c-sortable elements, as generalized to the Fuss-Catalan setting by Stump, Thomas, and Williams [14, Chapter 6].

A lot of interesting combinatorics of the usual ($m = 1$) weak order on a finite Coxeter group W is connected to lattice congruences of the weak order and their associated quotients. For basic facts about lattice congruences, from a combinatorial point of view, see [11, Section 9-5]. For the lattice congruences of the weak order specifically, see [6, 11, 10].

However, lattice congruences of the m-weak order do not appear to play an important role. Indeed, we prove in [12] that if W is finite and irreducible with $|S| > 1$, then there are no nontrivial lattice congruences of $\mathsf{Weak}^{(\infty)}(W)$. The main thesis of this paper is that *meet-congruences* of the m-weak order (as opposed to *lattice congruences*) play a crucial combinatorial role.

*Partially supported by the National Science Foundation under Grant Number DMS-2054489 and by the Simons Foundation under grant number SFI-MPS-TSM-00013393.

Acknowledgements

This is an extended abstract of a full paper [12] of the same title. The authors are grateful for helpful conversations with Vic Reiner, Christian Stump, and Nathan Williams.

2 Meet-congruences

We assume the most basic definitions about partially ordered sets, meet-semilattices, and lattices. Results in this section of the extended abstract are quoted from [13, Section 2].

We write $x \wedge y$ for the meet in a meet-semilattice L and $\bigwedge U$ for the meet of a finite, nonempty subset of L. Dually, we write $x \vee y$ and $\bigvee U$ for the join, when these exist. We will consider ***finitary*** meet-semilattices, meaning that for every $x \in L$, the set $\{y \in L : y \leq x\}$ is finite. A finitary meet-semilattice has a unique minimal element $\hat{0}$. In a finitary meet-semilattice, the join of a pair or of a set exists if and only if some upper bound exists. A finitary meet-semilattice L is ***upper-homogeneous***, or ***upho*** for short, if for every $x \in L$, the subposet induced by $\{y \in L : x \leq y\}$ is isomorphic to L.

The equivalence relation Θ is a ***meet-congruence*** if whenever $x_1 \equiv_\Theta x_2$ and $y_1 \equiv_\Theta y_2$, then also $x_1 \wedge y_1 \equiv_\Theta x_2 \wedge y_2$. A map η between two meet-semilattices is a ***meet-homomorphism*** if it satisfies $\eta(x \wedge y) = \eta(x) \wedge \eta(y)$. In this case, the set of nonempty fibers of η is a meet-congruence.

A subset $U \subseteq L$ is ***order-convex*** if whenever $x, y \in U$ have $x \leq y$, then $[x, y] \subseteq U$. The following fact was pointed out in [1, Proposition 3.3], [5, Proposition 37].

Proposition 2.1. *An equivalence relation Θ on a finitary meet-semilattice L is a meet-congruence if and only if*

(i) *Each Θ-class is order-convex in L and has a unique minimal element, and*

(ii) *The map $\pi_\downarrow^\Theta$ mapping each element to the minimal element of its Θ-class is order-preserving.*

If Θ is a meet-congruence of L, the ***quotient*** L/Θ is the set of Θ-classes. The meet of the class of x and the class of y is defined to be the class of $x \wedge y$. Write $\pi_\downarrow^\Theta(L)$ for the subposet of L induced by the image of $\pi_\downarrow^\Theta$.

Proposition 2.2. *Suppose L is a finitary meet-semilattice and Θ is a meet-congruence on L. The subposet $\pi_\downarrow^\Theta(L)$ is a meet-semilattice, isomorphic to the quotient L/Θ. The map $\pi_\downarrow^\Theta$ is a surjective meet-homomorphism from L to $\pi_\downarrow^\Theta(L)$.*

A subset U of L is a ***partial join-sublattice*** if whenever $x, y \in U$ and $x \vee y$ exists in L, then also $x \vee y \in U$. The following proposition is due to Freese and Nation [2].

Proposition 2.3. *Suppose L is a finitary meet-semilattice and $U \subseteq L$.*

a. *The following are equivalent.*

(i) *There exists a meet-congruence Θ such that $U = \pi_\downarrow^\Theta(L)$.*
(ii) *U is a partial join-sublattice of L and contains $\hat{0}$.*
(iii) *For all $x \in L$, the set $\{a \in U : a \leq x\}$ has a unique maximal element.*

b. *If Θ is a meet-congruence with $U = \pi_\downarrow^\Theta(L)$, the map $\pi_\downarrow^\Theta$ sends an element $x \in L$ to the largest element of U below x.*

The set of all meet-congruences of L, partially ordered by refinement, is the ***meet-congruence lattice*** $\mathsf{Con}_\wedge(L)$. The following fact is well known.

Proposition 2.4. *Suppose L is a meet-semilattice. Then $\mathsf{Con}_\wedge(L)$ is a complete lattice, and in fact a complete sublattice of the partition lattice of L.*

We refer to cover relations in L as ***edges*** of L. A meet-congruence Θ ***contracts*** an edge $a \lessdot b$ if $a \equiv_\Theta b$. For any edge $a \lessdot b$, there is a unique smallest meet-congruence $\mathsf{JIcon}_\wedge(a \lessdot b)$ contracting that edge. The map sending the edge $a \lessdot b$ to the meet-congruence $\mathsf{JIcon}_\wedge(a \lessdot b)$ is a bijection from the set of edges of L to the set of completely join-irreducible meet-congruences of L.

The following theorem can be taken as the definition of $\mathsf{MIcon}_\wedge(x)$, for any $x \in L \setminus \{\hat{0}\}$. To rephrase: $\mathsf{MIcon}_\wedge(x)$ is the coarsest (largest) meet-congruence Θ with $x \in \pi_\downarrow^\Theta$. Part of this theorem is due to Freese and Nation [2, Theorem 1].

