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Parking Spaces for Complex Reflection Groups

Jason Stack*[1]

[1]*School of Natural Sciences and Mathematics, University of Texas at Dallas, United States of America*

Abstract. For W a Coxeter group, Q its root lattice, and h its Coxeter number, we call $Q/(h+1)Q$ the standard W-parking space, which carries an action of W. The orbits of the W-action can be parametrized by the W-nonnesting partitions, defined as antichains in the root poset $(\Phi^+, \leq)$. In this setting, there is also the notion of a W-noncrossing partition. Replacing the W-nonnesting partitions with the W-noncrossing partitions, (Armstrong, Reiner, Rhoades 2015) define a new combinatorial parking space: the W-noncrossing parking functions. They further define an algebraic W-parking space, with both parking spaces carrying a $(W \times C)$-action, and give an explicit map between the two. Answering an open problem of theirs, we extend these definitions and results to irreducible well–generated complex reflection groups W. We define a combinatorial W-noncrossing parking space and an algebraic W-parking space for such W, and exhibit a $(W \times C)$-equivariant isomorphism between the two. As a consequence of this isomorphism, we enumerate the W-noncrossing parking functions. We prove the results for all such complex reflection groups except G_{34}, E_7, and E_8. A Fuss generalization of the real case is given by (Rhoades 2014), which we also extend to well–generated complex reflection groups.

Keywords: Coxeter-Catalan combinatorics, Fuss–Catalan numbers, noncrossing partitions, noncrossing parking functions

1 Introduction

1.1 Noncrossing Partitions

We first discuss the *W-noncrossing partitions* for W a Coxeter group. In type A, the symmetric group, these are partitions $\pi = \{B_1, \ldots, B_r\}$ of $[n] := \{1, \ldots, n\}$ such that there exists no $1 \leq a < b < c < d \leq n$ where $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. The term *noncrossing* comes from a visualization of these partitions by taking n nodes equidistant in a circle labelled by $[n]$. Taking the convex hull of each block of π, the noncrossing condition corresponds to the intersection of any two distinct convex hulls being empty, as shown in Figure 1.

*jcs220002@utdallas.edu. Partially supported by the National Science Foundation under Grant No. DMS-2246877

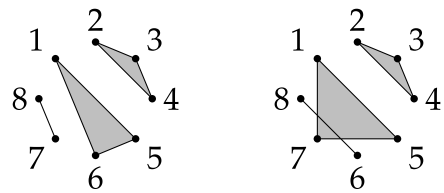


Figure 1: An A_7-noncrossing partition and a non-example.

Let (W, S) be a finite Coxeter system with reflection representation V and fix a Coxeter element $c \in W$, any element that can be written as a product of all elements in S. When we move to the complex setting, we will rely on the notion of *regular* elements to define the Coxeter elements, but we postpone the details until later. The poset of all *W-noncrossing partitions*, denoted $\mathrm{NC}(W, c)$, is defined as the interval $[e, c]_T$ in the absolute order (the Cayley graph of W generated by its reflections T). The W-noncrossing partitions are counted by the *Coxeter-Catalan number*

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{h + d_i}{d_i},$$

where $d_1 \leq d_2 \leq \cdots \leq d_n$ are the degrees of a set of algebraically independent homogeneous polynomials generating the algebra of invariants $\mathrm{Sym}(V^*)^W$ and $h = d_n$ is the Coxeter number of W [18, 5]. D. Armstrong defined a Fuss generalization of these noncrossing partitions, the *k-W-noncrossing partitions*, given by multichains in $\mathrm{NC}(W, c)$ of length k [1].

The noncrossing partitions and their Fuss generalizations are then used to define the *W-noncrossing parking functions* and the *k-W-noncrossing parking functions*, respectively, for real reflection groups W [2, 10, 20]. The object of this paper is to extend the definition of the noncrossing parking functions and related results to irreducible well–generated complex reflection groups.

