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Surprise polynomiality in algebraic combinatorics

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Abstract. We exposit a general method for explaining "surprise" polynomiality properties of sums in algebraic combinatorics. We present three applications; the first is classical, while the others are recent. Our third application answers a question of Pak–Robichaux, giving a conceptual proof that certain sums of Schubert structure constants are eventually polynomial in all Lie types and computing their lead terms.

Keywords: Schubert calculus, equivariant restriction, Weyl group, Hilbert polynomial, RSK correspondence

1 Introduction

The purpose of this extended abstract of [11] is to present a simple, flexible method for proving that certain types of sums in algebraic combinatorics are (eventually) polynomial. Our main point is that when the constants being summed have sufficiently strong symmetry properties, one may demonstrate polynomiality of their sums without computing the constants themselves. We will illustrate the principle with three examples: the first is classical, the second is from previous work joint with A. Yong [12], and the third is new work answering a recent question of Pak–Robichaux [7] in Schubert calculus.

Let $X = \bigsqcup_{\vec{a} \in \mathbb{Z}^n_{\geq 0}} X_{\vec{a}}$ be a multigraded collection of sets, with $f\colon X \to \mathbb{Z}$ a function assigning an integer to each element $x \in X$.

Definition 1.1. For each subset $Y = \bigsqcup_{\vec{a} \in \mathbb{Z}^n_{\geq 0}} Y_{\vec{a}}$ of X, define

$$\gamma_Y(\vec{a}) :- \sum_{y \in Y_{\vec{a}}} f(y).$$

Given an equivalence relation $\sim$ on X, let $[x]_\sim$ denote the $\sim$-equivalence class of an element $x \in X$. We say the function $f\colon X \to \mathbb{Z}$ *respects* $\sim$ if f is well-defined on equivalence classes $[x]_\sim \in X/\sim$, and we call $Y \subseteq X$ *$\sim$-finite* if Y only intersects finitely many $\sim$-equivalence classes. The following proposition is then immediate:

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Proposition 1.2. *If $f\colon X \to \mathbb{Z}$ respects $\sim$ and $Y \subseteq X$ is $\sim$-finite, then*

$$\gamma_Y(\vec{a}) = \sum_{\substack{[y]_\sim \\ y \in Y}} f(y) \cdot |[y] \cap Y_{\vec{a}}|.$$

The assumption that Y is $\sim$-finite guarantees that the sum in Proposition 1.2 is finite. When Proposition 1.2 applies, it allows one to describe the dependence of $\gamma_Y(\vec{a})$ on $\vec{a}$ without computing the constants $f(y)$. In practice, this is useful when the $f(y)$ are combinatorially complex and the structure of $\sim$-equivalence classes is simple. The following sections apply Proposition 1.2 to prove polynomiality results in three distinct settings.[1]

2 Hilbert series of coordinate rings

We recount a classical application of Proposition 1.2. Given an embedded projective variety $\mathfrak{Y} \hookrightarrow \mathbb{P}^N$, its *homogeneous coordinate ring* $\mathbb{C}[\mathfrak{Y}]$ is a graded quotient of the polynomial ring $\mathbb{C}[z_0, z_1, \ldots, z_N]$. The *Hilbert function* $HF_{\mathfrak{Y}}\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ of $\mathfrak{Y}$ is defined by

$$HF_{\mathfrak{Y}}(a) := \dim_{\mathbb{C}} \mathbb{C}[\mathfrak{Y}]_a.$$

Theorem 2.1 (Hilbert)**.** *The Hilbert function $HF_{\mathfrak{Y}}(a)$ is eventually polynomial in a.*

The theory of *Gröbner bases* (see, e.g., [4]) offers a combinatorial proof of Theorem 2.1. Gröbner basis theory constructs a monomial generating set $\mathcal{G} = \{m_1, \ldots, m_r\}$ for the *initial ideal* associated to $\mathfrak{Y}$ (with respect to some term order). The monomials in $z_0, \ldots, z_n$ not divisible by any element of $\mathcal{G}$ are then a graded vector space basis for $\mathbb{C}[\mathfrak{Y}]$ called the *standard monomials* of $\mathfrak{Y}$ [6, pg. 158]. Enumerating the standard monomials of $\mathfrak{Y}$ via inclusion-exclusion proves Theorem 2.1.

