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Multitriangulations on ciliated surfaces

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Abstract. We generalize the definition of k-triangulations to arbitrary ciliated surfaces, and prove that the simplicial complex $\Delta_{\mathcal{C}_n,2}$ is pure and a weak pseudomanifold of dimension $2(n-1)$, where $\Delta_{\mathcal{C}_n,2}$ is the simplicial complex associated with 2-triangulations on the half-cylinder, which is an annulus with n marked points on one boundary. This result generalizes the work of Vincent Pilaud and Francisco Santos for polygons and resolves a conjecture of Mathias Lepoutre and Vincent Pilaud for $k=2$. To achieve this, we show that 2-triangulations on the half-cylinder decompose as complexes of star polygons, and that 2-triangulations on the half-cylinder are in bijection with 2-triangulations on the $4n$-gon invariant under rotation by $\pi/2$ radians. Building on work by Vincent Pilaud and Christian Stump, we also introduce chevron pipe dreams, a new combinatorial model that more naturally captures the symmetries of k-triangulations. In addition, we propose some conjectures on k-triangulations of general ciliated surfaces.

Keywords: multitriangulations, flip graphs, pipe dreams, subword complexes

1 Introduction

A *k-triangulation*, or *multitriangulation of order k*, on the convex n-gon is a maximal set of edges (namely, internal diagonals) such that no $k+1$ of them mutually intersect inside the polygon. See Figure 1 for an example of a 2-triangulation on the 12-gon. When $k=1$, a 1-triangulation corresponds to the usual notion of an ideal triangulation.

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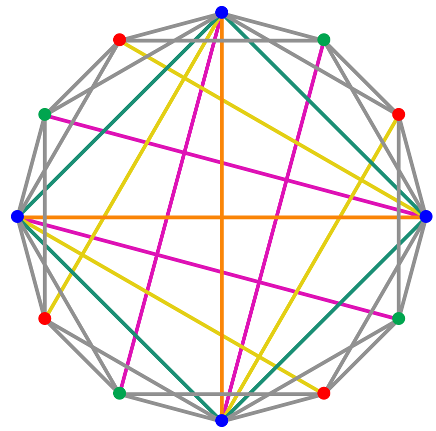


Figure 1: A 2-triangulation on the 12-gon invariant under rotation by $\pi/2$ radians.

Jakob Jonsson, Vincent Pilaud, Francisco Santos, Christian Stump, and many other authors have proved various structural results about k-triangulations on polygons, including: Each k-triangulation T on the n-gon can be regarded as a simplicial complex $\delta^T_{n,k}$, where each facet is a k-star [16]; All k-triangulations on the n-gon have the same number of edges, which is equal to $k(2n-2k-1)$ [2, 12, 6]; Every k-relevant edge e of a k-triangulation T can be "flipped" to a unique edge f such that $(T \setminus \{e\}) \cup \{f\}$ is a k-triangulation [12, 16]; There is a bijection between k-triangulations on the n-gon and reduced pipe dreams for the permutation $\pi_{n,k} = [1,\dots,k,n-k,n-k-1,\dots,k+1]$ [13, 18, 15]; The simplicial complex $\Delta_{n,k}$, whose vertices are the k-relevant edges of the n-gon and whose facets correspond to k-triangulations, is a vertex-decomposable sphere [9].

Furthermore, k-triangulations on polygons admit many natural interpretations, such as k-compatible split systems [5], pseudoline arrangements on the Möbius strip [13, 15], multi-cluster complexes of type A [3], and bases of the Pfaffian variety $\mathcal{P}f_k(n)$ of antisymmetric matrices of rank $\leq 2k$ [4].

In this paper, we generalize the notion of k-triangulations on polygons to k-triangulations on *ciliated surfaces*, or *bordered surfaces with marked points*.

Definition 1.1 (ciliated surface)**.** Let X be a connected 2-dimensional Riemann surface with boundary. Fix a finite set of marked points, denoted M, in the closure of X. The points in M do not need to be on the boundary of X. The pair (X, M) is called a *ciliated surface*.

We will use X to denote the ciliated surface (X, M) when there is no ambiguity.