1.2 Parking Functions

1.2.1 Parking Functions

Definition 1.1. A *(classical) parking function* is a sequence of positive integers, $(a_1, \ldots, a_n)$, such that their increasing rearrangement, $(b_1, b_2, \ldots, b_n)$, satisfies $b_i \leq i$ for all $i = 1, \ldots, n$. The set of all parking functions, which we denote Park_n, becomes the *classical parking space* when endowed with the action of $\mathfrak{S}_n$ via $(w \cdot f)(i) = f(w^{-1}(i))$ for $f \in \mathrm{Park}_n$ and $w \in \mathfrak{S}_n$. Note that Park_n is of size $(n+1)^{n-1}$ and has Catalan many $\mathfrak{S}_n$-orbits.

The action of $\mathfrak{S}_n$ on Park_n can be generalized to crystallographic real reflection groups, or *Weyl groups* W. Let Q be the root lattice of W and consider the quotient $Q/(h+1)Q$, which has cardinality $(h+1)^n$ and carries a W-action induced by the action of W on V. With Φ^+ the positive roots, the W-orbits on $Q/(h+1)Q$ can be parametrized by antichains in the root poset $(\Phi^+, \leq)$, known as the *W-nonnesting partitions*, which share a connection with the noncrossing partitions, notably in their enumeration by $\mathrm{Cat}(W)$ (see [3] [18] [9, Section 4], [23, Section 5] and [22]). Then, we define the space of *W-nonnesting parking functions* as the set of cosets

$$\mathrm{Park}^{\mathrm{NN}}(W) := \{wW_\pi : w \in W \text{ and } \pi \in \mathrm{NN}(W)\},$$

under left multiplication by elements of W with $W_\pi := \langle t_\alpha \in T : \alpha \in \pi \rangle$. $\mathrm{Park}^{\mathrm{NN}}(W)$ is in W-equivariant bijection with $Q/(h+1)Q$ [4, Lemma 4.1, Theorem 4.2].

1.2.2 Noncrossing Parking Functions

By simply replacing the nonnesting partitions with noncrossing partitions, we obtain the space of *W-noncrossing parking functions* [2] as the set of cosets

$$\mathrm{Park}^{\mathrm{NC}}(W) := \{wW_\pi : w \in W \text{ and } \pi \in \mathrm{NC}(W)\},$$

again under left multiplication of a coset by elements of W. Here, $W_\pi := \langle t \in T : t \leq \pi \rangle$. The W-noncrossing parking functions carry not just a W-action, but a $(W \times C)$-action, where $C = \langle c \rangle$. B. Rhoades defines a Fuss analogue of the noncrossing parking functions, the *k-W-noncrossing parking functions* [20]. The k-W-noncrossing parking space carries an action of $W \times \mathbb{Z}_{kh}$. We discuss this space in more detail in Subsection 2.2.

1.3 Algebraic Parking Space

Let W be a Coxeter group with reflection representation V. D. Armstrong, V. Reiner, and B. Rhoades define the *algebraic W-parking space* (or the *k-algebraic W-parking space* in the Fuss case) by a choice of a degree p *homogeneous system of parameters*, (Θ_p), and the quotient $\mathrm{Sym}\, V^*/(\Theta_p - \mathbf{x})$. We choose Θ_{h+1} of degree $h+1$ (or $kh+1$) and have that this quotient carries a $(W \times C)$-action (or $(W \times \mathbb{Z}_{kh})$-action). D. Armstrong, V. Reiner, and B. Rhoades construct bijections in each type between the W-noncrossing parking functions and the algebraic W-parking space that are invariant under these actions [2]. B. Rhoades does the same in the Fuss case [20].

We extend these results to well–generated complex reflection groups. Let W be an irreducible well–generated complex reflection group not of type G_{34}, E_7, or E_8. Let h be the Coxeter number of W and n its rank.

