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A bijection on balanced words reversing both des and maj

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Abstract. Balanced words on a finite alphabet are those words in which every letter of the alphabet occurs the same number of times. The notion of descents and major index extends in a natural way to words. It is known that the bivariate generating polynomials for descents and major index over balanced words on the alphabet $[k]$ with n occurrences each is palindromic, but a bijective proof has been missing even for balanced binary words. We give an explicit bijection proving this result. We also show that for balanced binary words, this bijection simultaneously flips the ascent and comajor index as well. For permutations (which are also examples of balanced words), we show that our bijection is different from the complementation map.

Keywords: balanced words, descent, major index, bijective proof

1 Introduction

The combinatorics of words is a fascinating topic with a long history and many elegant results. In this work, we are concerned with the descent and major index on balanced words, in which each letter occurs the same number of times; see below for the precise definitions. It is known due to work of Tielker [8, Proposition 4.3.21][1] that the bivariate generating polynomial of these statistics is palindromic. Carnevale–Voll conjectured [3, Conjectures A and B] further properties of this polynomial. Habsieger [6] and Carnevale [2] gave some partial results towards the conjectures. The main result of this paper is a bijective proof of this bivariate palindromicity.

Permutations are a special case of balanced words and there is a rich history of beautiful bijections, one of the most famous being Foata's bijection [4] exchanging the inversion number and the major index, both being Mahonian statistics. Here, we are concerned with descents, which is a so-called Eulerian statistic. The so-called 'fundamental bijection' due to Foata [5] exchanges the number of descents and excedances. Han [7] gave

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[1] the argument in the proof of [3, Proposition 2.12] contains a gap. We thank Angela Carnevale for pointing this out.

a similar bijection on words with fixed content exchanging the major index and another Mahonian statistic called the Z-statistic, introduced by Zeilberger–Bressoud [9] in their proof of Andrews' q-Dyson conjecture.

We now begin with the basic definitions and state our results. Let k and n be fixed positive integers. We will work with words over the alphabet $[k] = \{1, \dots, k\}$. A word w is said to be *balanced* if each letter in the alphabet occurs the same number of times in w. Let $\mathcal{W}_{n,k}$ be the set of balanced words on $[k]$ with n occurrences each, and let $\mathcal{W}_k = \cup_n \mathcal{W}_{n,k}$ be the set of all balanced words over $[k]$.

For any finite word w, balanced or otherwise, let $\ell(w)$ be its length. The *descent set* of a word w is defined by $\mathrm{Des}(w) = \{i \in [\ell(w) - 1] \mid w_i > w_{i+1}\}$, and the number of *descents* is $\mathrm{des}(w) = |\mathrm{Des}(w)|$. The *major index* of w is $\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$. We can similarly define *ascents* and the *comajor index*. It is easy to see that the ascents and descents are symmetric in $\mathcal{W}_k$ because for $w = w_1 \dots w_{kn}$, its complement $w' = (k + 1 - w_1) \dots (k + 1 - w_{2n})$ satisfies $\mathrm{asc}(w') = \mathrm{des}(w)$ and $\mathrm{des}(w') = \mathrm{asc}(w)$. It is quite clear that the minimum value of both des and maj over $\mathcal{W}_{n,k}$ is 0 corresponding to the unique word $w_{min} = 1 \dots 12 \dots 2k \dots k$. A little more work shows that the maximum values of des and maj in $\mathcal{W}_{n,k}$ are $(k - 1)n$ and $n^2\binom{k}{2}$ respectively, and both occur for the unique word $w_{max} = k(k-1) \cdots 1 \; k(k-1) \cdots 1 \; \cdots \; k(k-1) \cdots 1$.

The bivariate generating polynomial of des and maj on balanced words is

$$C_{n,k}(x, q) = \sum_{w \in \mathcal{W}_{n,k}} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}.$$

Theorem 1.1 ([3, Proposition 2.12])**.** *The bivariate generating polynomial of* des *and* maj *on balanced words is palindromic, i.e.*

$$C_{n,k}(x^{-1}, q^{-1}) = x^{-n(k-1)} q^{-n^2\binom{k}{2}} C_{n,k}(x, q).$$

For example, the coefficients of $C_{2,3}(x, q)$ can be written as the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 6 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 10 & 14 & 10 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & 6 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and it is clear that rotating the array by 180° leaves it invariant.

We will construct an explicit bijection proving this result in Section 3. The bijection builds on the one constructed for binary words, which we explain first in Section 2. We also show that the latter respects ascents and the comajor index, as well as a property we call mirror-symmetry. The detailed proofs appear in the `arXiv` version [1].

