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Plethystic lifts of q-binomial identities

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Abstract. We develop a framework to study plethysms of SL_2-representations over fields of arbitrary characteristic. Each representation of the type $\mathrm{Sym}_n\mathrm{Sym}^m\mathbb{F}^2$ is identified with a space of symmetric polynomials. A certain map on polynomials (defined by substitution) gives morphisms between these representations. We study the map combinatorially, and use it to categorify numerous q-binomial identities including the q-analogues of the team-and-leader identity $(n+1)\binom{n+d+1}{n+1} = \binom{n+d}{n}(n+d+1)$ and its generalisations. We conclude with a potential application to the generalised Foulkes Conjecture for $\mathrm{SL}_2(\mathbb{C})$, and some further conjectures and directions raised by this work.

Keywords: Symmetric polynomials, plethysm, modular representation, q-analogues

1 Introduction

A powerful 'metatheorem' in enumerative combinatorics says that every interesting identity has a q-analogue which refines it. Let q be an invertible variable. The q-analogue of the integer n (the *q-integer n*) is

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n},$$

and the q-analogue of the binomial coefficient (the *q-binomial*) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q}.$$

The q-binomials are omnipresent in the vast landscape of q-hypergeometric identities [17]. They can be interpreted combinatorially as the generating functions for k-subsets of $\mathbb{N}_0$ by the sum of their elements, partitions in an $(n-k) \times k$ rectangle, and inversions in permutations of n objects [25, Ch. 1]. Each q-binomial can be written uniquely as a linear combination of q-integers,

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = \sum_k a^k_{n,m} [k]_q. \tag{1.1}$$

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The coefficients in this expansion are the $\mathrm{SL}_2(\mathbb{C})$-*plethysm coefficients*: a *plethysm* is a representation of $\mathrm{SL}_2(\mathbb{C})$ of the form $\mathrm{Sym}^n\mathrm{Sym}^m\mathbb{C}^2$ and there is an $\mathrm{SL}_2(\mathbb{C})$-isomorphism

$$\mathrm{Sym}^n\mathrm{Sym}^m\mathbb{C}^2 \cong \bigoplus\nolimits_k (\mathrm{Sym}^{k-1}\mathbb{C}^2)^{\oplus a^k_{n,m}}\,. \tag{1.2}$$

One recovers (1.1) by taking characters. We say that (1.2) *lifts* or *categorifies* (1.1) over $\mathbb{C}$. Thus the fundamental bridge between combinatorics and algebra is that $\left[{n+m \atop m}\right]_q$ is the character of $\mathrm{Sym}^n\mathrm{Sym}^m\mathbb{C}^2$ on the complex diagonal matrix $\mathrm{diag}(q, q^{-1})$.

Despite the number of existing q-binomial identities, understanding $\mathrm{SL}_N(\mathbb{C})$-plethysm coefficients for general N is an elusive and hard problem. The problem dates to Littlewood in the 1930s and was identified by Stanley as one of the major problems in algebraic combinatorics [26, Problem 9]. It has deep connections with computational complexity [8]. The recent FPSAC abstracts [2, 12, 19] show that it is a highly active research topic.

Our research is motivated by the following overarching question:

Which q-binomial identities reflect a deeper property of plethysms?

Let us proceed by example. The *team-and-leader identity* mentioned in the abstract draws its name from the combinatorial proof, in which teams of size $n + 1$ with a single leader are counted in two ways, firstly by choosing $n + 1$ of the $n + d + 1$ people and having them elect a leader; and secondly by choosing a leader in $n + d + 1$ ways and allowing the leader to pick the rest of the team. More generally, we count teams of size $n + m$ with m leaders and n followers. One q-analogue of the team-and-leader identity is then

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q \begin{bmatrix} n+m+d \\ n+m \end{bmatrix}_q = \begin{bmatrix} n+d \\ n \end{bmatrix}_q \begin{bmatrix} n+m+d \\ m \end{bmatrix}_q. \tag{1.3}$$

Automatically from (1.1) and (1.2), we obtain an $\mathrm{SL}_2(\mathbb{C})$-isomorphism

$$\mathrm{Sym}^m\,\mathrm{Sym}^n\,\mathbb{C}^2 \otimes \mathrm{Sym}^{m+n}\,\mathrm{Sym}^d\,\mathbb{C}^2 \cong \mathrm{Sym}^n\,\mathrm{Sym}^d\,\mathbb{C}^2 \otimes \mathrm{Sym}^m\,\mathrm{Sym}^{d+n}\,\mathbb{C}^2$$

lifting (1.3) over $\mathbb{C}$. However, we will shortly see that there is a much more general result. Let $\mathbb{F}$ be a field. There are two different ways of constructing a symmetric power in the category of $\mathrm{SL}_2(\mathbb{F})$-representations, by taking invariants or coinvariants:

$$\mathrm{Sym}_n V = (V^{\otimes n})^{\mathfrak{S}_n} \quad \text{and} \quad \mathrm{Sym}^n V = V^{\otimes n} / \langle \cdots \otimes x \otimes y \otimes \cdots - \cdots \otimes y \otimes x \otimes \cdots \rangle.$$

When $\mathbb{F} = \mathbb{C}$ these are isomorphic $\mathrm{SL}_2(\mathbb{C})$-representations, but they are typically not isomorphic as $\mathrm{SL}_2(\mathbb{F})$-representations when the field has positive characteristic. This makes the following theorem from [13] notable.