Theorem 1.2. *For all such W, there is a $(W \times C)$-equivariant isomorphism between the W-noncrossing parking functions and the algebraic W-parking space.*

Theorem 1.3. *For $W = G(d,1,n)$ or $W = G(d,d,n)$, and $k \geq 1$, there is a $(W \times \mathbb{Z}_{kh})$-equivariant isomorphism between the k-W-noncrossing parking functions and the k-algebraic W-parking space.*

Theorem 1.4. *For all such W, there are $(h+1)^n$ W-noncrossing parking functions.*

Theorem 1.5. *For all such W, there are $(kh+1)^n$ k-W-noncrossing parking functions.*

We extend the models and proofs constructed from the real case into the complex setting. To do so, we restrict to well–generated complex reflection groups for which we have a notion of Coxeter element. Notably, $G(2,1,n) \cong W(B_n)$ and $G(2,2,n) \cong W(D_n)$, which informs the models and bijection for the two infinite subfamilies. We prove Theorems 1.4 and 1.5 by building the noncrossing elements as subspaces in [17, 12].

2 Background

2.1 Noncrossing Parking Functions in Real Reflection Groups

Let (W, S) be a finite Coxeter system, with simple generators $S = \{s_1, \ldots, s_n\}$ and fix a *Coxeter element*, c, a product of the simple reflections, in some order. Let T be the set of all reflections in W. For $w \in W$, the *reflection length*, $\ell_T(w)$, is the smallest $r \in \mathbb{Z}^+$ such that $w = t_1 \cdots t_r$ with $t_i \in T$. The *absolute order* is then defined by

$$u \leq_T v \iff \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).$$

Definition 2.1 ([5][8])**.** The poset of *W-noncrossing partitions*, $\mathrm{NC}(W,c)$, is the interval $[e,c]_T$ in the absolute order.

Note, there is an isomorphism between $\mathrm{NC}(W,c)$ and $\mathrm{NC}(W,c')$ for any Coxeter elements c, c' [19, Corollary 1.5], and as such we drop the dependence on c in the notation. We build upon these noncrossing partitions to define the noncrossing parking functions.

Definition 2.2 ([2])**.** The set of *W-noncrossing parking functions* is defined as

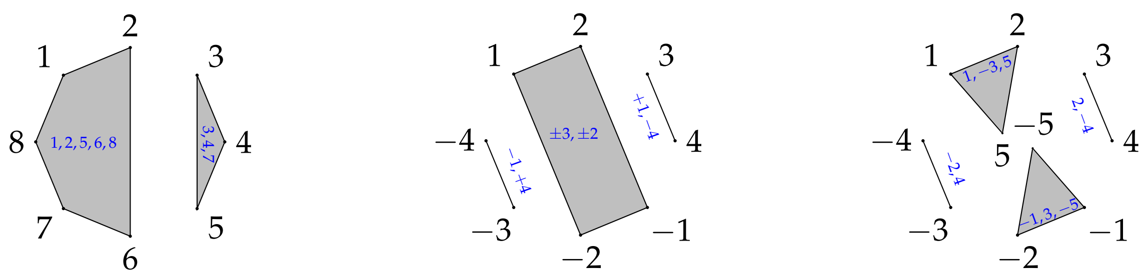
$$\mathrm{Park}^{\mathrm{NC}}(W) := \{wW_\pi : w \in W \text{ and } \pi \in \mathrm{NC}(W)\}.$$

We will use the notation $[w,\pi]$ to denote the element $wW_\pi \in \mathrm{Park}^{\mathrm{NC}(W)}$. This set carries a $(W \times C)$-action given by

$$(v, c^p)[w,\pi] = [vwc^{-p}, c^p \pi c^{-p}],$$

for any $p \in \mathbb{N}$ and any $v \in W$.

Example 2.3. In the classical (real) types A, B, and D, we can model the noncrossing parking functions pictorially by building upon models for our noncrossing partitions [2]. We will occasionally write $\bar{i} := -i$ for ease of notation. We provide examples for each type in Figure 2.



(a) Type A noncrossing parking function with $\pi = \{\{1,2,6,7,8\},\{3,4,5\}\}$ and $w = (125768)(34)$.

(b) Type B noncrossing parking function with $\pi = \{\pm\{3,4\},\{\pm 1,\pm 2\}\}$ and $w = (123)(\bar{4}4)$.