2 Bijection on binary words

A *boolean array* is a word in the letters T and F.[2] If μ is a boolean array, let $\bar{\mu}$ be its *complement*, i.e, the boolean array swapping the T's for F's and vice versa in μ. Let $w \in \mathcal{W}_{n,2}$. Define the boolean array μ_w with respect to the occurrences of 1 in w as

$$(\mu_w)_i = \begin{cases} \mathrm{T} & \text{if the } i\text{'th 1 in } w \text{ is immediately preceded by 2,} \\ \mathrm{F} & \text{otherwise.} \end{cases}$$

Similarly, define the boolean array ν_w with respect to the occurrences of 2 in w as

$$(\nu_w)_i = \begin{cases} \mathrm{T} & \text{if the } i\text{'th 2 in } w \text{ is followed by 1,} \\ \mathrm{F} & \text{otherwise.} \end{cases}$$

Let $B_n = \{\mathrm{T}, \mathrm{F}\}^n$ be the set of boolean arrays of length n. For $\mu \in B_n$, let $n_{\mathrm{T}}(\mu)$ (resp. $n_{\mathrm{F}}(\mu)$) be the number of T's (resp. F's) in μ. Let $B_n^k = \{\mu \in B_n \mid n_{\mathrm{T}}(\mu) = k\}$ be the set of boolean arrays having k T's. Define

$$P_n = \bigsqcup_{0 \le k \le n} B_n^k \times B_n^k.$$

It is clear that the cardinality of P_n is given by

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

which is also the cardinality of $\mathcal{W}_{n,2}$.

Given $w \in \mathcal{W}_{n,2}$ with $\mathrm{des}(w) = d$, it is clear that w can be uniquely written in the form

$$w = 1^{p_1} 2^{q_1}\ 21\ 1^{p_2} 2^{q_2}\ 21 \cdots 21\ 1^{p_{d+1}} 2^{q_{d+1}}.$$

The descents are then at positions $p_1 + q_1 + 1, p_1 + q_1 + 2 + p_2 + q_2 + 1, \ldots, p_1 + q_1 + 2 + \cdots + p_d + q_d + 1$. Define a map $\phi : \mathcal{W}_{n,2} \to P_n$ by setting $\phi(w) = (\mu_w, \nu_w)$ where

$$\mu_w = \mathrm{F}^{p_1}\,\mathrm{T}\,\mathrm{F}^{p_2}\,\mathrm{T} \cdots \mathrm{F}^{p_d}\,\mathrm{T}\,\mathrm{F}^{p_{d+1}},$$

and

$$\nu_w = \mathrm{F}^{q_1}\,\mathrm{T}\,\mathrm{F}^{q_2}\,\mathrm{T} \cdots \mathrm{F}^{q_d}\,\mathrm{T}\,\mathrm{F}^{q_{d+1}}.$$

Clearly ϕ is a bijection that takes $\mathrm{des}(w)$ to $n_T(\mu_w)$. Now define the map $f_2 : \mathcal{W}_2 \to \mathcal{W}_2$ by $f_2(w) = w'$ where w' is the unique word in $\mathcal{W}_{n,2}$ having $\mu_{w'} = \bar{\mu}_w$ and $\nu_{w'} = \bar{\nu}_w$. The main result of this section proves Theorem 1.1 for $k = 2$ and is the following.

[2] We use T and F instead of 1 and 0 respectively to avoid confusion with the symbols in the alphabet.

Theorem 2.1. *The map f_2 on $\mathcal{W}_2$ is an involution (and hence a bijection) satisfying* $\operatorname{des}(f_2(w)) = n - \operatorname{des}(w)$ *and* $\operatorname{maj}(f_2(w)) = n^2 - \operatorname{maj}(w)$ *for all $w \in \mathcal{W}_{n,2}$.*

Proof of Theorem 2.1. From the definitions above, it is clear that f_2 is a bijection. Let $w' = f_2(w)$. Then,

$$\operatorname{des}(w) + \operatorname{des}(w') = n_{\mathrm{T}}(\mu_w) + n_{\mathrm{T}}(\mu_{w'}) = n_{\mathrm{T}}(\mu_w) + n_{\mathrm{T}}(\bar{\mu}_w) = n.$$

Suppose w has d descents at positions $a_1 < \cdots < a_d$ so that $\operatorname{maj}(w) = a_1 + \cdots + a_d$. Then,

$$a_i = |\{w_j \mid j < a_i, w_j = 1\}| + |\{w_j \mid j < a_i, w_j = 2\}| + 1.$$

Now, write $\mu_w = \mathrm{F}^{p_1}\,\mathrm{T}\,\mathrm{F}^{p_2}\,\mathrm{T}\cdots\mathrm{F}^{p_d}\,\mathrm{T}\,\mathrm{F}^{p_{d+1}}$ and $\nu_w = \mathrm{F}^{q_1}\,\mathrm{T}\,\mathrm{F}^{q_2}\,\mathrm{T}\cdots\mathrm{F}^{q_d}\,\mathrm{T}\,\mathrm{F}^{q_{d+1}}$. Then

$$a_i = \left(\sum_{j=1}^{i-1}(p_j+1) + p_i\right) + \left(\sum_{j=1}^{i-1}(q_j+1) + q_i\right) + 1$$

and thus, a_i is the position of the i'th T in μ_w plus the position of the i'th T in ν_w minus 1. Said another way, we count each letter in μ_w and ν_w before the i'th T once and then add 1, to obtain a_i.