Proposition 1.1. *There is a characteristic-independent* $\mathrm{SL}_2(\mathbb{F})$*-isomorphism*

$$\mathrm{Sym}_m\mathrm{Sym}^n\,\mathbb{F}^2 \otimes \mathrm{Sym}_{m+n}\mathrm{Sym}^d\,\mathbb{F}^2 \cong \mathrm{Sym}_n\mathrm{Sym}^d\,\mathbb{F}^2 \otimes \mathrm{Sym}_m\mathrm{Sym}^{d+n}\,\mathbb{F}^2$$

lifting (1.3) *over* $\mathbb{F}$.

Isomorphisms that exist over $\mathbb{C}$ do not in general hold over $\mathbb{F}$: see for instance [22, Th. 1.6] for many counterexamples. This makes characteristic-independent isomorphisms a rarity. Their study is fast becoming a vibrant field. In [22], McDowell and the fourth author lifted *Hermite reciprocity*

$$\mathrm{Sym}_m \mathrm{Sym}^n \mathbb{F}^2 \cong \mathrm{Sym}^n \mathrm{Sym}_m \mathbb{F}^2 \tag{1.4}$$

and the *Wronskian isomorphism*

$$\textstyle\bigwedge^n \mathrm{Sym}^{n+m-1} \mathbb{F}^2 \cong \mathrm{Sym}_n \mathrm{Sym}^m \mathbb{F}^2. \tag{1.5}$$

The second and fourth authors lifted a special case of Stanley's q-Hook-Content Formula [21]. We generalise this in Proposition 5.1. Recently, Ikenmeyer, Omar, and Tsintsilidas lifted a correspondence of certain Kronecker and plethysm coefficients [16].

We construct a new framework for lifting q-binomial identities over $\mathbb{F}$. In §2 we give an isomorphism between each plethysm and a certain space of symmetric polynomials:

$$\mathrm{Sym}_m \mathrm{Sym}^n \mathbb{F}^2 \cong \Lambda_{\leqslant n}[x_1, \ldots, x_m]. \tag{1.6}$$

This brings symmetric polynomials into representation theory *without taking characters*, but instead by using symmetric polynomials as the *elements* of the representation. With the exception of (1.4), all of the above isomorphisms (both old and new) become simple maps between polynomial rings, defined by specialisations of certain variables to other variables. The canonical example of such a map is the k-fold plethystic substitution (or simply, the *k-fold map*) of symmetric functions

$$\begin{aligned} \Lambda[\mathbf{x}_\infty] &\to \Lambda[\mathbf{y}_\infty] \\ f(x_1, x_2, \ldots) = f(\mathbf{x}_\infty) &\mapsto f[k\mathbf{y}_\infty] = f(y_1, \overset{k}{\ldots}, y_1, y_2, \overset{k}{\ldots}, y_2, \ldots) \end{aligned} \tag{1.7}$$

defined in §3. In this abstract we give combinatorial interpretations for the structure constants of the k-fold map on the Schur and the monomial bases. We sketch a proof of Proposition 1.1 (lifting the q-team-and-leader identity) in §3 and Proposition 5.1 (lifting the q-Hook-Content Formula) in §5.

We finish by giving two conjectures. First, that a certain q-binomial identity implied by the q-Pfaff–Saalschütz identity [15] lifts to a filtration of $\mathrm{SL}_2(\mathbb{F})$-plethysms. This result would complete a crucial step in the categorification of Lusztig's integral form of the Cartan subalgebra of the quantum group $U_q(\mathfrak{sl}_2)$. See [20] for details on this programme, which predicted Propositions 1.1 and 5.1, and is heuristic evidence for Conjecture 6.1.

The second is the generalised Foulkes Conjecture for $\mathrm{SL}_2(\mathbb{C})$: if $a = \min\{a, b, c, d\}$ and $ab = cd$ then

$$\begin{bmatrix} c+d \\ c \end{bmatrix}_q \geqslant_q \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where $f \geqslant_q g$ if $f - g$ is a non-negative combination of q-integers. This conjecture was posed by Doran, Vessenes, Abdesselam and Chipalkatti, and Bergeron [6, 27, 1, 3]. We conjecture, moreover, that the k-fold map lifts this inequality (over $\mathbb{C}$) when a divides c.

2 Plethysms as spaces of symmetric polynomials

Let $\mathbb{F}_d[X,Y]$ be the ring of homogeneous polynomials in X and Y of degree d. The group $\mathrm{SL}_2(\mathbb{F})$ acts on $\mathbb{F}_d[X,Y]$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot P(X,Y) = P(\alpha X + \gamma Y,\ \beta X + \delta Y).$$

A classical result is that $\mathbb{C}_d[X,Y]$ is the irreducible $\mathrm{SL}_2(\mathbb{C})$-representation of dimension $d+1$. Moreover $\mathbb{C}_d[X,Y] \cong \mathrm{Sym}^d\mathbb{C}^2$, and more generally $\mathbb{F}_d[X,Y] \cong \mathrm{Sym}^d\mathbb{F}^2$.