(c) Type D noncrossing parking function with $\pi = \{\pm\{1,2,5\},\pm\{3,4\}\}$ and $w = (2\bar{3}4)(15)$.

Figure 2: Parking functions in real reflection groups for types A, B, and D.

2.2 Fuss Analogues

Recall that a *k-multichain* in a poset $(P,\leq)$ is a sequence of elements, $(x_1 \leq \cdots \leq x_k)$, in P.

Definition 2.4 ([20])**.** The set of *k-W-noncrossing parking functions* is defined as

$$\mathrm{Park}^k_{\mathrm{NC}}(W) := \{wW_{\pi_1} \subseteq wW_{\pi_2} \subseteq \cdots \subseteq wW_{\pi_k} : w \in W \text{ and } \pi_i \in \mathrm{NC}(W)\}.$$

We will use $[w,(\pi_1 \leq \cdots \leq \pi_k)]$ to denote an element of $\mathrm{Park}^k_{\mathrm{NC}}(W)$, where $(\pi_1 \leq \cdots \leq \pi_k)$ is a k-multichain in $\mathrm{NC}(W)$. The set $\mathrm{Park}^k_{\mathrm{NC}}(W)$ carries an action by $W \times \mathbb{Z}_{kh}$, which generalizes the action in the non–Fuss setting. In [20], B. Rhoades describes a combinatorial model in types B and D, which we extend to the two infinite families of irreducible well–generated complex reflection groups.

2.3 Complex Reflection Groups

We now discuss the necessary background for complex reflection groups, referring mostly to [16]. Let V be a finite dimensional complex vector space. We call a linear transformation $g \in \mathrm{GL}(V)$ a *reflection* if g has finite order and $\mathrm{Fix}(g)$ is of codimension 1. In this case, we call $\mathrm{Fix}(g)$ the *reflecting hyperplane* of g.

Definition 2.5. A *(finite) complex reflection group*, W, is a finite subgroup of $\mathrm{GL}(V)$ that is generated by reflections. We denote $\mathcal{R}$ by the set of all reflections in W and $\mathcal{A}$ the set of corresponding reflecting hyperplanes. We say W is *irreducible* if V is an irreducible W–module.

The missing piece to generalizing the W-noncrossing partitions to the complex case is the notion of a Coxeter element. For this, we restrict to *well–generated complex reflection* *groups*. The *degrees*, $d_1 \leq \cdots \leq d_n$, of W are defined as the degrees of the homogeneous polynomials generating $\mathrm{Sym}(V^*)^W \cong \mathbb{C}[x_1, \ldots, x_n]$, the symmetric algebra of V^*, where $x_1, \ldots, x_n$ is a basis for V^*.

Definition 2.6. Let W be an irreducible complex reflection group with reflection representation V of dimension n. Then, W is *well–generated* if W is generated by n reflections and we define the *Coxeter number* of W by $h := d_n$, the highest degree.

The irreducible well–generated complex reflection groups are classified [21]. They are the two infinite subfamilies—$G(d,1,n)$ and $G(d,d,n)$—as well as the exceptional groups G_i where $i \in \{4,5,6,8,9,10,14,16,17,18,20,21,23,\ldots,30,32,\ldots,37\}$.

In contrast to the real case, we have no simple reflections for irreducible well–generated complex reflection groups. Instead, we rely on certain regular elements to define the Coxeter elements in this setting [24]. A vector $v \in V$ is *regular* if it is not contained in any reflecting hyperplane $H \in \mathcal{A}$. For ζ a root of unity, an element $c \in W$ is *ζ-regular* if it has a regular eigenvector with eigenvalue ζ. In this case, we say that the order r of ζ is a *regular number* for W. Equivalently, r is a regular number for W if r divides the same number of degrees and *codegrees*, which for well–generated W we define by $d_1^* \leq \cdots \leq d_n^*$ such that $d_i^* := d_n - d_i$ [15, 14]. The Coxeter number $h := d_n$ is thus a regular number for W as it divides only d_n and d_1^*.

Definition 2.7. For W an irreducible well–generated complex reflection group, the *Coxeter elements* of W are the ζ-regular elements of W, for ζ an h^{th} root of unity.