Let $w' = f_2(w)$. By the definition of f_2, $\mu_{w'} = \bar{\mu}_w$ and $\nu_{w'} = \bar{\nu}_w$. Let $b_1 < \cdots < b_{n-d}$ be the positions of the $n-d$ descents of w'. By the same argument as above, b_i counts each letter in $\bar{\mu}_w$ and $\bar{\nu}_w$ before the i'th T once with a 1 added. Equivalently, b_i counts each letter in μ_w and ν_w before the i'th F once with a 1 added.

Thus, when we add $\operatorname{maj}(w)$ and $\operatorname{maj}(f_2(w))$, we are counting the j'th letter in both μ_w and ν_w exactly $(n-j)$ times, plus an extra 1. Therefore,

$$\operatorname{maj}(w) + \operatorname{maj}(f_2(w)) = \sum_{j=1}^{n}(j-1) + \sum_{j=1}^{n}(j-1) + n = n^2,$$

completing the proof. □

See Figure 1 for an example of computing $f_2(w)$ for a word $w \in \mathcal{W}_{5,2}$.

2.1 Ascents

While complementation does flip ascents and descents, it does not satisfy Theorem 2.1. However, we show that f_2 satisfies the ascent equivalents of the two lemmas as well, as illustrated in the example below.

Example 2.2. *For the example $w = 1222112112$ in Figure 1,* $\operatorname{asc}(w) = 3$ *and* $\operatorname{comaj}(w) = 16$. *Moreover,* $\operatorname{asc}(f_2(w)) = 5 - 3 = 2$ *and* $\operatorname{comaj}(f_2(w)) = 25 - 16 = 9$.

$$w = \phi(\mu_w, \nu_w) = 1222112112$$

$$\mu_w = \mathrm{F\,T\,F\,T\,F} \qquad\qquad \nu_w = \mathrm{F\,F\,T\,T\,F}$$

$$\bar{\mu}_w = \mathrm{T\,F\,T\,F\,T} \qquad\qquad \bar{\nu}_w = \mathrm{T\,T\,F\,F\,T}$$

$$f_2(w) = \phi^{-1}(\bar{\mu}_w, \bar{\nu}_w) = 2112112221$$

Figure 1: For $w = 1222112112$, $\operatorname{des}(w) = 2$, $\operatorname{maj}(w) = 11$, $\operatorname{des}(f_2(w)) = 5 - 2 = 3$, and $\operatorname{maj}(f_2(w)) = 25 - 11 = 14$. The T's and the descents in the words have both been written in red to make clear the construction of the boolean arrays.

We start by defining the dual of f_2, which we call f_2', that works on ascents instead of descents. Given $w \in \mathcal{W}_{n,2}$ with $\operatorname{asc}(w) = d$, we can write w uniquely in the form

$$w = 2^{q_1}1^{p_1}\ 12\ 2^{q_2}1^{p_2}\ 12 \ldots 12\ 2^{q_{d+1}}1^{p_{d+1}}.$$

Define the map $\phi' : \mathcal{W}_{n,2} \to P_n$ by setting $\phi'(w) = (\mu'_w, \nu'_w)$ where

$$\mu'_w = \mathrm{F}^{p_1}\,\mathrm{T}\,\mathrm{F}^{p_2}\,\mathrm{T} \ldots \mathrm{F}^{p_d}\,\mathrm{T}\,\mathrm{F}^{p_{d+1}},$$
$$\nu'_w = \mathrm{F}^{q_1}\,\mathrm{T}\,\mathrm{F}^{q_2}\,\mathrm{T} \ldots \mathrm{F}^{q_d}\,\mathrm{T}\,\mathrm{F}^{q_{d+1}}.$$

Finally, define the map $f_2' : \mathcal{W}_2 \to \mathcal{W}_2$ by $f_2'(w) = w'$ where w' is the word having $\mu'_{w'} = \bar{\mu}'_w$ and $\nu'_{w'} = \bar{\nu}'_w$. It should be clear that f_2' is a bijection and that the proof of Theorem 2.1 can clearly be repurposed to prove that $\operatorname{asc}(w) + \operatorname{asc}(f_2'(w)) = n$ and $\operatorname{comaj}(w) + \operatorname{comaj}(f_2'(w)) = n^2$ for all $w \in \mathcal{W}_{n,2}$. We now prove something a little more surprising.