Consider now the plethysm $\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2$. By definition,

$$\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2 = \left\{P \in \mathbb{F}[X_1,\ldots,X_n,Y_1,\ldots,Y_n] \mid \text{degree } d \text{ multi-homogenous in pairs}\right\}^{\mathfrak{S}_n},$$

where $\sigma \in \mathfrak{S}_n$ acts by sending X_i to $X_{\sigma(i)}$ and Y_i to $Y_{\sigma(i)}$ for all i. We remark that, using tensor products of algebras, the right-hand side becomes $(\mathbb{F}_d[X,Y]^{\otimes n})^{\mathfrak{S}_n}$.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. For ease of notation, we specialise $Y_i = 1$ for all i, to obtain polynomials in $X_1,\ldots,X_n$ of *degree at most d in each variable* and invariant in the $\mathfrak{S}_n$ action. This is a subspace of the space $\Lambda[\mathbf{X}]$ of symmetric polynomials in $\mathbf{X}$, which we shall call $\Lambda_{\leqslant d}[\mathbf{X}]$. We obtain an isomorphism

$$\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2 \cong \Lambda_{\leqslant d}[\mathbf{X}].$$

The action of an element $U = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F})$ inherited by $\Lambda_{\leqslant d}[\mathbf{X}]$ is given by

$$Uf(X_1,\ldots,X_n) = \prod_{i=1}^{n}(\beta X_i + \delta)^d \cdot f\left(\frac{\alpha X_1 + \gamma}{\beta X_1 + \delta},\ldots,\frac{\alpha X_n + \gamma}{\beta X_n + \delta}\right). \tag{2.1}$$

Alternatively, as we show in [13], the $\mathrm{SL}_2(\mathbb{F})$ action is completely determined by the action of the element $\mathbf{f} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathfrak{sl}_2(\mathbb{C})$ as the operator $\mathrm{D}_{\mathbf{X}} = \frac{\partial}{\partial X_1} + \cdots + \frac{\partial}{\partial X_n}$.

To illustrate the strength of this identification, we sketch a brief proof of (1.5).

Proposition 2.1 (Wronskian isomorphism [22, 11, 16])**.** *Let $n, d \in \mathbb{N}_0$. Let $V(\mathbf{x})$ be the Vandermonde determinant. The map*

$$\begin{aligned} \zeta_{n,d} : \Lambda_{\leqslant d}[\mathbf{x}] &\to V(\mathbf{x})\Lambda_{\leqslant d}[\mathbf{x}] \\ P(\mathbf{x}) &\mapsto V(\mathbf{x})P(\mathbf{x}) \end{aligned}$$

realises an isomorphism of $\mathrm{SL}_2(\mathbb{F})$*-representations* $\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2 \cong \bigwedge^n \mathrm{Sym}^{d+n-1}\mathbb{F}^2$.

Sketch. Since polynomial rings are integral domains, the map $\zeta_{n,d}$ is injective. Since $V(\mathbf{x})$ is a determinant, every element of $\mathrm{SL}_2(\mathbb{F})$ acts on it trivially, so $\zeta_{n,d}$ is $\mathrm{SL}_2(\mathbb{F})$-equivariant. Conclude by noting each side has dimension $\binom{d+n}{d}$. $\square$

Observe that $\{s_\lambda(\mathbf{X}) \mid \lambda \in L(n,d)\}$ and $\{m_\lambda(\mathbf{X}) \mid \lambda \in L(n,d)\}$ are bases of $\Lambda_{\leqslant d}[\mathbf{X}]$, where $L(n,d)$ is the set $\{\lambda \in \mathrm{Par} \mid \lambda_1 \leqslant d,\ \ell(\lambda) \leqslant n\}$ of partitions in an $n \times d$ rectangle and s_λ and m_λ are the Schur and monomial symmetric polynomials.

3 The k-fold map

Definition 3.1. *Suppose either $|\mathbf{x}| = k|\mathbf{y}|$ or $|\mathbf{x}| = \infty = |\mathbf{y}|$. The* k-fold map *is defined by*

$$\begin{gathered}\Lambda[\mathbf{x}] \to \Lambda[\mathbf{y}]\\ f(x_1, x_2, \ldots) = f(\mathbf{x}) \mapsto f[k\mathbf{y}] = f(y_1, \overset{k}{\ldots}, y_1, y_2, \overset{k}{\ldots}, y_2, \ldots).\end{gathered}$$

Equivalently, the k-fold map sends each x_i to $y_{\lceil i/k \rceil}$. Letting $\mathbf{x}_n$ denote an alphabet of size n, note that $\Lambda_{\leqslant m}[\mathbf{x}_{kn}]$ gets mapped to $\Lambda_{\leqslant km}[\mathbf{y}_n]$.