Theorem 2.5. *Suppose L is a finitary meet-semilattice.*

a. *If $x \in L \setminus \{\hat{0}\}$, the set $\left\{\Theta \in \mathsf{Con}_\wedge(L) : x = \pi_\downarrow^\Theta(x)\right\}$ has a unique maximal element $\mathsf{MIcon}_\wedge(x)$.*

b. *The map sending x to $\mathsf{MIcon}_\wedge(x)$ is a bijection from $L \setminus \{\hat{0}\}$ to the set of completely meet-irreducible meet-congruences of L.*

c. *A meet-congruence is completely meet-irreducible in $\mathsf{Con}_\wedge(L)$ if and only if it is a coatom of $\mathsf{Con}_\wedge(L)$.*

d. *If $\Theta \in \mathsf{Con}_\wedge(L)$, then* $\Theta = \displaystyle\bigwedge_{x \in \mathsf{JIrr}(\pi_\downarrow^\Theta(L))} \mathsf{MIcon}_\wedge(x)$.

e. *If $\Theta \in \mathsf{Con}_\wedge(L)$, then $\left\{\mathsf{MIcon}_\wedge(x) : x \in \mathsf{JIrr}(\pi_\downarrow^\Theta(L))\right\}$ is the unique minimal set of completely meet-irreducible meet-congruences whose meet is Θ.*

An interval in L is called a ***polygon*** if it is the union of two finite maximal chains from x to y, each chain having 3 or more elements, with the chains disjoint except at x and y. We say L is ***lower-polygonal*** if, for every pair x_1, x_2 of elements, both covered by some element y, the interval $[x_1 \wedge x_2, y]$ is a polygon. We say L is ***partial upper-polygonal*** for every pair y_1, y_2 of elements, both covering some element y, if $y_1 \vee y_2$ exists, then the interval $[x, y_1 \vee y_2]$ is a polygon. We say that L is ***upper-polygonal*** for every pair y_1, y_2 of elements, both covering some element y, the join $y_1 \vee y_2$ exists and the interval

$[x, y_1 \vee y_2]$ is a polygon. A finitary lattice L is ***polygonal*** if it is both upper-polygonal and lower-polygonal.

The edges in a polygon $[x, y]$ are ***top edges*** (incident to y) or ***bottom edges*** (incident to x), or ***side edges*** (incident to neither x nor y). Two edges in the polygon are ***opposite*** if they are not in the same chain. A set $\mathcal{E}$ of edges of L is closed under ***meet-forcing in polygons*** if for every $a \lessdot b$ in $\mathcal{E}$ and every polygon P such that $a \lessdot b$ is a top edge of P, the side edges and bottom edges of P that are opposite to $a \lessdot b$ are also in $\mathcal{E}$. A set $\mathcal{E}$ of edges of L is closed under ***meet-contraforcing in polygons*** if for every edge $a \lessdot b$ in $\mathcal{E}$ and every polygon P such that $a \lessdot b$ is a bottom or side edge of P, the opposite top edge of P is also in $\mathcal{E}$.

Given a meet-congruence Θ, define $\mathsf{edges}(\Theta)$ to be the set of edges contracted by Θ and define $\mathsf{nonedges}(\Theta)$ to be the set of edges of L *not* contracted by Θ.

3 The m-weak order

In our treatment of the m-weak order, we have followed and generalized [14], with a lot of input from [4]. We begin with the case $m = \infty$. We assume basic familiarity with Coxeter groups and the weak order.

Suppose (W, S) is a Coxeter system and for each $s, t \in S$, let $\mathsf{m}(s, t)$ be the order of st in W. Given an integer $k \geq 2$, let $(st)^{\frac{k}{2}}$ stand for the word that alternates $sts\cdots$ and has k letters. The ***Artin monoid*** $\mathbf{B}^+$ associated to W is the *monoid* generated by S subject to the braid relations $(st)^{\frac{\mathsf{m}(s,t)}{2}} = (ts)^{\frac{\mathsf{m}(s,t)}{2}}$ for all $s \neq t \in S$ with $\mathsf{m}(s, t) < \infty$. As usual, when $\mathsf{m}(s, t) = \infty$, no relation involving s and t is imposed. We say that $\mathbf{B}^+$ is of ***finite type*** when W is a finite Coxeter group. The ***∞-weak order*** $\mathsf{Weak}^{(\infty)}(W)$ is the prefix order on $\mathbf{B}^+$: the partial order with ground set $\mathbf{B}^+$ and with $u \leq w$ if and only if there exists a word for w with a prefix that is a word for u. Equivalently, every word for u is a prefix of some word for w. Most of the following theorem is a quotation of results from [4], and the remainder is proved using results of [4] and a result of [3].

Theorem 3.1. *The ∞-weak order* $\mathsf{Weak}^{(\infty)}(W)$ *is a finitary, lower-polygonal, partial upper-polygonal, upho meet-semilattice and is polygonal if* $\mathsf{m}(s, t) < \infty$ *for all* $s, t \in S$. *If W is finite, then* $\mathsf{Weak}^{(\infty)}(W)$ *is a finitary polygonal upho lattice.*

There is a well-defined injection $W \hookrightarrow \mathbf{B}^+$ given by taking any reduced word for an element of W and interpreting the word as a product in $\mathbf{B}^+$. We indicate this injection by $w \mapsto \boldsymbol{w}$ and write $\mathbf{B}^+_{\mathrm{red}}$ for the image of the injection. When W is finite, w_0 is the longest element of W and $\boldsymbol{w}_0$ is the corresponding element of $\mathbf{B}^+$. We write e for the identity element in W and thus $\boldsymbol{e}$ for the unique minimal element of $\mathbf{B}^+$.