We rephrase the argument of the preceding paragraph as an application of Proposition 1.2. Let $X = \bigsqcup_{a \geq 0} X_a$, where X_a is the set of degree-a monomials in $z_0, \ldots, z_N$, and let $Y \subseteq X$ denote the standard monomials of $\mathfrak{Y}$. Let $f\colon X \to \mathbb{Z}$ be the indicator function for Y, i.e., $f(x) = 1$ if $x \in Y$ and $f(x) = 0$ otherwise. Define an equivalence relation $\sim$ on X by setting $x \sim x'$ if they are divisible by the same monomials in $\mathcal{G}$. Then $Y = [1]_\sim$, so f respects $\sim$ and Y is $\sim$-finite. Thus Proposition 1.2 applies, showing that

$$HF_{\mathfrak{Y}}(a) = \gamma_Y(a) = |Y_a|.$$

In this example, the constants $f(x)$ are particularly simple and the setup of Proposition 1.2 is something of an overkill. Our other examples involve more complicated constants, making the invocation of Proposition 1.2 more significant.

[1] See Ramos–Speyer–White [8, Theorem 1.5] for a similar method from the theory of *representation stability*. Some cases of their method overlap with ours, but neither approach subsumes the other.

3 RSK as a linear operator

The *Robinson–Schensted–Knuth correspondence* (RSK) is a central bijection in algebraic combinatorics, equivalent to the *Cauchy identity* and the first fundamental theorem of invariant theory for GL_n. In recent joint work with A. Yong [12], we studied RSK as a linear operator transitioning between the *monomial* and *bitableau* bases for the vector space $R = \mathbb{C}[\mathsf{Mat}_{m,n}]$ of polynomials in an $m \times n$ matrix of variables $Z = [z_{ij}]$. This perspective was motivated by a question of Bruns–Conca–Raicu–Varbaro [3, Question 4.2.8]. The infinite-dimensional RSK operator restricts to finite-dimensional operators on the *weight spaces* of R. For each $d \geq 0$, let $R_d \subseteq R$ be the subspace spanned by degree-d monomials. For each pair of weak compositions $(\sigma, \pi) \in \mathbb{Z}^m_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$ such that $|\sigma| = |\pi|$, let $R_{\sigma,\pi} \subseteq R$ be the subspace spanned by monomials $\prod_{i,j} z_{ij}^{a_{ij}}$ such that for each $i \in [m]$ and $j \in [n]$,

$$\sum_{k=1}^{n} a_{ik} = \sigma_i \quad \text{and} \quad \sum_{k=1}^{m} a_{kj} = \pi_j.$$

Then the restrictions $\mathsf{RSK}_{m,n,d}$ and $\mathsf{RSK}_{\sigma,\pi}$ of the RSK operator to R_d and $R_{\sigma,\pi}$ are well-defined [12, Section 2.3]. The following theorem is a main result of [12]:

Theorem 3.1 ([12, Theorem 1.1(IV)]). *The trace of* $\mathsf{RSK}_{m,n,d}$ *is a polynomial in* $O(m^d n^d)$.

Here, we recall the proof of Theorem 3.1 from [12, Section 8] and interpret it via Proposition 1.2. Let $X_{m,n}$ denote the set of all weak compositions $(\sigma, \pi) \in \mathbb{Z}^m_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$ with $|\sigma| = |\pi|$, and set $X = \bigsqcup_{(m,n) \in \mathbb{Z}^2_{\geq 0}} X_{m,n}$. For a fixed choice of $d \geq 0$, let $Y^d \subseteq X$ consist of the pairs (σ, π) with $|\sigma| = |\pi| = d$. The function $f\colon X \to \mathbb{Z}$ maps each pair (σ, π) to the trace of $\mathsf{RSK}_{\sigma,\pi}$.

Definition 3.2. Define $\sim$ as the equivalence relation on X generated by:

(I) $(\sigma, \pi) \sim (\pi, \sigma)$.

(II) Let $\sigma^+ = (\sigma_1, \ldots, \sigma_k, 0, \sigma_{k+1}, \ldots, \sigma_m)$ for some k. Then $(\sigma, \pi) \sim (\sigma^+, \pi)$.

(III) Let $\vec{e}_k \in \mathbb{Z}^m_{\geq 0}$ and $\vec{e}_\ell \in \mathbb{Z}^n_{\geq 0}$ denote standard basis vectors. If $\sigma_k + \pi_\ell \geq |\sigma|$, then $(\sigma, \pi) \sim (\sigma + \vec{e}_k, \pi + \vec{e}_\ell)$.