With the definition of a Coxeter element for W, we may extend the definitions of the W-noncrossing partitions and then the W-noncrossing parking functions.

3 Results

In this section, we define the combinatorial and algebraic parking spaces for irreducible well–generated complex reflection groups W. With $\mathcal{R}$ the set of reflections in W, we analogously define the *reflection length* and *absolute order* as in the real case. Fix a Coxeter element $c \in W$. The poset $\mathrm{NC}(W,c)$ of *W-noncrossing partitions* is defined as the interval $[e,c]_{\mathcal{R}}$ in the absolute order. We will drop the c in the notation of $\mathrm{NC}(W,c)$ due to Corollary 1.5 of [19]. We build upon $\mathrm{NC}(W)$, to define the W-noncrossing parking functions.

Definition 3.1. For W an irreducible well–generated complex reflection group, the set of *W-noncrossing parking functions* is defined as

$$\mathrm{Park}^{\mathrm{NC}}(W) := \{wW_\pi : w \in W \text{ and } \pi \in \mathrm{NC}(W)\},$$

with $W_\pi := \langle t \in \mathcal{R} : t \leq \pi \rangle$.

Similarly to the real case, we use $[w, \pi]$ to denote an element of $\mathrm{Park}^{\mathrm{NC}}(W)$ and $\mathrm{Park}^{\mathrm{NC}}(W)$ carries a $(W \times C)$-action. We next extend the k-W-noncrossing parking functions to irreducible well–generated complex reflection groups.

Definition 3.2. The set of *k-W-noncrossing parking functions* is defined as

$$\mathrm{Park}^k_{\mathrm{NC}}(W) := \{wW_{\pi_1} \subseteq wW_{\pi_2} \subseteq \cdots \subseteq wW_{\pi_k} : w \in W \text{ and } \pi_i \in \mathrm{NC}(W)\}.$$

We will use $[w, (\pi_1 \leq \cdots \leq \pi_k)]$ to denote an element of $\mathrm{Park}^k_{\mathrm{NC}}(W)$, where $(\pi_1 \leq \cdots \leq \pi_k)$ is a k-multichain in $\mathrm{NC}(W)$. Similarly to the real case, $\mathrm{Park}^k_{\mathrm{NC}}(W)$ carries an action by $W \times \mathbb{Z}_{kh}$ extended from the non–Fuss case.

3.1 Algebraic Parking Space

In the real case, D. Armstrong, V. Reiner, and B. Rhoades rely on a choice of a degree p *homogeneous system of parameters (hsop)*, (Θ_p), to define the algebraic parking space. Due to a result of P. Etingof [2, Appendix], for $k \geq 1$, there exists a degree $kh+1$ hsop $(\Theta_{kh+1}) = (\theta_1, \ldots, \theta_n)$ carrying $V^*(-kh-1)$ such that the map $V^* \to S$ given by $x_i \to \theta_i$ is W-equivariant. In types B and D, the hsop can be computed easily as $\left(x_1^{kh+1}, \ldots, x_n^{kh+1}\right)$. The *algebraic W-parking space* is then defined as the quotient $S/\left(\Theta_{h+1} - \mathbf{x}\right)$ and the *algebraic k-W-parking space* by the quotient $S/\left(\Theta_{kh+1} - \mathbf{x}\right)$ for $\mathbf{x} := (x_1, \ldots, x_n)$ a basis of V^* and $S := \mathrm{Sym}(V^*)$.

We extend the algebraic parking space to irreducible well–generated complex reflection groups. Let W be an irreducible well–generated complex reflection group with Coxeter number h and reflection representation V. Let $S := \mathrm{Sym}(V^*)$ be the algebra of polynomial functions on V.

Definition 3.3 ([7], Definition 4.1)**.** Let U be an n-dimensional complex representation of W. We say that a collection $\Theta_p = (\theta_1, \ldots, \theta_n)$, with each $\theta_i \in S$ all homogeneous of degree p, forms a *homogeneous system of parameters (hsop) carrying $U(-p)$* if $(\theta_1, \ldots, \theta_n)$ are algebraically independent, the quotient $S/(\Theta_p)$ is finite dimensional over $\mathbb{C}$, and the span $\mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n$ is a W-stable subspace of the p^{th} homogeneous component of S and carries a W-representation equivalent to U.