For any $x \in \mathbf{B}^+$, there is a unique maximal element $\alpha(x)$ of $\{a \in \mathbf{B}^+_{\mathrm{red}} : a \leq x\}$ [4, Proposition 2.1]. When W is finite, $\alpha(x) = x \wedge \boldsymbol{w}_0$. Let $\omega(x)$ be the unique element

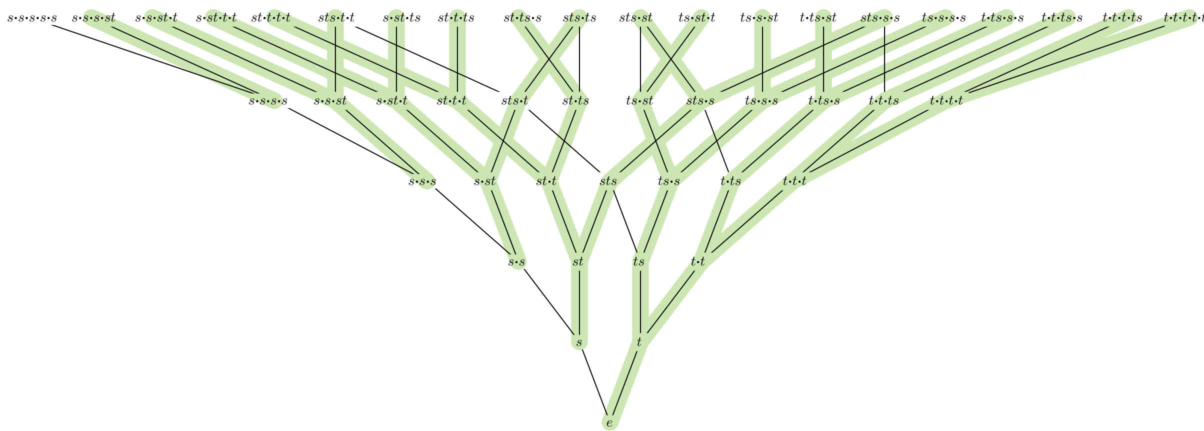

Figure 1: $\Theta^{(\infty)}_{\{s\}}$ on part of $\mathsf{Weak}^{(\infty)}(W)$ for W of type A_2

of $\mathbf{B}^+$ such that $x = \alpha(x)\omega(x)$. The ***Garside factorization*** or ***normal form*** of x is the unique factorization of the element as $\boldsymbol{x}_1 \cdots \boldsymbol{x}_k$ defined recursively as follows: If $x = \boldsymbol{e}$, then $k = 0$, so that the factorization is empty. If $x > \boldsymbol{e}$, then $\boldsymbol{x}_1 = \alpha(x)$ and $\boldsymbol{x}_2 \cdots \boldsymbol{x}_k$ is the Garside factorization of $\omega(x)$. Each $\boldsymbol{x}_i$ is in $\mathbf{B}^+_{\text{red}}$ and thus is the image of an element $x_i \in W$ with $x_i \neq e$. The factors $\boldsymbol{x}_i$ are called the ***Garside factors*** of x.

For $0 \leq m \leq \infty$, the ***m-weak order*** or ***m-eralized weak order*** $\mathsf{Weak}^{(m)}(W)$ is the subposet of $\mathbf{B}^+$ consisting of elements with at most m Garside factors. The map $w \mapsto \boldsymbol{w}$ is an isomorphism from the usual (right) weak order $\mathsf{Weak}(W)$ on W to the 1-weak order $\mathsf{Weak}^{(1)}(W)$. When W is finite and $m < \infty$, [14, Proposition 2.11.4] says that $\mathsf{Weak}^{(m)}(W)$ is the interval $[\boldsymbol{e}, \boldsymbol{w}_0^m]$ in $\mathsf{Weak}^{(\infty)}(W)$, which is finite because $\mathsf{Weak}^{(\infty)}(W)$ is finitary.

Example 3.2. Figures 1 and 2 show $\mathsf{Weak}^{(\infty)}(W)$ and (2 copies of) $\mathsf{Weak}^{(3)}(W)$ for W of type A_2, with elements labeled by their Garside factorizations. The shading of certain edges in the figure will be explained later.

Theorem 3.3. $\mathsf{Weak}^{(m)}(W)$ *is a finitary, lower polygonal, partial upper-polygonal meet-semilattice. If W is finite and $m < \infty$, then* $\mathsf{Weak}^{(m)}(W)$ *is a finite polygonal lattice.*

Remark 3.4. The basic structure of the m-weak order is closely related to meet-congruences and meet-homomorphisms. In the full version [12], we construct a meet-congruence on $\mathsf{Weak}^{(\infty)}(W)$ whose associated meet-quotient is the m-weak order $\mathsf{Weak}^{(m)}(W)$. For any $k < m$, the analogous meet-congruence Θ^m_k realizes $\mathsf{Weak}^{(k)}(W)$ as a meet-quotient of $\mathsf{Weak}^{(m)}(W)$. We also realize $\mathsf{Weak}^{(k)}(W)$ as a meet-quotient of $\mathsf{Weak}^{(m)}(W)$ in a different way. Figure 2 shows these two meet-congruences for $k = 2$, $m = 3$, and W of type A_2. The meet-congruence is indicated by shading contracted edges. Furthermore, we show, for $0 \leq m \leq \infty$ and any standard parabolic subgroup W_J of W, that the map