Theorem 2.3. *For $w \in \mathcal{W}_{n,2}$, $\operatorname{asc}(w) + \operatorname{asc}(f_2(w)) = n$ and $\operatorname{comaj}(w) + \operatorname{comaj}(f_2(w)) = n^2$.*

Sketch of proof. We can show a reversible mapping between (μ_w, ν_w) and (μ'_w, ν'_w). In particular, $(\nu'_w)_1 = (\mu_w)_1^c$ and $(\mu'_w)_n = (\nu_w)_n^c$. For instance, if

$$w = 1^{p_1}2^{q_1}\ 21\ 1^{p_2}2^{q_2}\ 21 \cdots 21\ 1^{p_{d+1}}2^{q_{d+1}}$$

and $p_1 > 0$ i.e. $(\mu_w)_1 = \mathrm{F}$ and $q_{d+1} > 0$ i.e. $(\nu_w)_n = \mathrm{F}$, we have

$$w = 1^{p_1-1}122^{q_1}1^{p_2}122^{q_2}1^{p_3}12 \ldots 2^{q_d}1^{p_{d+1}}122^{q_{d+1}-1},$$
$$\mu' = \mathrm{F}^{p_1-1}\,\mathrm{T}\,\mathrm{F}^{p_2}\,\mathrm{T} \ldots \mathrm{F}^{p_{d+1}}\,\mathrm{T}\,\mathrm{F}^0, \quad \nu' = \mathrm{F}^0\,\mathrm{T}\,\mathrm{F}^{q_1}\,\mathrm{T} \ldots \mathrm{F}^{q_d}\,\mathrm{T}\,\mathrm{F}^{q_{d+1}-1}.$$

□

Remark 2.4. *For $k > 2$, the ascents and descents do not behave well jointly. For instance, $w = 111222333 \in \mathcal{W}_{3,3}$ is the only word having $\operatorname{des}(w) = 0, \operatorname{maj}(w) = 0, \operatorname{asc}(w) = 2$, and $\operatorname{comaj}(w) = 9$ but there are no words having $\operatorname{des}(w) = 6, \operatorname{maj}(w) = 27, \operatorname{asc}(w) = 4$, and $\operatorname{comaj}(w) = 18$.*

2.2 Mirror-symmetric words

A balanced word $w \in \mathcal{W}_{n,2}$ can be drawn as a lattice path in an $n \times n$ grid that starts at $(0,0)$ and ends at (n,n) taking north and east steps. One way to do this is to let a 1 in w be an east step in the lattice path and to let 2 to be a north step. A lattice path is *mirror-symmetric* if it is reflectively symmetric across the NW-SE diagonal. Let $\operatorname{rev}(w) = w_{\ell(w)} \dots w_1$ be the reverse of a word w. A word w is mirror-symmetric if

$$w_{n+1} \dots w_{2n} = \overline{\operatorname{rev}(w_1 \dots w_n)}.$$

For example, $w = 1^n 2^n$ is one such path. Note that $f_2(w) = (21)^n$ is also mirror-symmetric. It is not difficult to see that there are 2^n mirror-symmetric words in $\mathcal{W}_{n,2}$.

Theorem 2.5. *Let $w \in \mathcal{W}_2$ be mirror-symmetric. Then, so is $f_2(w)$.*

Example 2.6. *The word $w = 12222122212111211112$ is mirror-symmetric. One can verify that $f_2(w) = 21111212112212122221$ is also mirror-symmetric. Note that $f_2(w)$ is not obtained from w by interchanging 1's and 2's.*

3 Bijection on $\mathcal{W}_k$

We will construct a bijective map f_k on $\mathcal{W}_{n,k}$ to prove Theorem 1.1 in this section. The map will be constructed inductively using f_{k-1}.

3.1 Various maps

Here, we will define the maps needed to construct the bijection. We will consider the running example $w = 325544135121432 \in \mathcal{W}_{3,5}$ throughout this section to illustrate the notation and ideas.

Recall that in Section 2, a word $w \in \mathcal{W}_{n,2}$ is reversibly encoded as a pair of boolean arrays (μ_w, ν_w). For $w \in \mathcal{W}_{n,k}$, define the word $w_{-k} \in \mathcal{W}_{n,k-1}$ as the subword of w with the k's deleted. We will reversibly encode w as a triplet (w_{-k}, μ_w, ν_w) for some boolean arrays μ_w and ν_w.

Notation 3.1. *For $w \in \mathcal{W}_{n,k}$, let $\underline{w}$ be the subword of w obtained by removing all occurrences of k which do not lead to a change in the number of descents. To be precise, let $w_i = 0$ for $i \leq 0$ and*

$i > n$. In particular, let w_i to w_j be a sequence of k's for $1 \leq i \leq j \leq n$. Then, delete $w_i \dots w_j$ if one of the following is true:

1. *$w_{i-1} > w_{j+1}$ and both $w_{i-1}, w_{j+1} < k$, or*
2. *$w_{j+1} = k$, $w_{i-1} \leq w_{j+2}$, and both $w_{i-1}, w_{j+2} < k$.*

Notice that the order in which disjoint blocks of k's are deleted does not matter.