Proposition 3.2. *The k-fold map $\Lambda_{\leqslant m}[\mathbf{x}_{kn}] \to \Lambda_{\leqslant km}[\mathbf{y}_n]$ is $\mathrm{SL}_2(\mathbb{F})$-equivariant.*

Proof. Let κ denote the k-fold map defined above, and let $U = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F})$. Following (2.1), we compute explicitly that both $\kappa(Uf(\mathbf{x}_{kn}))$ and $U\kappa(f(\mathbf{x}_{kn}))$ equal

$$\prod_{i=1}^{n} ((\beta y_i + \delta)^m)^k \cdot f\left(\frac{\alpha y_1 + \gamma}{\beta y_1 + \delta}, \overset{k}{\ldots}, \frac{\alpha y_1 + \gamma}{\beta y_n + \delta}, \ldots, \frac{\alpha y_n + \gamma}{\beta y_n + \delta}, \overset{k}{\ldots}, \frac{\alpha y_n + \gamma}{\beta y_n + \delta}\right). \qquad \square$$

3.1 The k-fold map on the Schur basis

Definition 3.3. *For each $\lambda, \mu \in \mathrm{Par}$, define $d_\lambda^\mu(k)$ by $s_\lambda[k\mathbf{y}_\infty] = \sum_\mu d_\lambda^\mu(k) s_\mu(\mathbf{y}_\infty)$.*

We study $D(k) = \left(d_\lambda^\mu(k)\right)_{\lambda,\mu \in \mathrm{Par}}$. Observe first that this matrix is block-diagonal.

Lemma 3.4. *If $|\lambda| \neq |\mu|$ then $d_\lambda^\mu(k) = 0$.*

Our first formula expresses $d_\lambda^\mu(k)$ in terms of k-multi-Littlewood–Richardson coefficients, defined by $c_{\boldsymbol{\nu}}^\lambda = \langle s_{\nu^{(1)}} \cdots s_{\nu^{(k)}}, s_\lambda \rangle$, where $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)}) \in \mathrm{Par}^k$. The k-multi-Littlewood–Richardson coefficient $c_{\boldsymbol{\nu}}^\lambda$ counts the number of lattice k-multi-tableaux of multi-shape $\boldsymbol{\nu}$ and content λ [28, §3 and Lemma 6.1].

Proposition 3.5. *For all $\lambda, \mu \in \mathrm{Par}$, we have $d_\lambda^\mu(k) = \sum_{\boldsymbol{\nu} \in \mathrm{Par}^k} c_{\boldsymbol{\nu}}^\lambda c_{\boldsymbol{\nu}}^\mu$.*

The symmetries of Littlewood–Richardson coefficients [4] endow $D(k)$ with a rich combinatorial structure.

Lemma 3.6.

(i) *The matrix $D(k)$ is symmetric: $d_\lambda^\mu(k) = d_\mu^\lambda(k)$.*

(ii) *The blocks of $D(k)$ are conjugation invariant: $d_\lambda^\mu(k) = d_{\lambda'}^{\mu'}(k)$.*

(iii) *We have $d_\lambda^\mu(k) = d_{\lambda^\square}^{\mu^\square}(k)$, where the box-complement of λ (resp. μ) is taken inside $L(b, a)$ (resp. $L(d, c)$), for all choices of a, b, c, d such that $c/a = b/d = k$.*

Proposition 3.7 (Cauchy formula)**.** *For all* $\lambda, \mu \in \mathrm{Par}$*, we have*

$$d^{\mu}_{\lambda}(k) = \Big\langle s_\lambda(\mathbf{y}_\infty)s_\mu(\mathbf{z}_\infty),\ \prod_{i,j}(1-y_jz_i)^{-k}\Big\rangle.$$

Let $f * g$ denote the Kronecker product of symmetric polynomials. The Kronecker coefficients are defined by $g_{\lambda\mu\nu} = \langle s_\lambda,\ s_\mu * s_\nu\rangle$.

Proposition 3.8 (Kronecker formula)**.** *For all* $\lambda, \mu \in \mathrm{Par}$*, we have* $d^{\mu}_{\lambda}(k) = (s_\lambda * s_\mu)(1, \overset{k}{\dots}, 1)$.

Corollary 3.9. *For all* $\lambda, \mu \in \mathrm{Par}$*, we have* $d^{\mu}_{\lambda}(k) = \sum_\nu g_{\lambda\mu\nu}\ \#\mathrm{SSYT}_k(\nu)$.

Proof. Expand the Kronecker product to obtain

$$s_\lambda * s_\mu(1, \overset{k}{\dots}, 1) = \sum_\nu g_{\lambda\mu\nu}\ s_\nu(1, \overset{k}{\dots}, 1) = \sum_\nu g_{\lambda\mu\nu}\ \#\mathrm{SSYT}_k(\nu). \qquad \square$$

We believe this new connection between Kronecker coefficients and the structure constants of the k-fold map will repay further investigation.