Lemma 3.1 and Theorem 1.5 of [12] prove that $\mathsf{RSK}_{\sigma,\pi}$ and $\mathsf{RSK}_{\sigma',\pi'}$ are similar matrices whenever $(\sigma, \pi) \sim (\sigma', \pi')$. It follows that f respects $\sim$. Moreover, relation (II) in Definition 3.2 implies that each Y^d is $\sim$-finite, and [12, Theorem 3.22] computes each cardinality $N_{\sigma,\pi}(m,n,d) := \left|[(\sigma,\pi)]_\sim \cap Y^d_{m,n}\right|$, showing in particular that they are polynomial in m and n. Theorem 3.1 immediately follows by applying Proposition 1.2.

4 Schubert structure constants of bounded length

Our final application of Proposition 1.2 answers a recent question of Pak–Robichaux [7].

Let $\{G_n\}_{n\geq 1} \in \{\{SL_{n+1}\}, \{SO_{2n+1}\}, \{Sp_{2n}\}, \{SO_{2n}\}\}_{n\geq 1}$ be one of the families of complex classical Lie groups, each with maximal Borel subgroup B and maximal torus T. Each *generalized flag variety* G_n/B has a cell decomposition into *Schubert cells*, which are the B_--orbits of elements of the *Weyl group* $W_n :- N(T)/T$. The *Schubert varieties* $\mathfrak{X}_w$ for $w \in W_n$ are the Zariski closures of these cells; the codimension of $\mathfrak{X}_w$ is given by the *Coxeter length* $\ell(w)$. Via Poincaré duality, the Schubert varieties yield cohomology classes $\sigma_w \in H^\star(G_n/B)$. These *Schubert classes* in fact form a $\mathbb{Z}$-linear basis for $H^\star(G_n/B)$. Thus there exist integers, the *Schubert structure constants* $c_{u,v}^w$, such that

$$\sigma_u \smile \sigma_v = \sum_{w\in W_n} c_{u,v}^w \sigma_w.$$

The Schubert structure constants are known to be nonnegative for geometric reasons, and it is a major open problem to give a positive combinatorial rule for them.

For a fixed choice of classical type, consider the sum of Schubert structure constants

$$\gamma_k(n) := \sum_{\substack{u,v,w\in W_n \\ \ell(w)=k}} c_{u,v}^w.$$

Pak–Robichaux introduced a signed puzzle rule [7, Theorem 1.1] for $c_{u,v}^w$ in the case where $G_n = SL_{n+1}$ is the special linear group and $W_n = \mathcal{S}_{n+1}$ is the symmetric group. They proved the following consequence:

Theorem 4.1 ([7, Theorem 1.2])**.** *If $\{G_n\} = \{SL_{n+1}\}$, then $\gamma_k(n)$ is a polynomial in n.*

In [7, Section 8.3], the authors state that it would be interesting to find a more conceptual argument for Theorem 4.1, especially one which explicitly determines the degree (or more) of $\gamma_k(n)$. We give such an argument in all classical Lie types via Proposition 1.2:

Theorem 4.2. *Let $\{G_n\} \in \{\{SL_{n+1}\}, \{SO_{2n+1}\}, \{Sp_{2n}\}, \{SO_{2n}\}\}$.*

(I) For sufficiently large n, $\gamma_k(n)$ is a polynomial in n.

(II) The lead term of $\gamma_k(n)$ is

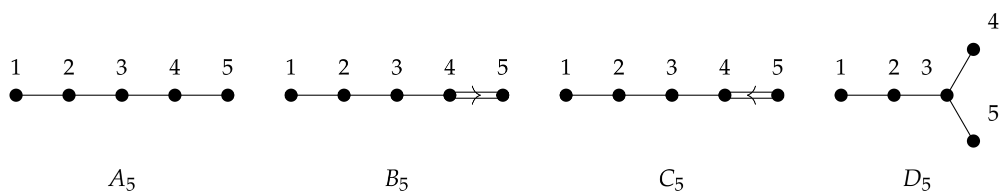
$$LT(\gamma_k(n)) = \frac{(2n)^k}{k!}.$$

To apply Proposition 1.2 in this context, let $X = \bigsqcup_{n\geq 1} W_n$ be the union of all Weyl groups in a fixed choice of Lie type. For each $k \geq 0$, the subset $Y^k \subseteq X$ consists of all Weyl group elements w with Coxeter length k. The map $f\colon X \to \mathbb{Z}$ sends each Weyl group element w to the (necessarily finite) sum of Schubert structure constants $\sum_{u,v} c_{u,v}^w$.