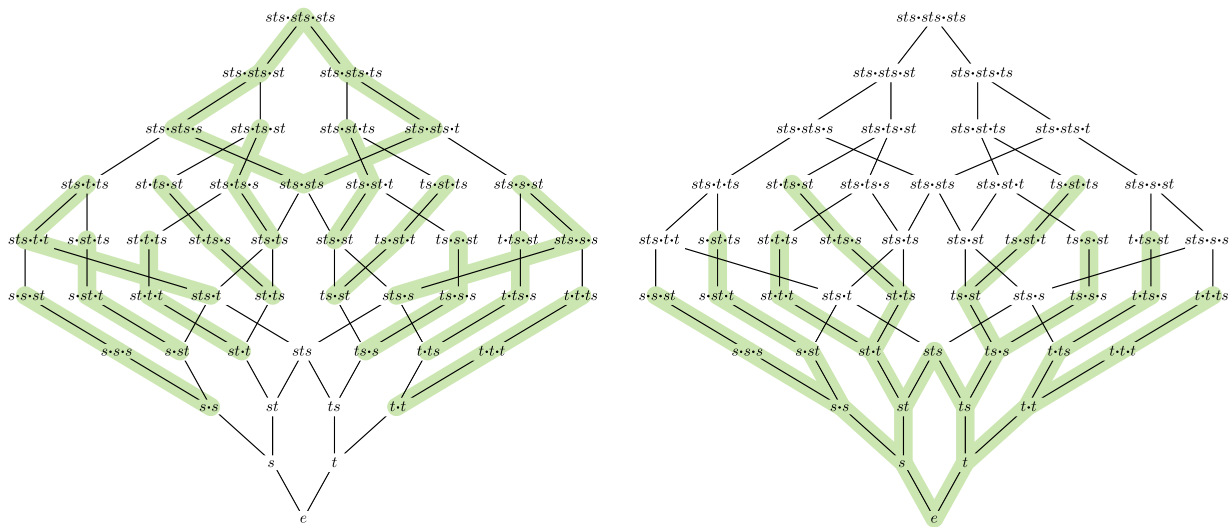

Figure 2: Θ_2^3 and $\Theta_{w_0}^{\delta}$ on $\mathsf{Weak}^{(3)}(W)$ for W of type A_2

from $\mathsf{Weak}^{(m)}(W)$ to $\mathsf{Weak}^{(m)}(W_J)$ is a surjective meet-homomorphism. Figure 1 shows the corresponding meet-congruence, for $m = \infty$, W of type A_2, $S = \{s, t\}$, and $J = \{s\}$.

4 m-Cambrian meet-congruences

The m-Cambrian lattice was defined by Stump, Thomas, and Williams [14] as a sublattice of the m-weak order. We construct a meet-congruence $\Theta_c^{(m)}$ on the m-weak order such that the m-Cambrian lattice is the set of bottom elements of $\Theta_c^{(m)}$-classes.

Fix a Coxeter group W and Coxeter element c. Stump, Thomas, and Williams [14] extended the definition [8, Section 2] of c-sorting words and c-sortable elements in W to c-sortable elements of $\mathsf{Weak}^m(W)$ for $m \leq \infty$ and W finite.

Fix some reduced word $s_1 \cdots s_n$ for c and let $c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \cdots$. The symbols "$|$" are for convenience, allowing us to identify subwords of c^∞ with sequences of subsets of S. The ***c-sorting word*** for $w \in \mathbf{B}^+$ is the subword of c^∞ that is lexicographically leftmost (as a sequence of positions in c^∞) among subwords that are words for w. An element $w \in \mathbf{B}^+$ is ***c-sortable*** if and only if the sequence of subwords corresponding to its c-sorting word is nested, with each later subset in the sequence contained in or equal to the previous subset. The property of being c-sortable depends (as the name suggests) only on c, not on the choice of reduced word for c chosen to define c^∞.

The recursive definition of c-sortable elements by induction on length and rank extends to c-sortable elements in $\mathbf{B}^+$ [14, Proposition 6.2.1]. Using a similar recursion, we now define a map $\pi_\downarrow^c$ on $\mathbf{B}^+$ that extends the domain of the map in [9, (3.1)] from W to $\mathbf{B}^+$. We say that $s \in S$ is ***initial*** in c if there is a reduced word for c having s as its first letter (or equivalently, $s \leq c$ in the weak order). Let $\langle s\rangle$ stand for $S \setminus \{s\}$. If s is initial in c, then sc is a Coxeter element of $W_{\langle s\rangle}$ and scs is a Coxeter element of W. As a base case, define $\pi_\downarrow^c(\boldsymbol{e}) = \boldsymbol{e}$. If s is initial in c, define

$$\pi_\downarrow^c(w) = \begin{cases} s \cdot \pi_\downarrow^{scs}(w') & \text{if } w = sw', \text{ or} \\ \pi_\downarrow^{sc}(w_{\langle s\rangle}) & \text{if } s \not\leq w. \end{cases}$$

Since sc is a Coxeter element of the parabolic subgroup $W_{\langle s\rangle}$, the notation $\pi_\downarrow^{sc}$ refers to the map on $\mathbf{B}^+_{\langle s\rangle}$. The following proposition extends [9, Proposition 3.2] from W to $\mathbf{B}^+$.

Proposition 4.1. *Let W be a finite Coxeter group and fix a Coxeter element c. For each $x \in \mathbf{B}^+$, $\pi_\downarrow^c(x)$ is the unique maximal c-sortable element weakly below x.*

The following theorem serves to define the ***m*-eralized *c*-Cambrian meet-congruence** $\Theta_c^{(m)}$ for $m \leq \infty$ and justifies the term ***m*-eralized *c*-Cambrian lattice** for the subposet $\mathsf{Camb}^{(m)}(W,c)$ of $\mathsf{Weak}^{(m)}(W)$ induced by the c-sortable elements in $\mathsf{Weak}^{(m)}(W)$.