Example 3.10. *The following is the top-left corner of the infinite matrix* $D(2)$*.*

	$\varnothing$	(1)	(2)	$(1,1)$	(3)	$(2,1)$	$(1,1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$\varnothing$	1	·	·	·	·	·	·	·	·	·	·	
(1)	·	2	·	·	·	·	·	·	·	·	·	·
(2)	·	·	3	1	·	·	·	·	·	·	·	·
$(1,1)$	·	·	1	3	·	·	·	·	·	·	·	·
(3)	·	·	·	·	4	2	·	·	·	·	·	·
$(2,1)$	·	·	·	·	2	6	2	·	·	·	·	·
$(1,1,1)$	·	·	·	·	·	2	4	·	·	·	·	·
(4)	·	·	·	·	·	·	·	5	3	1	·	·
$(3,1)$	·	·	·	·	·	·	·	3	9	3	4	·
$(2,2)$	·	·	·	·	·	·	·	1	3	6	3	1
$(2,1,1)$	·	·	·	·	·	·	·	·	4	3	9	3
$(1,1,1,1)$	·	·	·	·	·	·	·	·	·	1	3	5

3.2 The k-fold map on the monomial basis

We now turn to the study of the coefficients of the k-fold map in the monomial basis. Let $K_{\lambda,\mu} = [m_\mu]s_\lambda$ be the *Kostka coefficient* and let $K = (K_{\lambda,\mu})_{\lambda,\mu\in\mathrm{Par}}$ be the *Kostka matrix*. We study the block-diagonal matrix $B(k) = K^{-1} \cdot D(k) \cdot K$.

Definition 3.11. *For each* $\lambda, \mu \in \mathrm{Par}$*, define* $b^{\mu}_{\lambda}(k)$ *by* $m_\lambda[k\mathbf{y}_\infty] = \sum_\mu b^{\mu}_{\lambda}(k) m_\mu(\mathbf{y}_\infty)$*.*

Definition 3.12. *Given a multiset $A = \{\{a_1, a_2, \ldots, a_n\}\}$, a* multiset partition *of A is a collection of multisets* (parts) *$A^{(1)}, A^{(2)}, \ldots, A^{(\ell)}$ such that the union (with multiplicity) of the parts is A. The* weight *of a multiset partition $A^{(1)}, A^{(2)}, \ldots, A^{(\ell)}$ is the tuple $w = (w_1, w_2, \ldots, w_\ell)$, where w_i is the sum of elements of $A^{(i)}$ (with multiplicity). When $k \mid n$, let $\mathcal{P}_k(A, w)$ be the set of multiset partitions of A of weight w, and in which each part has k elements (with multiplicity).*

Proposition 3.13. *Let $|\lambda| = |\mu|$ and $A_\lambda = \{\{\lambda_i \mid i = 1, \ldots, k\ell(\mu)\}\}$. Then*

$$b^\mu_\lambda(k) = \sum_{\mathcal{P}_k(A,\mu)} \prod_{i=1}^{\ell(\mu)} \frac{k!}{m_0(A^{(i)}_\lambda)!\; m_1(A^{(i)}_\lambda)! \cdots},$$

where $m_j(A^{(i)}_\lambda)$ is the multiplicity of j in $A^{(i)}_\lambda$.

If $b^\mu_\lambda(k) \neq 0$ then $\mathcal{P}_k(A_\lambda, \mu) \neq \varnothing$, so μ coarsens λ. In particular, $\mu \geqslant_{\text{lex}} \lambda$, so $B(k)$ is lower-triangular. The diagonal is $(k^{\ell(\lambda)})_{\lambda \in \mathrm{Par}}$ since $B(k)$ is conjugate to the matrix of the k-fold map in the power sum basis, and $p_\lambda[k\mathbf{y}_\infty] = k^{\ell(\lambda)} p_\lambda(\mathbf{y}_\infty)$.

Example 3.14. *The following is the top-left corner of the infinite matrix $B(2)$.*

	$\varnothing$	(1)	(2)	$(1,1)$	(3)	$(2,1)$	$(1,1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$\varnothing$	1	·	·	·	·	·	·	·	·	·	·	·
(1)	·	2	·	·	·	·	·	·	·	·	·	·
(2)	·	·	2	·	·	·	·	·	·	·	·	·
$(1,1)$	·	·	1	4	·	·	·	·	·	·	·	·
(3)	·	·	·	·	2	·	·	·	·	·	·	·
$(2,1)$	·	·	·	·	2	4	·	·	·	·	·	·
$(1,1,1)$	·	·	·	·	·	2	8	·	·	·	·	·
(4)	·	·	·	·	·	·	·	2	·	·	·	·
$(3,1)$	·	·	·	·	·	·	·	2	4	·	·	·
$(2,2)$	·	·	·	·	·	·	·	1	·	4	·	·
$(2,1,1)$	·	·	·	·	·	·	·	·	4	4	8	·
$(1,1,1,1)$	·	·	·	·	·	·	·	·	·	1	4	16

Again, we can obtain a formula in terms of Kronecker products.