The equivalence relation $\sim$ on X we use to prove Theorem 4.2 is defined in terms of induced subgraphs of *Dynkin diagrams*. After our original preprint [11] appeared on the arXiv, we learned that our equivalence relation is the same as the *Cartan equivalence* employed by Richmond–Slofstra [9] to solve the isomorphism problem for Schubert varieties in the flag variety. We thank Edward Richmond for showing this connection to us. Here, we give an independent argument using the *equivariant restriction* formula of Andersen–Jantzen–Soergel [1] to prove that f respects $\sim$ and each Y^k is $\sim$-finite, and to enumerate elements of the sets $[w]_\sim \cap Y^k_n$. The (eventual) polynomiality of $\gamma_k(n)$ then follows by Proposition 1.2, and we show moreover that the $c^w_{u,v}$ contributing to the lead term of this polynomial are the easiest Schubert structure constants to compute.

4.1 Background on equivariant restrictions

We give a brief overview of the results in Lie theory and equivariant cohomology used in our proof of Theorem 4.2, following the exposition and notation of [10]. See [10] for further reading and references. The four families of classical complex Lie groups, SL_{n+1}, SO_{2n+1}, Sp_{2n}, and SO_{2n}, are characterized by certain (multi)graphs, the *Dynkin diagrams* of types A_n, B_n, C_n, and D_n respectively. Each node in a Dynkin diagram denotes a simple reflection generating the associated Weyl group. Edges represent commutation relations between these generators, with non-adjacent nodes corresponding to commuting pairs of generators. Examples of the classical Dynkin diagrams for $n = 5$ are below.



For our purposes, we only need to know that Dynkin diagrams encode the generators and relations of Weyl groups. In particular, call a map $\iota\colon \Delta \to \Delta'$ of Dynkin diagrams an *embedding* if it realizes Δ as an induced subgraph of Δ'. Then ι induces an embedding $W \hookrightarrow W'$ of the corresponding Weyl groups. For example, the fact that B_n, C_n, and D_n contain embedded copies of A_{n-1} indicates that the associated Weyl groups contain symmetric groups as subgroups.

Now, let G_n/B be a generalized flag variety with Weyl group W_n and simple roots $\{\alpha_1, \ldots, \alpha_n\}$. There is a torus T acting on G_n/B on the left, which stabilizes each Schubert variety $\mathfrak{X}_w$ ($w \in W_n$). Thus each $\mathfrak{X}_w$ gives rise to an *equivariant Schubert class* ξ_w in the T-equivariant cohomology $H^\star_T(G_n/B)$. The equivariant Schubert classes form a basis for $H^\star_T(G_n/B)$ as a $H^\star_T(pt)$-module, where we identify $H^\star_T(pt)$ with the polynomial ring $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$. Thus we obtain *equivariant Schubert structure constants* $C^w_{u,v} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, defined by the equation

$$\xi_u \smile \xi_v = \sum_{w \in W_n} C_{u,v}^w \xi_w.$$

Each $C_{u,v}^w$ is a homogeneous polynomial of degree $\ell(w) - \ell(u) - \ell(v)$. Setting $\alpha_i = 0$ for all i recovers ordinary cohomology; in particular, $C_{u,v}^w = c_{u,v}^w$ whenever $\ell(w) = \ell(u) + \ell(v)$ (i.e., whenever $c_{u,v}^w \neq 0$). The advantage in working with $H_T^\star(G_n/B)$ comes from *GKM theory* [5]: the set $(G_n/B)^T$ of T-fixed points of G_n/B is indexed by W_n (hence finite), and it turns out that the map

$$H_T^\star(G_n/B) \to H_T^\star((G_n/B)^T) \cong \bigoplus_{w \in W_n} \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \tag{4.1}$$

on cohomology induced by the inclusion $(G_n/B)^T \hookrightarrow G_n/B$ is also an injection. Thus the product structure on $H_T^\star(G_n/B)$ is given by pointwise products of lists of polynomials.

Definition 4.3. The image of ξ_v under the embedding (4.1) is denoted $(\xi_{v|w})_{w \in W_n}$. The polynomial $\xi_{v|w} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ is called the *equivariant restriction* of ξ_v at w.

Andersen–Jantzen–Soergel [1] gave a formula for $\xi_{v|w}$ involving the combinatorics of W_n and its associated root system, independently discovered by Billey [2]. Let s_{α_i} denote the simple reflection through the hyperplane orthogonal to α_i. Let $I = s_{\beta_1} \ldots s_{\beta_{\ell(w)}}$ be a reduced word for $w \in W_n$. For each subword $J \subseteq I$ and $i \in [\ell(w)]$, let

$$\gamma_{J,i} := \begin{cases} \beta_i s_{\beta_i} & \beta_i \in J, \\ s_{\beta_i} & \beta_i \notin J. \end{cases}$$

Theorem 4.4 (Equivariant restriction, [1]). *For v, w, I, J, and $\gamma_{J,i}$ as above, the equivariant restriction of ξ_v at w is the polynomial*

$$\xi_{v|w} = \sum_{J \subseteq I} \prod_{i=1}^{\ell(w)} \gamma_{J,i} \cdot 1 \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n],$$

where the sum is over subwords $J \subseteq I$ that are reduced words for v.