Theorem 4.2. *Let W be a finite Coxeter group, fix a Coxeter element c, and take $0 \leq m \leq \infty$.*

- **a.** *There exists a meet-congruence $\Theta_c^{(m)}$ on $\mathsf{Weak}^{(m)}(W)$ such that $\pi_\downarrow^{\Theta_c^{(m)}}(\mathsf{Weak}^{(m)}(W))$ is the set of c-sortable elements in $\mathsf{Weak}^{(m)}(W)$.*
- **b.** *The map $\pi_\downarrow^{\Theta_c^{(m)}}$ coincides with the restriction of $\pi_\downarrow^c$ to $\mathsf{Weak}^{(m)}(W)$.*
- **c.** *The subposet $\mathsf{Camb}^{(m)}(W,c)$ of $\mathsf{Weak}^{(m)}(W)$ induced by the c-sortable elements is a lattice, and in fact, a join-sublattice of $\mathsf{Weak}^{(m)}(W)$.*
- **d.** *The restriction of $\pi_\downarrow^c$ is a meet-homomorphism from the m-weak order $\mathsf{Weak}^{(m)}(W)$ to the m-eralized c-Cambrian lattice $\mathsf{Camb}^{(m)}(W,c)$.*

Example 4.3. Figure 3 shows $\Theta_c^{(3)}$ on $\mathsf{Weak}^{(3)}(W)$ for W of type A_2 and $c = st$.

Remark 4.4. Theorem 4.2.c is already known, by results of [14]. In fact, Stump, Thomas, and Williams show further that $\mathsf{Camb}^{(m)}(W,c)$ is a sublattice of $\mathsf{Weak}^{(m)}(W)$.

When W is finite, the c-Cambrian meet-congruence $\Theta_c^{(1)}$ on $\mathsf{Weak}^{(1)}(W) = W$ coincides, as an equivalence relation, with the c-Cambrian lattice congruence on W. The c-Cambrian lattice congruence on W has a simple characterization as a join (in the lattice of lattice congruences on W) of join-irreducible lattice congruences. Here, we will not go into details on the lattice of lattice congruences on W, but the following statement is a version of the characterization: The Cambrian lattice congruence on W is the smallest lattice congruence on W that contracts all edges $(ts)^{\frac{k-1}{2}} \lessdot (ts)^{\frac{k}{2}}$ such that s is before t in a reduced word for c and $2 \leq k < \mathsf{m}(s,t)$.

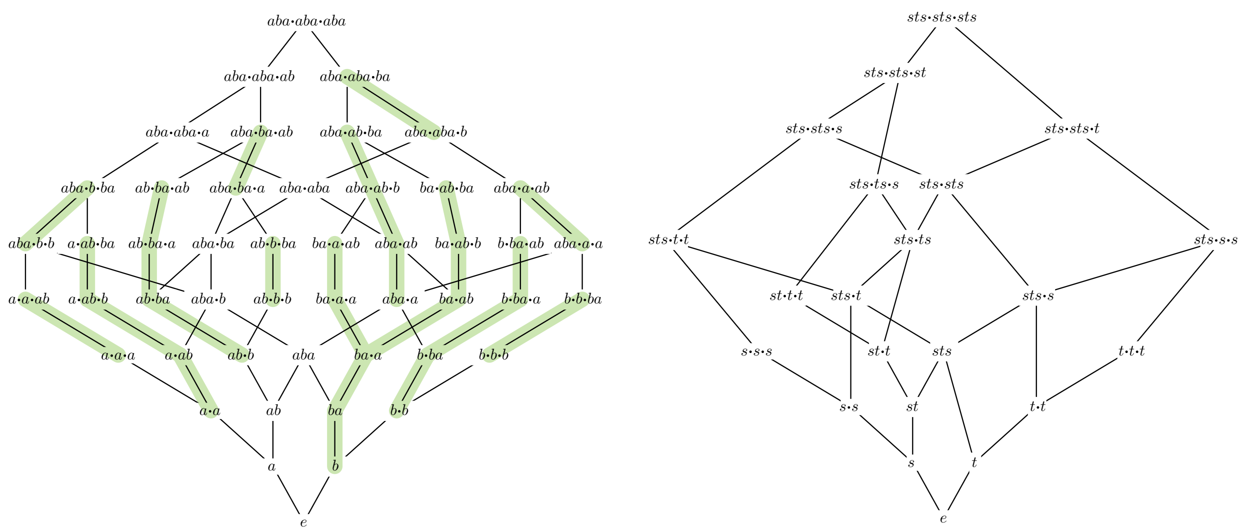

Figure 3: Cambrian meet-congruence and quotient on $\mathsf{Weak}^{(3)}(W)$ for W of type A_2

It appears that there is no analogous description of the meet-congruence $\Theta_c^{(m)}$ by specifying a small set of edges to contract (or equivalently, as a join of join-irreducible meet-congruences). On the other hand, Theorem 2.5 makes it possible to describe $\Theta_c^{(m)}$ as the meet of meet-irreducible meet-congruences. The case $m = 1$ of the following theorem is [13, Theorem 4.1.2].