Proposition 3.15. *For all $\lambda, \mu \in \mathrm{Par}$, we have $b^\mu_\lambda(k) = (m_\lambda * h_\mu)(1, \overset{k}{\ldots}, 1)$.*

4 Lifting the q-team-and-leader identity

To establish Proposition 1.1, we translate both the left- and the right-hand side of the desired isomorphism through (1.6) — Proposition 1.1 then becomes

$$\Lambda_{\leqslant n}[\mathbf{z}_m] \otimes \Lambda_{\leqslant d}[\mathbf{x}_n, \mathbf{y}_m] \cong \Lambda_{\leqslant d}[\mathbf{x}_n] \otimes \Lambda_{\leqslant n+d}[\mathbf{y}_m].$$

This framework allows us to construct a simple map from the left- to the right-hand side of this expression as follows. We state the result for a larger class of maps, which we use in the next section.

Proposition 4.1. *Suppose $\delta \leqslant |\mathbf{x}|$, $\varepsilon \in \mathbb{N}_0$, and $|\mathbf{y}| = |\mathbf{z}|$. Then the following map is injective:*

$$\begin{aligned}\pi_\delta : \Lambda_{\leqslant\delta}[\mathbf{z}] \otimes \Lambda_{\leqslant\varepsilon}[\mathbf{x},\mathbf{y}] &\to \Lambda_{\leqslant\varepsilon}[\mathbf{x}] \otimes \Lambda_{\leqslant\varepsilon+\delta}[\mathbf{y}] \\ P(\mathbf{x},\mathbf{y},\mathbf{z}) &\mapsto P(\mathbf{x},\mathbf{y},\mathbf{y}).\end{aligned}$$

The proof of this proposition uses a novel application of *Lagrange interpolation*, which almost miraculously recovers each variable in the alphabet $\mathbf{z}$ that disappeared in the substitution. The details of the proof can be found in [13, §4].

The map π_δ above is an extension of the 1-fold map, so it is $\mathrm{SL}_2(\mathbb{F})$-equivariant by Proposition 3.2. When $\delta = |\mathbf{x}| = n$, $|\mathbf{y}| = |\mathbf{z}| = m$, and $\varepsilon = d$, dimension counting shows that the map is an isomorphism. This concludes our new proof of Proposition 1.1.

5 Lifting Stanley's q-Hook-Content Formula for hooks

A (non-obvious) application of Stanley's q-Hook-Content Formula for partitions of hook shape gives the following factorisation

$$s_{(m+1,1^{n-1})}(q^{-d}, q^{2-d}, \ldots, q^d) = \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q \begin{bmatrix} m+d+1 \\ m+n \end{bmatrix}_q.$$

To lift this identity over $\mathbb{F}$, we construct an $\mathrm{SL}_2(\mathbb{F})$-representation $\Delta^{(m+1,1^{n-1})}\mathrm{Sym}^d\mathbb{F}^2$ whose character when $\mathbb{F} = \mathbb{C}$ is the left-hand side. Sparing the details for this abstract, we show in [13] that this representation is $\mathrm{SL}_2(\mathbb{F})$-isomorphic to the intersection

$$V(\mathbf{y}_m)\Lambda_{\leqslant d-n+1}[\mathbf{y}_m] \otimes \Lambda_{\leqslant d}[\mathbf{x}_n, \mathbf{y}_m] \quad \cap \quad \ker \mu, \tag{5.1}$$

where $V(\mathbf{y}_m)$ is the Vandermonde determinant and μ is a map on polynomials given by the action of $1 - \sum_{i=1}^n (x_i, y_1) \in \mathbb{F}\mathfrak{S}_{n+1}$ (where (x_i, y_1) swaps x_i with y_1).

Proposition 5.1. *There is a characteristic-independent $\mathrm{SL}_2(\mathbb{F})$-isomorphism*

$$\Delta^{(m+1,1^{n-1})}\mathrm{Sym}^d\mathbb{F}^2 \cong \mathrm{Sym}_m\mathrm{Sym}^{n-1}\mathbb{F}^2 \otimes \mathrm{Sym}_{m+n}\mathrm{Sym}^{d-n+1}\mathbb{F}^2.$$

Sketch. Let π_{n-1} be the injection from Proposition 4.1 (with $\varepsilon = d - n + 1$). Let $\zeta_{n,d-n+1}$ be defined as in Proposition 2.1. Calculation shows that the image of the right-hand side under the map $(\zeta_{n,d-n+1} \otimes 1) \circ \pi_{n-1}$ is inside the intersection (5.1) and hence inside $\Delta^{(m+1,1^{n-1})}\mathrm{Sym}^d\mathbb{F}^2$. Conclude by counting dimensions. □

The case $m = 0$ of this proposition is the Wronskian isomorphism (1.5).

6 Towards a categorification of $U_q^0(\mathfrak{sl}_2)$

The $\mathrm{SL}_2(\mathbb{F})$-plethysms $\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2$ appear to be intimately linked to the Cartan subalgebra of the quantum group $U_q(\mathfrak{sl}_2)$. Recall the triangular decomposition

$$U_q(\mathfrak{sl}_2) \cong U_q^+(\mathfrak{sl}_2) \otimes U_q^0(\mathfrak{sl}_2) \otimes U_q^-(\mathfrak{sl}_2).$$

The Cartan subalgebra $U_q^0(\mathfrak{sl}_2)$ has an integral form spanned over $\mathbb{Z}[q,q^{-1}]$ by Lusztig's elements $\left[\begin{smallmatrix}K;i\\t\end{smallmatrix}\right] \in \mathbb{Z}[q,q^{-1}](K)$, which are a q-deformation of the elements $\binom{H+i}{t}$ in Kostant's $\mathbb{Z}$-form of the enveloping algebra $U(\mathfrak{sl}_2)$.