Example 4.5. Let $w = 3241 \in \mathcal{S}_4$ and choose the reduced word $I = s_1 s_2 s_3 s_1$ for w. Let $v = 2143$, which has reduced words $s_1 s_3$ and $s_3 s_1$. Using the type-A formula

$$s_{\alpha_i} \cdot \alpha_j = \begin{cases} -\alpha_i & i = j, \\ \alpha_i + \alpha_j & |i - j| = 1, \\ \alpha_j & |i - j| \geq 2, \end{cases}$$

we compute $\xi_{v|w}$ using Theorem 4.4:

$$\xi_{v|w} = \alpha_1 s_1 s_2 \alpha_3 s_3 s_1 \cdot 1 + s_1 s_2 \alpha_3 s_3 \alpha_1 s_1 \cdot 1 = \alpha_1 s_1 s_2 \alpha_3 + s_1 s_2 \alpha_3 \alpha_1 = (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3).$$

The restriction formula gives a slick, type-independent method for proving facts about Schubert structure constants without identifying explicit representatives for the cohomology classes. We record some well-known consequences for use in our proofs.

Lemma 4.6 ([10, pg. 2])**.** *The equivariant restriction $\xi_{v|w} = 0$ unless $v \leq w$ in Bruhat order. The equivariant structure constant $C^w_{u,v} = 0$ unless $u \leq w$ and $v \leq w$ in Bruhat order.*

Proposition 4.7 ([10, Theorem 2.1])**.** *If $\iota\colon \Delta \to \Delta'$ is an embedding of Dynkin diagrams, then $C^w_{u,v}$ and $C^{\iota(w)}_{\iota(u),\iota(v)}$ agree up to labelling of the variables. In particular, $c^w_{u,v} = c^{\iota(w)}_{\iota(u),\iota(v)}$.*

Proposition 4.8. *If $w = w' \times w''$ factors over a parabolic subgroup $W' \times W'' \leq W$, then $C^w_{u,v} = 0$ unless $u = u' \times u''$ and $v = v' \times v''$ also factor. In this case,*

$$C^w_{u,v} = C^{w'}_{u',v'} C^{w''}_{u'',v''}.$$

Proof. The first sentence is immediate from Lemma 4.6 and the fact that Bruhat order respects factorizations. Now, if I is a reduced word for w, then using the commutation relations for Weyl groups we may choose I to be of the form $I'I''$, where I' is a reduced word for w' and I'' is a reduced word for w''. We claim first that for $v = v' \times v'' \leq w$,

$$\xi_{v|w} = \xi_{v'|w'} \cdot \xi_{v''|w''}.$$

Note that by our choice of $I = I'I''$, each J appearing in the formula for $\xi_{v|w}$ must be of the form $J'J''$, where J' is a reduced word for v' and J'' is a reduced word for v''. Moreover, the simple reflections $s_{\beta_{i'}}$ associated to $\beta_{i'} \in I'$ fix all the roots $\beta_{i''} \in J''$, since these roots are orthogonal in the root system for $W' \times W''$. Our claim follows:

$$\xi_{v|w} = \sum_{J \subseteq I'I''} \prod_{i=1}^{\ell(w)} \gamma_{J,i} \cdot 1 = \left(\sum_{J' \subseteq I'} \prod_{i'=1}^{\ell(w')} \gamma_{J',i'} \cdot 1 \right) \left(\sum_{J'' \subseteq I''} \prod_{i''=1}^{\ell(w'')} \gamma_{J'',i''} \cdot 1 \right) = \xi_{v'|w'} \xi_{v''|w''}.$$

Now, let x denote the longest element in $W' \times W''$. Then by definition

$$\xi_{u|x} \cdot \xi_{v|x} = \sum_{w \in W} C^w_{u,v} \xi_{w|x}.$$

By Lemma 4.6, the only nonzero terms in the sum come from w such that $u, v \leq w \leq x$, and all such w factor as $w' \times w'' \in W' \times W''$. But we also have

$$\begin{aligned} \xi_{u|x}\xi_{v|x} &= \xi_{u'|x'}\xi_{u''|x''}\xi_{v'|x'}\xi_{v''|x''} \\ &= \sum_{w' \in W', w'' \in W''} C^{w'}_{u',v'} \xi_{w'|x'} C^{w''}_{u'',v''} \xi_{w''|x''} = \sum_{w = w' \times w''} C^{w'}_{u',v'} C^{w''}_{u'',v''} \xi_{w|x}. \end{aligned}$$

The proposition follows by uniqueness of equivariant Schubert structure constants. $\square$

4.2 Application of Proposition 1.2

Fix $X := \bigsqcup_{n\geq 1} W_n$ to be the sequence of Weyl groups associated to one of the classical familes of complex Lie groups. The Dynkin diagram for W_n will be denoted Δ_n.