Theorem 4.5. *Let W be a finite Coxeter group, fix a Coxeter element c, and take $0 \le m \le \infty$.*

- **a.** *The set $\left\{\mathsf{MIcon}_\wedge(x) : x \in \mathsf{JIrr}(\mathsf{Weak}^{(m)}(W)) \cap \mathsf{Camb}^{(m)}(W,c)\right\}$ is the unique minimal set of completely meet-irreducible meet-congruences on $\mathsf{Weak}^{(m)}(W)$ whose meet is $\Theta_c^{(m)}$.*
- **b.** *The meet-congruence $\Theta_c^{(m)}$ is the largest meet-congruence not contracting any edges $j_* \lessdot j$ such that j is a join-irreducible c-sortable element of $\mathsf{Weak}^{(m)}(W)$.*
- **c.** *The set $\mathsf{nonedges}(\Theta_c^{(m)})$ is the smallest set of edges of $\mathsf{Weak}^{(m)}(W)$ that is closed under meet-contraforcing in polygons and contains all edges $j_* \lessdot j$ such that j is a join-irreducible c-sortable element of $\mathsf{Weak}^{(m)}(W)$.*

As part of the proof of Theorem 4.5 in [12], we give a precise and surprisingly simple description of the join-irreducible elements of the m-Cambrian lattice (Theorem 4.6) and we conjecture a related description of all cover relations in the m-Cambrian lattice.

Theorem 4.6. *Let W be a finite Coxeter group, fix a Coxeter element c, and take $0 \le m \le \infty$.*

- **a.** *A c-sortable element of $\mathsf{Weak}^{(m)}(W)$ is join-irreducible as an element of $\mathsf{Camb}^{(m)}(W,c)$ if and only if it is join-irreducible as an element of $\mathsf{Weak}^{(m)}(W)$.*
- **b.** *An element of $\mathsf{Weak}^{(m)}(W)$ is c-sortable and join-irreducible if and only if its Garside factorization is $\boldsymbol{x}_1 \cdots \boldsymbol{x}_k$ (necessarily with $k \le m$) such that x_1 is a c-sortable join-irreducible element of W, s is the unique right descent of x_1, and $x_i = s$ for $i = 2, \dots, k$.*

Conjecture 4.7. *Let W be a finite Coxeter group, fix a Coxeter element c, and take $0 \leq m \leq \infty$. If $v \in \mathsf{Camb}^{(m)}(W,c)$, then the map $\pi_\downarrow^c$ restricts to a bijection from the set of elements covered by v in $\mathsf{Weak}^{(m)}(W)$ to the set of elements covered by v in $\mathsf{Camb}^{(m)}(W,c)$.*

The case of Conjecture 4.7 where $m = 1$ and W is finite is a special case of a fact that holds for all lattice congruences on finite lattices. (See [11, Proposition 9-5.8].)

For emphasis, we restate Theorem 4.6 as an alternative definition of the m-eralized c-Cambrian lattice.

Corollary 4.8. *Let W be a finite Coxeter group, fix a Coxeter element c, and take $0 \leq m \leq \infty$. Then $\mathsf{Camb}^{(m)}(W,c)$ is the join-sublattice of $\mathsf{Weak}^{(m)}(W)$ generated by elements whose Garside factorization is $\boldsymbol{x}_1 \cdots \boldsymbol{x}_k$ such that x_1 is a c-sortable join-irreducible element of W, s is the unique right descent of x_1, and $x_i = s$ for $i = 2,\ldots,k$.*

Surprisingly, the proof of Theorem 4.6, and thus the proof of Theorem 4.5, begins with an *enumerative* result of Stump, Thomas, and Williams [14] that gives the m-Fuss-Narayana refinement of the enumeration of c-sortable elements.

Example 4.9. Take W of type A_2 and $c = st$. The left picture in Figure 4 shows $\mathsf{Weak}^{(m)}(W)$ for $m \in \{2,3\}$ with the edges $j_* \lessdot j$ shaded for c-sortable join-irreducible elements j. Theorem 4.6.a characterizes these elements j. Theorem 4.5.b says that the m-eralized c-Cambrian meet-congruence is the largest meet-congruence not contracting the highlighted edges. Theorem 4.5.c says that $\mathsf{nonedges}(\Theta_c^{(m)})$ is obtained from the highlighted edges by meet-contraforcing in polygons.

5 The m-descent map and its meet-congruence

In the case $m = 1$, the lattice congruence that is the join of the Cambrian lattice congruences Θ_c and $\Theta_{c^{-1}}$ consists of the fibers of the map sending an element to its left descent set. Consideration of the analogous meet-congruence on the m-weak order leads to an m-analog of the descent map, as we now explain.

Let W be a finite Coxeter group, fix a Coxeter element c, and let Θ be the meet-congruence $\Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$. In [12], we prove that the join-sublattice $\pi_\downarrow^\Theta(\mathsf{Weak}^{(m)}(W))$ of $\mathsf{Weak}^{(m)}(W)$ is the set of all elements $\boldsymbol{w}_0(J_1)\cdots\boldsymbol{w}_0(J_m)$ such that $S \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m$. We also show that the join-irreducible elements of $\pi_\downarrow^\Theta(\mathsf{Weak}^{(m)}(W))$ are the elements of the form $\boldsymbol{s}^k$ for $s \in S$ and $1 \leq k \leq m$. In particular, the meet-congruence $\Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$ is independent of the choice of Coxeter element c. We will consider a generalization of this meet-congruence, removing the requirement that W is finite.

Accordingly, we define $\Theta_{\mathrm{des}}^{(m)}$ to be the meet-congruence such that the set of all bottom elements of $\Theta_{\mathrm{des}}^{(m)}$-classes is the partial join-sublattice generated by $\{\boldsymbol{s}^k : s \in S, 1 \leq k \leq m\}$.