While categorification of $U_q^+(\mathfrak{sl}_2)$ (or $U_q^-(\mathfrak{sl}_2)$) is well understood [18], describing a monoidal categorification of the Cartan subalgebra $U_q^0(\mathfrak{sl}_2)$ remains an open problem.

In $U_q^0(\mathfrak{sl}_2)$, we have the multiplication rule

$$\begin{bmatrix}K\\t\end{bmatrix}\begin{bmatrix}K\\s\end{bmatrix} = \sum_{i=0}^{\min(s,t)} \begin{bmatrix}t\\i\end{bmatrix}_q \begin{bmatrix}s\\i\end{bmatrix}_q \begin{bmatrix}K;i\\s+t\end{bmatrix} \tag{6.1}$$

where we write $\left[\begin{smallmatrix}K\\t\end{smallmatrix}\right] := \left[\begin{smallmatrix}K;0\\t\end{smallmatrix}\right]$. Evaluated at $K = q^d$, this identity is a special case of the *q-Pfaff–Saalschütz identity*, which is one of the many names given to q-binomial identities expressing the product of two q-binomials as a sum of triple products of q-binomials. We refer to our article [15] for the full statement and a literature review; see also [17].

We have used (6.1) to predict several new infinite families of characteristic-free isomorphisms, including Propositions 1.1 and 5.1. As an example, take $\Delta^{(2,1)}\mathrm{Sym}^d\mathbb{F}^2$. Letting $t = 2$ and $s = 1$, the multiplication rule reads

$$\begin{bmatrix}K\\2\end{bmatrix}\begin{bmatrix}K\\1\end{bmatrix} = \begin{bmatrix}K\\3\end{bmatrix} + [2]_q\begin{bmatrix}K;1\\3\end{bmatrix}.$$

By the "translation" described below, this lifts to a short exact sequence

$$0 \to \mathbb{F}^2 \otimes \textstyle\bigwedge^3 \mathrm{Sym}^{d+1}\mathbb{F}^2 \to \bigwedge^2 \mathrm{Sym}^d\mathbb{F}^2 \otimes \bigwedge^1 \mathrm{Sym}^d\mathbb{F}^2 \to \bigwedge^3 \mathrm{Sym}^d\mathbb{F}^2 \to 0,$$

which, since $\Delta^{(2,1)}\mathrm{Sym}^d\mathbb{F}^2$ is the kernel of $\bigwedge^2 \mathrm{Sym}^d\mathbb{F}^2 \otimes \bigwedge^1 \mathrm{Sym}^d\mathbb{F}^2 \to \bigwedge^3 \mathrm{Sym}^d\mathbb{F}^2$, predicts the isomorphism $\Delta^{(2,1)}\mathrm{Sym}^d\mathbb{F}^2 \cong \mathbb{F}^2 \otimes \bigwedge^3 \mathrm{Sym}^{d+1}\mathbb{F}^2$. This prediction is confirmed by Proposition 5.1. More generally, (6.1) predicts the isomorphisms

$$\begin{aligned}
\Delta^{(1,1,1)}\mathrm{Sym}^d\mathbb{F}^2 &\cong \mathrm{Sym}_3\mathrm{Sym}^{d-2}\mathbb{F}^2\\
\Delta^{(2,1)}\mathrm{Sym}^d\mathbb{F}^2 &\cong \mathbb{F}^2 \otimes \mathrm{Sym}_3\mathrm{Sym}^{d-1}\mathbb{F}^2\\
\Delta^{(3)}\mathrm{Sym}^d\mathbb{F}^2 &= \mathrm{Sym}_3\mathrm{Sym}^d\mathbb{F}^2
\end{aligned}$$

which are all instances of Proposition 5.1. Observe that the tensor factors on each right-hand side are plethysms of the form $\mathrm{Sym}_n\mathrm{Sym}^m\mathbb{F}^2$.

The "translation" procedure to obtain these conjectures is made more precise in the second author's doctoral thesis [20], where a programme to categorify the Cartan subalgebra $U_q^0(\mathfrak{sl}_2)$ by means of $\mathrm{SL}_2(\mathbb{F})$-plethysms is outlined.

Two major steps towards the programme are Propositions 1.1 and 5.1. A complete lifting of the general multiplication formula for Lusztig's elements remains open, involving two-row partition plethysms. The q-binomial identity to be lifted is

$$s_{(a+b,\ a)} \circ s_d(q^{-1}, q) = \sum_{k=0}^{a-1} \begin{bmatrix} a+b+d+k \\ 2a+b \end{bmatrix}_q s_{(k+b,\ k)} \circ s_{a-k}(q^{-1}, q),$$

which may be proved using q-Pfaff–Saalschütz and a Jacobi–Trudi identity [14]. The lift of this identity is conjectured below.