By construction, $\operatorname{des}(\underline{w}) = \operatorname{des}(w)$. Define the set of *non-descents* of a (not necessarily balanced) word $\operatorname{NDes}(w)$ in w as

$$\operatorname{NDes}(w) = \{0\} \cup \{1 \leq i \leq \ell(w) - 1 \mid w_i \leq w_{i+1}\}. \tag{3.1}$$

The size of $\operatorname{NDes}(w)$ is $|\operatorname{NDes}(w)| = \ell(w) - \operatorname{des}(w)$. For our running example, we obtain $\underline{w} = 3254413121432$ and $\operatorname{NDes}(w_{-5}) = \{0, 2, 3, 5, 7, 9\}$.

Lemma 3.2. *For $w \in \mathcal{W}_{n,k}$, suppose the positions of k in $\underline{w}$ are $a_1 < \cdots < a_r$. Then $\{a_i - i \mid 1 \leq i \leq r\} \subseteq \operatorname{NDes}(w_{-k})$.*

Proof. Note that $a_i - i$ is the position, in w_{-k}, of the letter preceding the i'th k in $\underline{w}$. The definition of $\underline{w}$ ensures that it has no consecutive k's. By Notation 3.1, the removal of any k that appears in $\underline{w}$ must necessarily decrease descents by 1. Therefore, the positions $a_i - i$ in w_{-k} are non-descents. □

Let $\mathcal{W}^d_{n,k} = \{w \in \mathcal{W}_{n,k} \mid \operatorname{des}(w) = d\}$ be the set of balanced words having d descents.

Notation 3.3. *For $w \in \mathcal{W}^d_{n,k}$, let $a_1 < \cdots < a_r$ be the positions of k in $\underline{w}$ and $\operatorname{NDes}(w_{-k}) = \{b_1 < \cdots < b_s\}$. Denote by $\mu_w \in B^r_{(k-1)n-d+r}$ the boolean array given by*

$$(\mu_w)_j = \begin{cases} \mathrm{T} & \text{if } b_j \in \{a_i - i \mid 1 \leq i \leq r\}, \\ \mathrm{F} & \text{otherwise.} \end{cases}$$

Note that $\ell(\mu_w) = |\operatorname{NDes}(w_{-k})| = (k-1)n - d + r$.

In our running example, $w \in \mathcal{W}^7_{3,5}, r = 1, a_1 = 3$ and so $\mu_w = \mathrm{FTFFFF} \in B^1_6$. For any subset $W \subseteq \mathcal{W}_{n,k}$, let $\underline{W} = \{\underline{w} \mid w \in W\}$. Also, let $\mathcal{W}^{d_1,d_2}_{n,k}$ be the subset of $\mathcal{W}^d_{n,k}$ such that $d_1 = \operatorname{des}(w_{-k})$ for $w \in \mathcal{W}^d_{n,k}$ and $d_2 = d - d_1$. Then, the set $\mathcal{W}_{n,k}$ is partitioned as

$$\mathcal{W}_{n,k} = \bigsqcup_{\substack{0 \leq d_1 \leq (k-2)n \\ 0 \leq d_2 \leq n}} \mathcal{W}^{d_1,d_2}_{n,k}.$$

Notation 3.4. *For $u \in \mathcal{W}^d_{n,k-1}$, write $\operatorname{NDes}(u) = \{b_1 < \cdots < b_{n(k-1)-d}\}$. Let $g : \mathcal{W}^{d_1}_{n,k-1} \times B^{d_2}_{(k-1)n-d_1} \to \underline{\mathcal{W}}^{d_1,d_2}_{n,k}$ be given by $g(u, \mu) = v$ where v is constructed from u by placing one k after every u_{b_i} whenever $\mu_i = \mathrm{T}$.*

Let $u = 324413121432 \in \mathcal{W}^6_{3,4}$ and $\mu = \mathrm{F\,T\,F\,F\,F\,F}$. Then $v = 3254413121432$ which is exactly $\underline{w}$ from our running example. It should be clear from Lemma 3.2 that the following proposition is true.

Proposition 3.5. *The map g defined in Notation 3.4 is a bijection. In particular, given $u \in \underline{\mathcal{W}}^{d_1,d_2}_{n,k}$, $g^{-1}(u) = (u_{-k}, \mu_u)$.*

Next, we define the *modified descent set* $\mathrm{Des}_+(w)$ for any word w as $\mathrm{Des}_+(w) = \mathrm{Des}(w) \cup \{\ell(w)\}$, whose size is $|\mathrm{Des}_+(w)| = \mathrm{des}(w) + 1$.