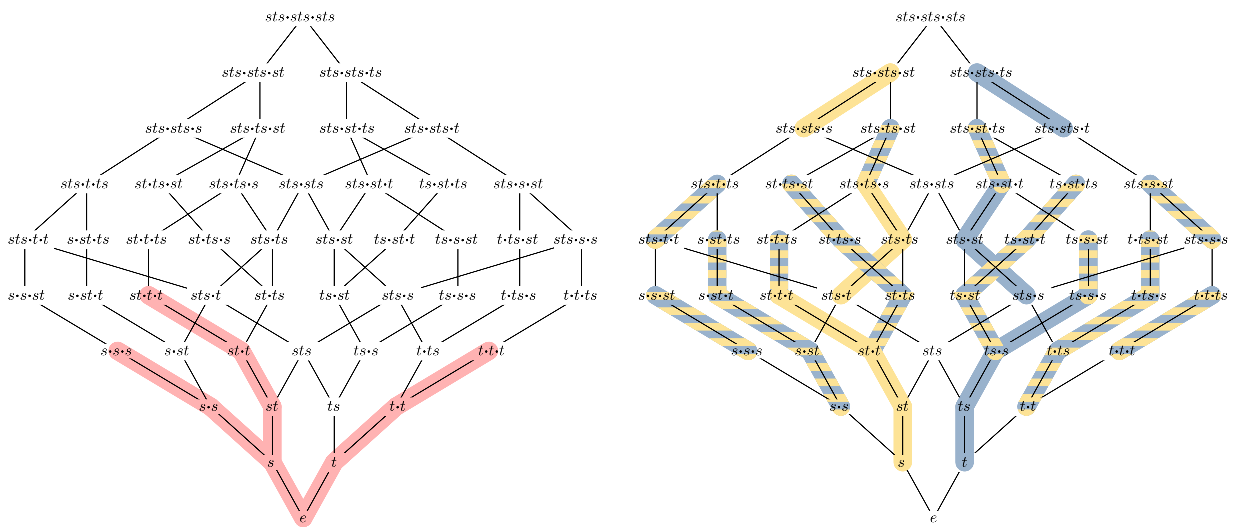

Figure 4: Two meet-congruences of $\mathsf{Weak}^{(3)}$ for W of type A_2. Left: The smallest set of non-edges that determines $\mathsf{Camb}^{(3)}(W,c)$ for $c = st$. Right: $\Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$ on $\mathsf{Weak}^{(3)}(W)$.

Thus, $\Theta_{\mathrm{des}}^{(m)} = \Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$ when W is finite. The existence of $\Theta_{\mathrm{des}}^{(m)}$ is guaranteed by Theorem 2.3. We write $\pi_{\downarrow(m)}^{\mathrm{des}}$ for the associated downward projection map.

We define, for $0 \le m \le \infty$, an m-eralization of the (left) descent set of an element of W. Given $x \in \mathsf{Weak}^{(m)}(W)$ and $1 \le i \le m$, define $D_i(x) = \{s \in S : \boldsymbol{s}^i \le x\}$. Equivalently, $D_i(x) = \{s \in S : \boldsymbol{s}^i \le \pi_{\downarrow(m)}^{\mathrm{des}}(x)\}$. Define $\mathrm{des}_L^{(m)}(x)$ to be the sequence $(D_1(x), \ldots, D_m(x))$, understood to be an infinite tuple $(D_1(x), D_2(x), \ldots)$ when $m = \infty$, and call $\mathrm{des}_L^{(m)}$ the ***m*-(left) descent map**.

Let $\widetilde{\mathbf{D}}^{(m)}$ be the set of sequences $(D_1, \ldots, D_m)$ with $S \supseteq D_1 \supseteq D_2 \supseteq \cdots \supseteq D_m$, partially ordered by entrywise inclusion. Again, $(D_1, \ldots, D_m)$ means $(D_1, D_2, \ldots)$ when $m = \infty$. This is a finitary distributive lattice isomorphic to the product of $|S|$ copies of the chain $0 \lessdot 1 \lessdot \cdots \lessdot m$ (understood to be an infinite chain $0 \lessdot 1 \lessdot \cdots$ when $m = \infty$). The isomorphism sends $(D_1, \ldots, D_m)$ to the tuple $(k_s : s \in S)$ such that k_s is the number of entries of $(D_1, \ldots, D_m)$ that contain s.

Let $\mathbf{D}^{(m)}$ be the downset of $\widetilde{\mathbf{D}}^{(m)}$ consisting of sequences $(D_1, \ldots, D_m)$ such that W_{D_i} is finite for all $1 \le i \le m$. Thus, $\mathbf{D}^{(m)}$ is a finitary meet-semilattice. It is a distributive lattice if and only if it is a lattice if and only if W is finite.

Define a map $\zeta : \mathbf{D}^{(m)} \to \pi_{\downarrow(m)}^{\mathrm{des}}(\mathsf{Weak}^{(m)}(W))$ recursively as follows. Let ϕ_{D_1} be the map on $\mathsf{Weak}^{(m)}(W)$ that conjugates each Garside factor by $w_0(D_1)$ and set

$$\zeta(D_1, \ldots, D_m) = \boldsymbol{w}_0(D_1) \cdot \phi_{D_1}(\zeta(D_2, \ldots, D_m)),$$

with ζ of the empty sequence understood to be $\boldsymbol{e}$.