Definition 4.9. Let I be a reduced word for $w \in W_n$. The *Dynkin support* of w is the Dynkin diagram $\Delta(w)$ isomorphic to the subgraph of W_n induced by the vertices corresponding to those s_{α_j} appearing in I.

Definition 4.10. We say two elements $w, w' \in X$ are *equivalent* and write $w \sim w'$ if there is an isomorphism of Dynkin supports $\Delta(w) \cong \Delta(w')$ sending w to w'.

Since all reduced words for w involve the same set of simple reflections, $\Delta(w)$ depends only on w and not on the choice of I. Coxeter length and Dynkin support are invariant under $\sim$, so we may refer to $\ell([w]_\sim)$ or $\Delta([w]_\sim)$. The next proposition sets up the application of Proposition 1.2 in the proof of Theorem 4.2. Recall that the function $f\colon X \to \mathbb{Z}$ is defined by mapping each $w \in X$ to $\sum_{u,v} c^w_{u,v}$, and $Y^k \subseteq X$ consists of all Weyl group elements with Coxeter length k.

Proposition 4.11. *With X, f, Y^k, and $\sim$ as above, f respects $\sim$ and Y^k is $\sim$-finite. In particular, Y^k is a finite union of $\sim$-equivalence classes, with each class represented by an element of W_{2k}.*

Proof. The claim that f respects $\sim$ is exactly the statement of Proposition 4.7. To see that Y^k is finite, let $w \in Y^k$ be arbitrary. Since $\ell(w) = k$, a reduced word I for w uses at most k distinct simple reflections. Thus $\Delta(w)$ has at most k vertices, and any such diagram embeds as an induced subgraph of any classical Dynkin diagram with $2k$ vertices. This proves that $w \sim w'$ for some $w' \in W_{2k}$, so Y^k is $\sim$-finite as claimed. □

Proposition 4.11 shows that the data $(X, f, Y^k, \sim)$ satisfy the hypotheses of Proposition 1.2. We may therefore rewrite $\gamma_k(n)$ as the finite sum

$$\gamma_k(n) = \sum_{\substack{[w]_\sim \\ w\in W_{2k}, \ell(w)=k}} \left(\sum_{u,v\leq w} c^w_{u,v} \right) |[w]_\sim \cap W_n|. \tag{4.2}$$

In (4.2) we use the fact that $[w]_\sim \cap Y^k_n = [w]_\sim \cap W_n$, since Y^k is a union of $\sim$-equivalence classes. To prove Theorem 4.2, it remains to compute the cardinality of each set $[w]_\sim \cap W_n$. We now turn our attention to this problem.

4.3 Enumerating equivalent elements

The enumeration of elements in $[w]_\sim \cap W_n$ depends slightly on our choice of classical Lie type. Since $\Delta(w)$ is (isomorphic to) an induced subgraph of Δ_n, we see that $\Delta(w)$

has at most one component of type B, C, or D (when Δ_n is also of this type) and all other components are of type A. Writing $w = \prod_{i=1}^{p} w_i$ as an element of the Weyl group associated to $\Delta(w)$, for $1 \leq j \leq n$ let $a_w(j)$ denote the number of permutations of type-A_j components of $\Delta(w)$ that preserve w.

Lemma 4.12. *Let Δ be a Dynkin diagram with k' vertices and c components, all of type A. Then for all $k \geq k'-1$, the number of embeddings $\Delta \hookrightarrow A_k$ is*

$$\frac{(k+1-k')!}{(k+1-k'-c)!} = \prod_{i=0}^{c-1}(k+1-k'-i),$$

a polynomial in k with lead term k^c.

Proof. An embedding $\iota \colon \Delta \hookrightarrow A_k$ is equivalent to an ordering of the c (labelled) factors of Δ and the $(k-k')$ (unlabelled) vertices of $A_k \setminus \iota(\Delta)$ in which no two components of Δ appear consecutively (see Figure 1). Such orderings are equinumerous with orderings of c labelled objects and $(k+1-k'-c)$ unlabelled objects (with no adjacency restrictions), which are enumerated by the formula in the lemma statement. □

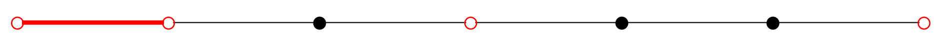


Figure 1: An embedding of $\Delta = A_1 \times A_1 \times A_2$ into A_7.