Let $(V^*)^{\sigma_p}$ be the *Galois twist* of V^* corresponding to p—that is, an automorphism of $\mathbb{C}$ mapping $e^{\frac{2\pi\sqrt{-1}}{h}}$ to $e^{\frac{2\pi p\sqrt{-1}}{h}}$.

Theorem 3.4 ([13],[11])**.** *If* $\gcd(p, h) = 1$*, then S contains a degree p hsop Θ_p carrying $(V^*)^{\sigma_p}$.*

The proof follows from the work of I. Gordon and S. Griffeth (see [13], Theorem 2.11 and Subsection 2.18) combined with the freeness theorem for finite Hecke algebras [11][1].

[1]The author would like to thank Stephen Griffeth for pointing out this result and Vic Reiner for suggesting to reach out to Stephen.

Let $x_1, \ldots, x_n$ be a basis for V^* and let Θ_{h+1} be an hsop of degree $h+1$ carrying $V^*(-h-1)$ such that the map $x_i \mapsto \theta_i$ is W-equivariant.

Definition 3.5. Let W be an irreducible well–generated complex reflection group. Let $\Theta_{h+1} = (\theta_1, \ldots, \theta_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be as above. The *algebraic W-parking space* is

$$\mathrm{Park}^{alg}(W, \Theta_{h+1}) := S/(\Theta_{h+1} - \mathbf{x}).$$

Similarly for the Fuss case, let $\Theta_{kh+1} = (\theta_1, \ldots, \theta_n)$ be an hsop of degree $kh+1$ carrying $V^*(-kh-1)$ such that $x_i \mapsto \theta_i$ is W-equivariant.

Definition 3.6. Let $k \geq 1$. Let $\Theta_{kh+1} = (\theta_1, \ldots, \theta_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be as above. Define the *k-algebraic W-parking space* by

$$\mathrm{Park}^{k}_{alg}(W, \Theta_{kh+1}) := S/(\Theta_{kh+1} - \mathbf{x}).$$

The ideals $(\Theta_{h+1} - \mathbf{x})$ and $(\Theta_{kh+1} - \mathbf{x})$ are $(W \times C)$-stable and $(W \times \mathbb{Z}_{kh})$-stable, respectively, with the actions of linear substitutions and root of unity scaling. We are able to drop the dependence on Θ_p from the definition by [2, Proposition 2.11]. Similarly to real types B and D, $\left(x_1^{h+1}, \ldots, x_n^{h+1}\right)$ (or x_i^{kh+1} in the Fuss case) for $G(d,1,n)$ and $G(d,d,n)$ gives an hsop carrying $V^*(-h-1)$ (or $V^*(-kh-1)$) [18].

Now that we have defined combinatorial and algebraic parking spaces for any irreducible well–generated complex reflection groups, we enumerate by providing an explicit $(W \times C)$-equivariant map (or $(W \times \mathbb{Z}_{kh})$ in the Fuss case) between the two parking spaces for the infinite families $G(d,1,n)$ and $G(d,d,n)$. For the exceptional groups, we compute the W-noncrossing parking space $(W \times C)$-character and compare it to the formula for the algebraic W-parking space $(W \times C)$-character extended from [2, Proposition 2.15]. We perform the character computation and enumeration via computer using [17, 12]. This character computation is infinite in the Fuss case and thus only conjectural.

4 $G(d,1,n)$

The group $G(d,1,n)$ is the group of $n \times n$ monomial matrices with nonzero entries being d^{th} roots of unity. Note $G(2,1,n) \cong W(B_n)$ and we think of $G(d,1,n)$ as permuting the set $[n]^d := \{1^1, \ldots, n^1, \ldots, 1^d, \ldots, n^d\}$. As such, the central symmetry in type B translates into a rotational symmetry in $G(d,1,n)$. D. Bessis and R. Corran show that the $G(d,1,n)$-noncrossing partitions are those of type A that are additionally invariant under $\frac{2\pi}{n}$–rotation [6]. From this, we build our noncrossing parking function model by labelling the blocks of the noncrossing partition of their image under w.