Notation 3.6. *Let $w \in \mathcal{W}^{d_1,d_2}_{n,k}$ with $d = d_1 + d_2$, and $\mathrm{Des}_+(\underline{w}) = \{j_1, \dots, j_{d+1}\}$. We can write w in terms of $\underline{w}$ as*

$$w = \underline{w}_1 \dots \underline{w}_{j_1} k^{p_1} \underline{w}_{j_1+1} \dots \underline{w}_{j_2} k^{p_2} \underline{w}_{j_2+1} \dots \underline{w}_{kn-d_2} k^{p_{d+1}}$$

for a unique assignment of integers $p_1, \dots, p_{d+1} \geq 0$ that add up to $n - d_2$. Now, let $\nu_w \in B^{n-d_2}_{n+d_1}$ be

$$\nu_w = \mathrm{T}^{p_1}\,\mathrm{F}\,\mathrm{T}^{p_2} \dots \mathrm{F}\,\mathrm{T}^{p_d}\,\mathrm{F}\,\mathrm{T}^{p_{d+1}}.$$

From our running example, we see that $\mathrm{Des}_+(\underline{w}) = \{1, 3, 5, 7, 9, 11, 12, 13\}$ and $\nu_w = \mathrm{F\,T\,F\,F\,T\,F\,F\,F\,F}$.

Notation 3.7. *Write $\nu \in B^{n-d_2}_{n+d_1}$ as $\nu = \mathrm{T}^{p_1}\,\mathrm{F}\,\mathrm{T}^{p_2} \dots \mathrm{T}^{p_d}\,\mathrm{F}\,\mathrm{T}^{p_{d+1}}$. Then define $h : \underline{\mathcal{W}}^{d_1,d_2}_{n,k} \times B^{n-d_2}_{n+d_1} \to \mathcal{W}^{d_1,d_2}_{n,k}$ by $h(u, \nu) = v$, where v is constructed from u by placing p_j k's in u immediately after each position $j \in \mathrm{Des}_+(u)$.*

Applying $h(3254413121432, \mathrm{F\,T\,F\,F\,T\,F\,F\,F\,F})$ gives back our running example $w = 325544135121432$.

Proposition 3.8. *The map h defined in Notation 3.7 is a bijection. In particular, given $u \in \mathcal{W}^{d_1,d_2}_{n,k}$, $h^{-1}(u) = (\underline{u}, \nu_u)$.*

Notation 3.9. *For $k \geq 3$, $n \geq 1$, $0 \leq d_1 \leq (k-2)n$, and $0 \leq d_2 \leq n$, let*

$$R^{d_1,d_2}_{n,k} = \mathcal{W}^{d_1}_{n,k-1} \times B^{d_2}_{(k-1)n-d_1} \times B^{n-d_2}_{n+d_1} \quad \text{and} \quad R_{n,k} = \bigsqcup_{\substack{0 \leq d_1 \leq (k-2)n \\ 0 \leq d_2 \leq n}} R^{d_1,d_2}_{n,k}.$$

Now let $\Gamma_k : \mathcal{W}_{n,k} \to R_{n,k}$ be defined by $\Gamma_k(w) = (w_{-k}, \mu_w, \nu_w)$.

Lemma 3.10. *The function Γ_k is a bijection.*

Proof. Suppose $\Gamma_k(w) = (w_{-k}, \mu_w, \nu_w)$. Then, if $w \in \mathcal{W}^d_{n,k}$, let $d_1 = \mathrm{des}(w_{-k})$ and $d_2 = d - d_1$. Then $\mu_w \in B^{d_2}_{(k-1)n-d_1}$, and so $g(w_{-k}, \mu_w)$ is well-defined and lives in $\underline{\mathcal{W}}^{d_1,d_2}_{n,k}$. Moreover, $\nu_w \in B^{n-d_2}_{n+d_1}$ and therefore, $h(g(w_{-k}, \mu_w), \nu_w)$ is well-defined and belongs to $\mathcal{W}^d_{n,k}$. One then verifies that $w = h(g(w_{-k}, \mu_w), \nu_w)$. $\square$

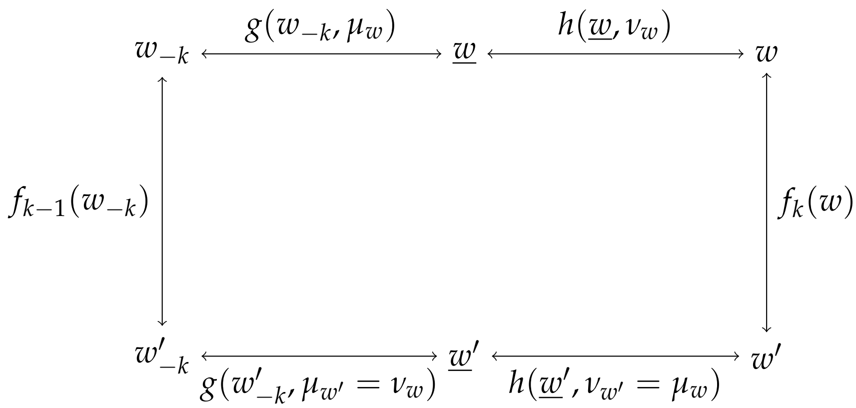

Figure 2: Schematic for the construction of f_k for $k \geq 3$. In particular, $f_k(w) = \Gamma_k^{-1}((f_{k-1}(w_{-k}), \nu_w, \mu_w))$.