Proposition 4.13. *Fix $w \in W_m$, let n' denote the number of vertices in $\Delta(w)$, and let p denote the number of type-A components in $\Delta(w)$. Then for*

$$n \geq \begin{cases} n'-1 & \text{in type } A, \\ n' & \text{in types } B \text{ and } C, \\ n'+1 & \text{in type } D, \end{cases}$$

$|[w]_\sim \cap W_n|$ is a polynomial with lead term

$$\frac{n^p}{\prod_{j=1}^{m} a_w(j)}.$$

Proof. Let $N_w(n)$ denote the number of induced subgraphs of Δ_n isomorphic to $\Delta(w)$. Then

$$|[w]_\sim \cap W_n| = \frac{N_w(n)}{\prod_{j=1}^{m} a_w(j)},$$

so it suffices to show that $N_w(n)$ is a polynomial with lead term n^p for sufficiently large n. In type A, this follows immediately by Lemma 4.12.

In types B, C, and D, the analysis is similar. If $\Delta(w)$ contains a component that is not of type A (say with r vertices), then this component is unique and has a unique embedding into each W_n. The remaining components of $\Delta(w)$ are then embedded into a copy of A_{n-r-1}. Applying Lemma 4.12 with $k = n - r - 1$, $k' = n' - r$, and $c = p$ shows that these embeddings are enumerated by a polynomial with lead term n^p for $n \geq n'$.

The enumeration of embeddings is slightly more complicated when all components of $\Delta(w)$ are of type A. Consider first the Dynkin diagrams of type B_n or C_n, which each contain one copy of A_{n-1}. By Lemma 4.12 (with $k = n - 1$, $k' = n'$, $c = p$), embeddings of $\Delta(w)$ into this embedded type-A subdiagram are enumerated by a polynomial with lead term n^p for $n \geq n'$. There are no further embeddings $\Delta(w) \hookrightarrow \Delta_n$ unless $\Delta(w)$ contains a component isomorphic to A_1. In this case, additional embeddings are obtained by mapping a type-A_1 component of $\Delta(w)$ to the final vertex of Δ_n, then embedding the rest of $\Delta(w)$ into a copy of $A_{n-2} \hookrightarrow \Delta_n$. By Lemma 4.12 (with $k = n - 2$, $k' = n' - 1$, $c = p - 1$), such "exceptional" embeddings are enumerated by a polynomial of degree $p - 1$ for $n \geq n'$. Thus in types B and C, $N_w(n)$ is a polynomial with lead term n^p for all $n \geq n'$ as desired.

Consider now the case where $\Delta_n = D_n$. The Dynkin diagram D_n contains two embedded copies of A_{n-1}, which intersect in a copy of A_{n-2}. By inclusion-exclusion and three applications of Lemma 4.12 (twice with $k = n - 1$, $k' = n'$, $c = p$ and once with $k = n - 2$, $k' = n'$, $c = p$), for $n \geq n' + 1$ the number of embeddings of $\Delta(w)$ into one of these type-A subdiagrams is enumerated by

$$2 \cdot \frac{(n - n')!}{(n - n' - p)!} - \frac{(n - 1 - n')}{(n - 1 - n' - p)},$$

which is a polynomial with lead term n^p. Moreover, all embeddings $\Delta(w) \hookrightarrow D_n$ are embeddings into these type-A subdiagrams unless $\Delta(w)$ contains at least two A_1 components or at least one A_3 component (see Figure 2 for an example). As in the previous case, Lemma 4.12 (with $(k, k', c) = (n - 3, n' - 2, p - 2)$ and $(n - 4, n' - 3, p - 1)$ respectively) shows that for $n \geq n'$ these "exceptional" embeddings are enumerated by polynomials of degree strictly less than p. Thus the lead term of $N_w(n)$ is n^p in all cases, completing the proof. □

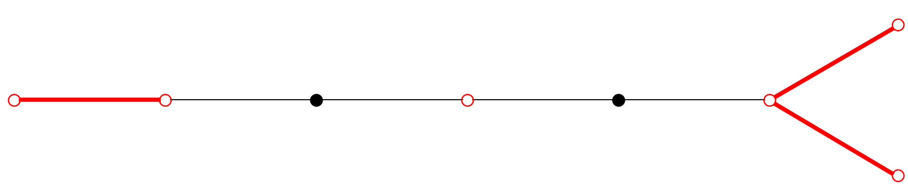


Figure 2: An "exceptional" embedding of $\Delta = A_1 \times A_2 \times A_3$ into D_8.