Conjecture 6.1. *Let $a, b \in \mathbb{N}$. The $\mathrm{SL}_2(\mathbb{F})$-plethysm $\Delta^{(a+b,\ a)}\mathrm{Sym}^d\mathbb{F}^2$ has the filtration:*

$$\begin{array}{c}
\Delta^{(b)}\,\mathrm{Sym}^a\,\mathbb{F}^2 \otimes \mathrm{Sym}_{2a+b}\mathrm{Sym}^{d-a}\,\mathbb{F}^2 \\
\Delta^{(1+b,\ 1)}\,\mathrm{Sym}^{a-1}\,\mathbb{F}^2 \otimes \mathrm{Sym}_{2a+b}\mathrm{Sym}^{d-a+1}\,\mathbb{F}^2 \\
\vdots \\
\Delta^{(a+b-2,\ a-2)}\,\mathrm{Sym}^2\,\mathbb{F}^2 \otimes \mathrm{Sym}_{2a+b}\mathrm{Sym}^{d-2}\,\mathbb{F}^2 \\
\Delta^{(a+b-1,\ a-1)}\,\mathrm{Sym}^1\,\mathbb{F}^2 \otimes \mathrm{Sym}_{2a+b}\mathrm{Sym}^{d-1}\,\mathbb{F}^2.
\end{array}$$

Besides the outstanding question of why this connection between $\mathrm{SL}_2(\mathbb{F})$-plethysms and $U_q^0(\mathfrak{sl}_2)$ exists, another challenge is computing homomorphism spaces between $\mathrm{SL}_2(\mathbb{F})$-plethysms. The characteristic-free setup appears to be quite rigid, in the sense that isomorphisms are rare, and the ones that we find seem to be unique up to scalar multiplication, as expected for "irreducible objects" by Schur's Lemma. This motivates the problem of determining the (integral) endomorphism algebras of $\mathrm{Sym}_n\mathrm{Sym}^d\mathbb{F}^2$.

7 Towards the generalised Foulkes Conjecture for $\mathrm{SL}_2(\mathbb{C})$

In this section we return to working in $\mathbb{C}$. The problem of understanding $\mathrm{SL}_N(\mathbb{C})$-plethysm coefficients is difficult even in its simplest cases. The *Foulkes Conjecture* [9] has been widely open for decades [23, 7, 5].

Conjecture 7.1 (Foulkes)**.** *Let $a, b \in \mathbb{N}$ with $a \leqslant b$. Then $\mathrm{Sym}^a\mathrm{Sym}^b\mathbb{C}^N$ is an $\mathrm{SL}_N(\mathbb{C})$-subrepresentation of $\mathrm{Sym}^b\mathrm{Sym}^a\mathbb{C}^N$.*

The conjecture does not hold over arbitrary characteristic [24, Ch. 3]. For $N = 2$ the conjecture holds trivially, since both plethysms are isomorphic by Hermite reciprocity (1.4). In contrast, several generalisations of the Foulkes Conjecture remain open for $\mathrm{SL}_2(\mathbb{C})$. The following is a special case of conjectures found in [6, 27, 1, 3] and [10], where the motivation is the particle entanglement spectrum of certain quantum states.

Conjecture 7.2. *Let $a, b, c, d \in \mathbb{N}$, with $a = \min\{a, b, c, d\}$ and $ab = cd$. Then* $\mathrm{Sym}^a\mathrm{Sym}^b\mathbb{C}^2$ *is an* $\mathrm{SL}_2(\mathbb{C})$*-subrepresentation of* $\mathrm{Sym}^c\mathrm{Sym}^d\mathbb{C}^2$*. Equivalently,*

$$\begin{bmatrix} c+d \\ c \end{bmatrix}_q \geqslant_q \begin{bmatrix} a+b \\ a \end{bmatrix}_q .$$

If a divides c, then the (c/a)-fold map is an $\mathrm{SL}_2(\mathbb{C})$-homomorphism $\mathrm{Sym}^b\mathrm{Sym}^a\mathbb{C}^2 \to \mathrm{Sym}^d\mathrm{Sym}^c\mathbb{C}^2$ by Proposition 3.2. Applying Hermite reciprocity to both sides, and using our new model for plethysms, we obtain a conjecture which implies Conjecture 7.2.

Conjecture 7.3. *Let $a, b, c, d \in \mathbb{N}$, with $a = \min\{a, b, c, d\}$ and $c/a = b/d \in \mathbb{N}$. Then the (c/a)-fold map $\Lambda_{\leqslant a}[\mathbf{x}_b] \to \Lambda_{\leqslant c}[\mathbf{y}_d]$ is injective.*

Using the MAGMA code `kFoldMatrices.m` available from the fourth author's website `www.ma.rhul.ac.uk/~uvah099/`, the conjecture has been verified for all $a, b, c, d \leqslant 10$.

Setting $c = b$ and $d = a$ in the conjecture predicts a new *explicit* isomorphism for Hermite reciprocity (1.4) when a divides b.

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