3.2 Proof of the main result

The idea of the proof is given in Figure 2. Starting with $w \in \mathcal{W}_{n,k}$, we obtain $\underline{w}$ and w_{-k} as well as μ_w and ν_w. We then apply f_{k-1} to get w'_{-k}. Finally, we use the g and h maps along with $\mu_{w'} = \nu_w$ and $\nu_{w'} = \mu_w$ to construct the word $w' \in \mathcal{W}_{n,k}$. Formally, define the map $f_k : \mathcal{W}_{n,k} \to \mathcal{W}_{n,k}$ for $k \geq 3$ by

$$f_k(w) = \Gamma_k^{-1}((f_{k-1}(w_{-k}), \nu_w, \mu_w)).$$

The base case of this recursion is the function f_2 defined in Section 2. This is the main result.

Theorem 3.11. *For* $k \geq 3$*,* f_k *satisfies* $\text{des}(w) + \text{des}(f_k(w)) = (k-1)n$ *and* $\text{maj}(w) + \text{maj}(f_k(w)) = n^2\binom{k}{2}$ *for all* $w \in \mathcal{W}_k$*.*

For our running example, one can check that $f_4(w_{-5}) = 142422331134$. Recalling that $\mu_w = \text{FTFFFF}$ and $\nu_w = \text{FTFFTFFFF}$, we get $\underline{f_5(w)} = 15424225331134$ and $f_5(w) = 154524225331134$. It is verified that $\text{des}(f_5(w)) = 12 - 7 = 5$ and $\text{maj}(f_5(w)) = 90 - 58 = 32$. It might help to see how Γ and f act on all of S_3 in Table 1.

Lemma 3.12. *Let* $w \in \mathcal{W}_{n,k}$ *for* $k \geq 3$*. Then,*

$$\text{maj}(w) - \text{maj}(w_{-k}) + \text{maj}(f_k(w)) - \text{maj}(f_k(w)_{-k}) = (k-1)n^2.$$

Proof of Theorem 3.11. We will prove this by induction on k. For $k = 2$, the result is proved in Theorem 2.1. So suppose $k > 2$ and let $w \in \mathcal{W}_{n,k}^{d_1,d_2}$. Then $\text{des}(w) = d_1 + d_2$,

$w \in S_3$	μ_w	w_{-3}	$g(w_{-3}, \mu_w)$	ν_w	$h(g(w_{-3}, \mu_w))$	des	maj
123	F F	12	12	T	123	0	0
132	F T	12	132	F	132	1	2
312	T F	12	312	F	312	1	1
321	T	21	321	F F	321	2	3
213	F	21	21	F T	213	1	1
231	F	21	21	T F	231	1	2

Table 1: The action of the map Γ_3 on S_3. The permutations are ordered so that f_3 maps the i'th row above the bar to the i'th row below the bar. Notice that the roles of μ_w and ν_w get swapped in these rows.

$\Gamma_k(w) = (w_{-k}, \mu_w, \nu_w)$ from Notation 3.9, and $f_{k-1}(w_{-k}) \in \mathcal{W}_{n,k-1}^{(k-2)n-d_1}$ by the induction hypothesis.

$$(f_{k-1}(w_{-k}), \nu_w, \mu_w) \in R_{n,k}^{(k-2)n-d_1,n-d_2} = \mathcal{W}_{n,k-1}^{(k-2)n-d_1} \times B_{n+d_1}^{n-d_2} \times B_{(k-1)n-d_1}^{d_2}$$

from Notations 3.3 and 3.6. By Lemma 3.10, $f_k(w) = \Gamma_k^{-1}((f_{k-1}(w_{-k}), \nu_w, \mu_w))$ and $f_k(w) \in \mathcal{W}_{n,k}^{(k-2)n-d_1,n-d_2}$. Therefore, $\operatorname{des}(f_k(w)) = (k-1)n - d_1 - d_2$, proving the first part.

Using Lemma 3.12 and by the induction hypothesis,

$$\begin{aligned}\operatorname{maj}(w) + \operatorname{maj}(f(w)) &= \operatorname{maj}(w_{-k}) + \operatorname{maj}(f(w)_{-k}) \\ &\quad + \operatorname{maj}(w) - \operatorname{maj}(w_{-k}) + \operatorname{maj}(f(w)) - \operatorname{maj}(f(w)_{-k}) \\ &= \binom{k-1}{2} n^2 + (k-1)n^2 = \binom{k}{2} n^2,\end{aligned}$$

proving the second part. □

From the proof, we immediately deduce the following.

Corollary 3.13. *The bijections satisfy the property* $f_{k-1}(w_{-k}) = f_k(w)_{-k}$ *for all* $w \in \mathcal{W}_{n,k}$ *for all* $n \geq 1, k \geq 3$.