Proof of Theorem 4.2(I). Fix k and a choice of classical Lie type. Equation (4.2) expresses $\gamma_k(n)$ as a finite sum of constant multiples of terms $|[w]_\sim \cap W_n|$, which Proposition 4.13 shows are polynomial for sufficiently large n. More specifically, Proposition 4.13 implies that $\gamma_k(n)$ is a polynomial in n for

$$n \geq \begin{cases} k-1 & \text{in type } A, \\ k & \text{in types } B \text{ and } C, \\ k+1 & \text{in type } D. \end{cases}$$

□

We can now prove Theorem 4.2(II) after computing only the simplest $c^w_{u,v}$.

Lemma 4.14. *If $w \in W_n$ has Dynkin support $\Delta(w) = A_1^k := \underbrace{A_1 \sqcup \cdots \sqcup A_1}_{k \text{ factors}}$, then*

$$\sum_{u,v \in W_n} c^w_{u,v} = 2^k.$$

Proof. In the case where $k = 1$, we compute directly: $c^{s_1}_{s_1,id} = c^{s_1}_{id,s_1} = 1$ and $c^{s_1}_{u,v} = 0$ otherwise, so the sum is 2 as claimed. For general k, if $\Delta(w) = A_1^k$, then w factors as $s_1 \times \cdots \times s_1$. Thus by Proposition 4.11 and Proposition 4.8 we have

$$\sum_{u,v \in W_n} c^w_{u,v} = \left(\sum_{u,v \in \mathcal{S}_2} c^{s_1}_{u,v} \right)^k = 2^k.$$

□

Proof of Theorem 4.2(II). By Equation (4.2),

$$LT(\gamma_k(n)) = \sum_{[w]} \left(\sum_{u,v \leq w} c^w_{u,v} \right) |[w]_\sim \cap W_n|,$$

where the sum is over equivalence classes $[w]$ of length k maximizing the degree of $|[w]_\sim \cap W_n|$. By Proposition 4.13, the degree of $|[w]_\sim \cap W_n|$ as a polynomial in n is the number of components in $\Delta(w)$. Among $w \in W_{2k}$ of length k, the number of components of $\Delta(w)$ is maximized when $\Delta(w) = A_1^k$, i.e., when w is a product of k mutually orthogonal simple reflections. By Lemma 4.14 and Proposition 4.13 it follows that

$$LT(\gamma_k(n)) = \frac{2^k}{k!} LT(N_w(n)) = \frac{(2n)^k}{k!},$$

completing the proof. □

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References

[1] H. H. Andersen, J. C. Jantzen, and W. Soergel. "Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p". *Astérisque* 220 (1994), p. 321.

[2] S. C. Billey. "Kostant polynomials and the cohomology ring for G/B". *Duke Math. J.* **96**.1 (1999), pp. 205–224. DOI.

[3] W. Bruns, A. Conca, C. Raicu, and M. Varbaro. *Determinants, Gröbner bases and cohomology*. Springer Monographs in Mathematics. Springer, Cham, [2022] ©2022, pp. xiii+507. DOI.

[4] D. A. Cox, J. B. Little, and D. O'Shea. *Using Algebraic Geometry*. Second. Vol. 185. Graduate Texts in Mathematics. Springer, 2005. DOI.

[5] M. Goresky, R. Kottwitz, and R. MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". *Invent. Math.* **131**.1 (1998), pp. 25–83. DOI.

[6] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417.

[7] I. Pak and C. Robichaux. "Signed puzzles for Schubert coefficients". 2025. arXiv:2504.17734.

[8] E. Ramos, D. Speyer, and G. White. "FI-sets with relations". *Algebr. Comb.* **3**.5 (2020), pp. 1079–1098. DOI.

[9] E. Richmond and W. Slofstra. "The isomorphism problem for Schubert varieties". 2022. arXiv:0911.4941.

[10] C. Robichaux, H. Yadav, and A. Yong. "The A·B·C·Ds of Schubert calculus". *Sém. Lothar. Combin.* **85** ([2020–2021]), Art. B85a, 12.

[11] A. Stelzer. "Sums of Schubert structure constants with bounded Coxeter length". To appear in *Algebraic Combinatorics*. arXiv:2506.13684.

[12] A. Stelzer and A. Yong. "RSK as a linear operator". To appear in *Selecta Mathematica*. arXiv:2410.23009.