Example 4.1. Take $\pi := \{\{1^1, 4^1, 5^1\}^j, \{2^1, 3^1\}, \{6^j\}\}$, where the exponent j outside the set indicates $d-1$ copies of the partition with element's exponent increased by j modulo

d for $j = 1, \ldots, d-1$. The exponent j on a number within a block indicates that number in every exponent is an element of said block. We show $[w, \pi]$ in Figure 3 with

$$w = \begin{pmatrix} 1^1 & 2^1 & 3^1 & 4^1 & 5^1 & 6^1 \\ 2^2 & 3^3 & 1^1 & 5^2 & 6^2 & 4^2 \end{pmatrix}. \tag{4.1}$$

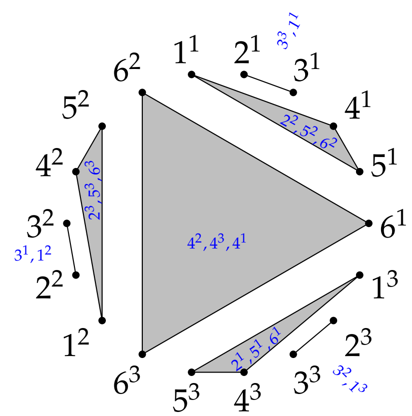


Figure 3: A $G(3,1,6)$-parking function example.

We conclude this section by constructing an explicit bijection between the combinatorial parking space and the algebraic via our combinatorial models. By our choice of hsop, the bijection shown gives an enumeration for the $G(d,1,n)$-noncrossing parking functions. In the next section, we discuss the k-$G(d,1,n)$-noncrossing parking functions. We begin with an initial model for $(\pi_1 \leq \cdots \leq \pi_k)$, simply k noncrossing partition models in the manner previously discussed. There is a map, ∇, due to D. Armstrong [1] that translates these k models into one. We then extend the procedure of B. Rhoades in type B, which relies on ∇, to get a singular model of our k-W-noncrossing parking function [20].

Example 4.2. Consider the following multichain. We show $\nabla\left([w, (\pi_1 \leq \pi_2)]\right)$ in Figure 4.

$$\{\{1^1, 2^1\}^j\} \leq \{\{1^1, 2^1, 3^1\}^j\} \text{ and } w = \begin{pmatrix} 1^1 & 2^1 & 3^1 \\ 2^1 & 1^2 & 3^2 \end{pmatrix}.$$

We conclude this section by giving an explicit bijection relying on the k-W-noncrossing parking function model. By our choice of hsop, the bijection shown gives an enumeration for the k-$G(d,1,n)$-noncrossing parking functions.

5 $G(d,d,n)$

The group $G(d,d,n)$ is the subgroup of $G(d,1,n)$ with the added condition that the nonzero entries in the matrix representation of any $w \in G(d,d,n)$ multiply to 1. Note $G(2,2,n) \cong W(D_n)$ and parallel to type B in the last section, we extend type D.