3.3 Permutations

Note that permutations are a special case of balanced words with $n = 1$. Thus, one can study the map f_k on the set of permutations, $S_k = \mathcal{W}_{1,k}$. We note that the complementation map sending each letter $a \mapsto k+1-a$ is also sufficient to bijectively prove Theorem 1.1. However, unlike f_k, this map does not reduce to S_{k-1} correctly, i.e. does not satisfy Corollary 3.13. For example, $f_8(46782513) = 76158342$ and $f_7(4672513) = 7615342$. We now show that f_k inverts the set of descents, just like complementation.

Theorem 3.14. *Let $w \in S_k$. Then $\mathrm{Des}(f_k(w)) = [k-1] \setminus \mathrm{Des}(w)$.*

Proof. This is clearly the case when $k = 2$. Now suppose $k > 2$. Note that $n_{\mathrm{T}}(\mu_w) + n_{\mathrm{T}}(\nu_w) = 1$. Without loss of generality, suppose $n_{\mathrm{T}}(\mu_w) = 1$ and let $w_i = k$ (if not, we can work with $f_k(w)$ instead). Let $u = w_{-k}$, $u' = f_{k-1}(w_{-k})$, and $w' = f_k(w)$. We have

$$\mathrm{Des}(w) = \mathrm{Des}(u_1 \dots u_{i-1}) \cup \{i-1+d \mid d \in \mathrm{Des}(ku_i \dots u_{k-1})\}.$$

Suppose T is at position t in μ_w, w is constructed from u by inserting k after the $(t-1)$'th non-descent (we insert at the beginning when $t = 1$). On the other hand, let j be the position of k in w', then we construct w' from u' by inserting k after the t'th descent in u' since $\nu_{f_k(w)} = \mu_w$. However, by our inductive assumption, the position $j-1$ in u is the location of its t'th non-descent. Hence, we can deduce that $i < j$, $\mathrm{Des}(u_i \dots u_j) = \{1, \dots, j-i-1\}$, and $\mathrm{Des}(u'_i \dots u'_j) = \emptyset$. Thus,

$$\begin{aligned}
\mathrm{Des}(w') &= \mathrm{Des}(u'_1 \dots u'_{j-1}) \cup \{j-1+d \mid d \in \mathrm{Des}(ku'_j \dots u'_n)\} \\
&= \mathrm{Des}(u'_1 \dots u'_{i-1}) \cup \{i-1+d \mid d \in \mathrm{Des}(u'_i \dots u'_j k u'_{j+1} \dots u'_n)\} \\
&= \mathrm{Des}(u'_1 \dots u'_{i-1}) \cup \big(\{i-1+d \mid d \in \mathrm{Des}(ku'_i \dots u'_n)\} \setminus \{i\}\big) \\
&= \{1, \dots, k-1\} \setminus \mathrm{Des}(w).
\end{aligned}$$

□

In the case of permutations, we get a simpler way to construct $f_n(w)$. The definitions of NDes in (3.1) and Des_+, as well as Theorem 3.14 lead to the following observations. (*1*) If n is right after the i'th non-descent in w_{-n}, then the position of n in $f_n(w)$ is one more than the position of the $(i+1)$'th non-descent in w_{-n}. If i is the last non-descent, then n is at the end of $f_n(w)$. (*2*) If n is right after the i'th descent in w_{-n}, then the position of n in $f_n(w)$ is one more than the position of the $(i-1)$'th descent in w_{-n}. If i is the first descent, then n is at the beginning of $f_n(w)$.

As an example, let us construct $f_8(w)$, where $w = 46872513$. Start with eight blanks $--------$. We see that 8 occurs after the third non-descent in $w_{-8} = 4672513$, where we recall from (3.1) that 0 is the first non-descent. Since the fourth non-descent is at position 4, 8 must be added to the fifth blank: $----8---$. Next, ignoring 8 in w, 7 occurs after the first descent in w with $7, 8$ removed. Hence, it must occur at the first blank: $7---8---$. Likewise, ignoring elements larger than 5 in w, 6 occurs after the first descent. Hence, it must occur at the first of the remaining blanks: $76--8---$. Next, 5 occurs at the second descent. Hence, it must occur at the second blank: $76-58---$. Continuing this way, we get $f_8(46872513) = 76158342$. We have not found this map in the literature.

4 Discussion

In this work, we have given a bijection on balanced words that simultaneously flips both the number of descents and the major index. The bijection is recursive on the size of the alphabet. Our bijection has many nice properties. For example, for balanced binary words, it also flips the number of ascents and the comajor index, as well as preserving mirror-symmetric words. For permutations, the bijection behaves nicely with respect to the restriction map.

We note that the inversion generating function of balanced words is also palindromic with an easy bijective proof. However, our bijection does not behave nicely with inversions.

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