Theorem 5.1.

a. $\mathrm{des}_L^{(m)}$ *is a surjective meet-homomorphism from the m-weak order to* $\mathbf{D}^{(m)}$.

b. $\Theta_{\mathrm{des}}^{(m)}$ *is the corresponding meet-congruence on* $\mathsf{Weak}^{(m)}(W)$ *(so that* $\pi_{\downarrow(m)}^{\mathrm{des}}$ *is the projection to bottom elements of* $\Theta_{\mathrm{des}}^{(m)}$*-classes).*

c. $\pi_{\downarrow(m)}^{\mathrm{des}}(\mathsf{Weak}^{(m)}(W))$ *is the set of elements* $\boldsymbol{w}_0(J_1)\cdots\boldsymbol{w}_0(J_k)$ *with* $S \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k$ *such that* W_{J_1} *is finite and* $k \le m$.

d. *The map* ζ *is an isomorphism from* $\mathbf{D}^{(m)}$ *to* $\pi_{\downarrow(m)}^{\mathrm{des}}(\mathsf{Weak}^{(m)}(W))$ *and* $\zeta \circ \mathrm{des}_L^{(m)} = \pi_{\downarrow(m)}^{\mathrm{des}}$.

e. $\{\mathsf{MIcon}_\wedge(\boldsymbol{s}^k) : s \in S,\ 0 < k \le m\}$ *is the unique minimal set of completely meet-irreducible meet-congruences on* $\mathsf{Weak}^{(m)}(W)$ *whose meet is* $\Theta_{\mathrm{des}}^{(m)}$.

f. $\Theta_{\mathrm{des}}^{(m)}$ *is the largest meet-congruence not contracting any edges* $\boldsymbol{s}^k \lessdot \boldsymbol{s}^{k+1}$ *such that* $s \in S$ *and* $0 \le k < m$.

g. $\mathsf{nonedges}(\Theta_{\mathrm{des}}^{(m)})$ *is the smallest set of edges that contains all edges* $\boldsymbol{s}^k \lessdot \boldsymbol{s}^{k+1}$ *for all* $s \in S$ *and* $0 \le k < m$ *and is closed under meet-contraforcing in polygons.*

Example 5.2. The picture on the right in Figure 4 shows the meet-congruence $\Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$ of $\mathsf{Weak}^{(3)}(W)$ for W of type A_2 and $c = st$. Edges contracted by $\Theta_c^{(m)}$ are shown in blue (or darker gray), edges contracted by $\Theta_{c^{-1}}^{(m)}$ are shown in yellow (or lighter gray), and edges contracted by both are shown striped. The meet-congruence $\Theta_c^{(m)} \vee \Theta_{c^{-1}}^{(m)}$ coincides with $\Theta_{\mathrm{des}}^{(3)}$.

Remark 5.3. It is tempting to confuse the sequence $(D_1(x), \ldots, D_m(x))$ in the definition of $\mathrm{des}_L^{(m)}(x)$ with the sequence $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k$ appearing in Assertion c of Theorem 5.1. However, they are not usually the same. For example, suppose $S = \{s, t\}$ and $\mathsf{m}(s,t) = 3$ and take x to be the element with Garside factorization $\boldsymbol{sts} \cdot \boldsymbol{s}$. Thus, $x = \pi_{\downarrow(m)}^{\mathrm{des}}(x) = \boldsymbol{w}_0(\{s,t\}) \cdot \boldsymbol{w}_0(\{s\})$. However, $\boldsymbol{sts} \cdot \boldsymbol{s} = \boldsymbol{s} \vee (\boldsymbol{t} \cdot \boldsymbol{t})$, and therefore $\mathrm{des}_L^{(m)}(x)$ is $(\{s,t\}, \{t\})$, not $(\{s,t\}, \{s\})$. Correspondingly, $w_0(\{s,t\}) \cdot t \cdot w_0(\{s,t\})$ is $sts \cdot t \cdot sts = s$, so $\zeta(\{s,t\}, \{t\}) = \boldsymbol{w}_0(\{s,t\}) \cdot \boldsymbol{s} = \boldsymbol{sts} \cdot \boldsymbol{s}$.

References

[1] E. Barnard and T. McConville. "Lattices from graph associahedra and subalgebras of the Malvenuto-Reutenauer algebra". *Algebra Universalis* **82**.1 (2021), Paper No. 2, 53. DOI.

[2] R. Freese and J. B. Nation. "Congruence lattices of semilattices". *Pacific J. Math.* **49** (1973), pp. 51–58. Link.

[3] Y. Gao, J. Guo, K. Seetharaman, and I. Seidel. "The rank-generating functions of upho posets". *Discrete Math.* **345**.1 (2022), Paper No. 112629, 14 pp. DOI.

[4] J. Michel. "A note on words in braid monoids". *J. Algebra* **215**.1 (1999), pp. 366–377. DOI.

[5] V. Pilaud and D. Poliakova. "Hochschild polytopes". *Math. Ann.* **392**.2 (2025), pp. 2395–2441. DOI.

[6] N. Reading. "Lattice congruences of the weak order". *Order* **21**.4 (2004), pp. 315–344. DOI.

[7] N. Reading. "Cambrian lattices". *Adv. Math.* **205**.2 (2006), pp. 313–353. DOI.

[8] N. Reading. "Clusters, Coxeter-sortable elements and noncrossing partitions". *Trans. Amer. Math. Soc.* **359**.12 (2007), pp. 5931–5958. DOI.

[9] N. Reading. "Sortable elements and Cambrian lattices". *Algebra Universalis* **56**.3-4 (2007), pp. 411–437. DOI.

[10] N. Reading. "Finite Coxeter groups and the weak order". *Lattice theory: special topics and applications. Vol. 2*. Birkhäuser/Springer, Cham, 2016, pp. 489–561.

[11] N. Reading. "Lattice theory of the poset of regions". *Lattice theory: special topics and applications. Vol. 2*. Birkhäuser/Springer, Cham, 2016, pp. 399–487.

[12] N. Reading and T. Thummar. "*m* is for meet-congruence". In preparation. 2026.

[13] N. Reading and T. Thummar. "Meet the congruences: meet-congruences in algebraic combinatorics". In preparation. 2026.

[14] C. Stump, H. Thomas, and N. Williams. "Cataland: why the Fuß?" *Mem. Amer. Math. Soc.* **305**.1535 (2025), pp. vii+143. DOI.