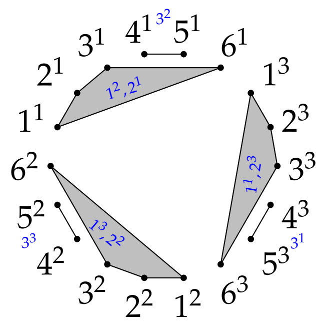


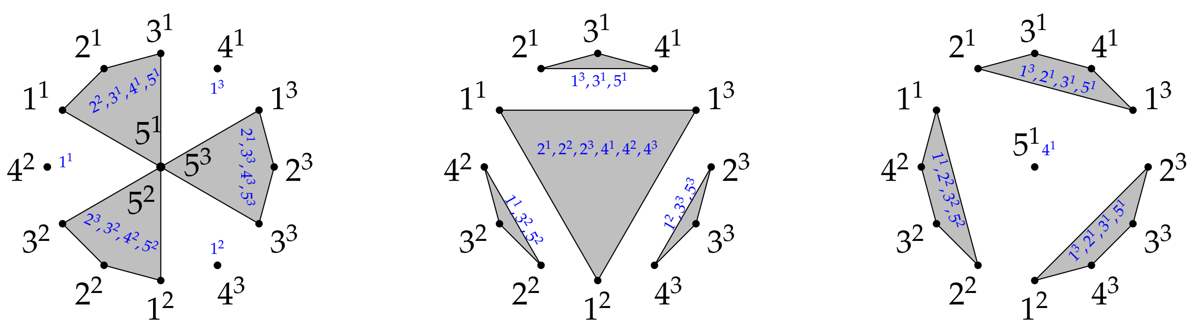
Figure 4: A single model of a 2-$G(3,1,6)$-noncrossing parking function.

D. Bessis and R. Corran demonstrate the $G(d,d,n)$-noncrossing partitions are those of $G(d,1,n)$ with the added condition that if there exists a zero block, B_0, then $n^j \in B_0$ [6]. As such, the models here are those of $G(d,1,n)$ with a central node added, similar to type D, representing the elements n^j, which we exclude in the presence of a zero block.

Example 5.1. Consider

$$w = \begin{pmatrix} 1^1 & 2^1 & 3^1 & 4^1 & 5^1 \\ 2^2 & 3^1 & 5^1 & 1^3 & 4^1 \end{pmatrix}.$$

We give three W-noncrossing parking functions models all labelled by w in Figure 5.

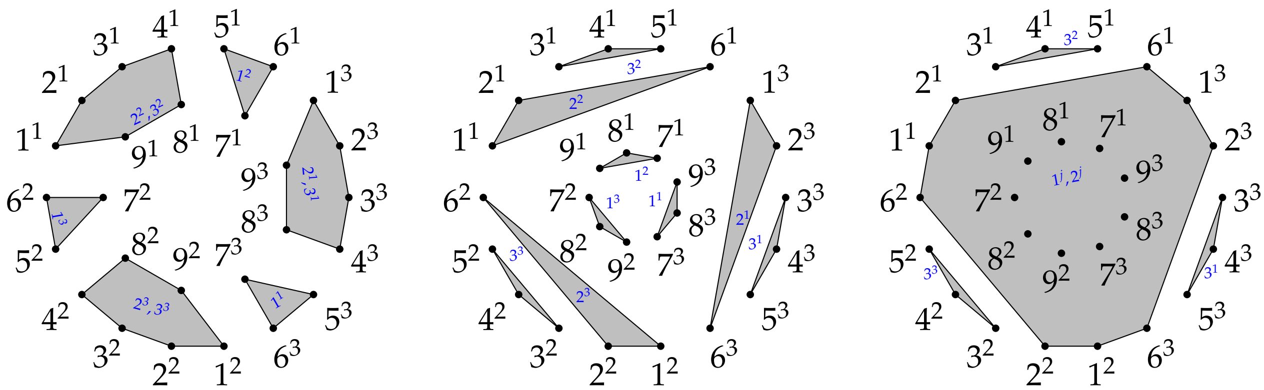


(a) (b) (c)

Figure 5: Three noncrossing parking functions in $G(3,3,5)$

We conclude this section by going through the explicit bijection. The proof splits itself into three cases given by the differing examples in Figure 5. By our choice of hsop, the bijection gives an enumeration for the $G(d,d,n)$-noncrossing parking functions.

In the next section, we discuss the Fuss case for $G(d,d,n)$. Similarly to our treatment of $G(d,1,n)$, we extend a map in type D due to B. Rhoades [20]. We obtain a singular model on an annulus. We conclude this section again by providing an explicit map which splits itself into the three cases shown in Figure 6. By our choice of hsop, the bijection shown gives an enumeration for the k-$G(d,d,n)$-noncrossing parking functions.



(a) (b) (c)

Figure 6: Three example models of three 3-$G(3,3,3)$-noncrossing partitions.

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