Definition 1.2 (k-triangulations)**.** Let X be a ciliated surface, let $\overline{X}$ be its universal cover, and let $\pi : \overline{X} \to X$ be an associated covering map. A *(weak) k-triangulation* on X is a maximal subset of edges e so that the set of preimages $\pi^{-1}(e) \in \overline{X}$ does not contain any $(k+1)$-*crossings* (namely, no $k+1$ of them mutually intersect inside the polygon).

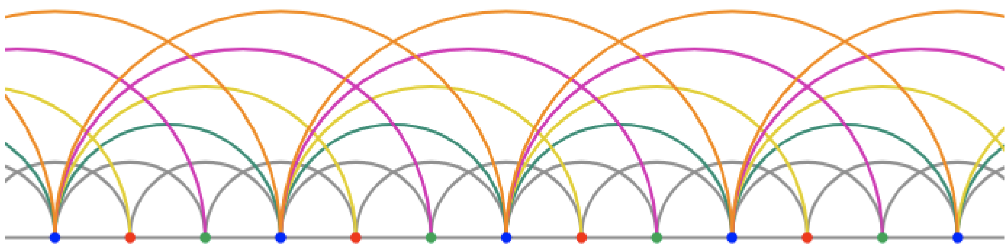


Figure 2: A 2-triangulation on the half-cylinder with 3 marked points, lifted to the universal cover.

Figure 2 is an example of a 2-triangulation on the half-cylinder with 3 marked points. Our main result is the following.

Theorem 1.3. Let $\mathcal{C}_n$ denote the annulus with n marked points on one of its boundary components. The flip graph of 2-triangulations on $\mathcal{C}_n$ is regular. Equivalently, the associated simplicial complex $\Delta_{\mathcal{C}_n,2}$ is pure and a weak pseudomanifold.

1.1 Amending a previous definition of k-triangulations on surfaces.

For readers familiar with the k-triangulation literature, we summarize in the next six paragraphs some previous attempts to define k-triangulations on ciliated surfaces and explain the advantages of our definition. In 2019, Mathias Lepoutre and Vincent Pilaud proposed generalizing k-triangulations on polygons to arbitrary surfaces by studying k-triangulations on their universal covers [11, Part III]. Their k-triangulations are "the projections of **periodic maximal** $(k+1)$-crossing-free edge sets on the universal cover" [11, Definition 1.1.4 and Definition 1.2.2 of Part III]. We call such objects *strong k-triangulations*.

In contrast, our k-triangulations are "the projections of **maximal periodic** $(k+1)$-crossing-free edge sets on the universal cover." We call these objects *weak k-triangulations*.[1]

In [11, Remark 12 of Part III], it is conjectured that on any surface, the notions of strong k-triangulation and weak k-triangulation are equivalent. However, this conjecture is incorrect. In fact, on many naturally arising surfaces, weak k-triangulations exist and strong k-triangulations do not. We decided to study weak k-triangulations because they have a richer structure than strong k-triangulations in the following two ways.

First, weak 1-triangulations always coincide with *ideal triangulations* (maximal collection of disjoint non-null-homotopic arcs that are pairwise non-homotopic) whereas strong 1-triangulations do not. For example, $\mathcal{C}_n$ has ideal triangulations but no strong 1-triangulations.

[1] We conceived of this definition before discovering [11].

Remark 1.4. There is a problem with the proof of [11, Proposition 1.3.3 of Part III], namely that there are no strong k-triangulations on the half-cylinder for any k. (Though [11, Figure 1.11 of Part III] seems to suggest otherwise, the figure does not correctly depict a strong k-triangulation, as a single edge connecting u_0 and u_4 can be added without creating a 4-crossing).

Second, weak k-triangulations lend themselves to an underexplored analog of [16, Theorem 4.1], which we call the Star Decomposition Theorem for polygons. While the Star Decomposition Theorem for the strong k-triangulations on any finite-type surface follows from the corresponding result for infinite tidy polygons (see [11, Section 1.2 of Part III]), the Star Decomposition Conjecture for the weak k-triangulations is far more subtle. In this paper, we prove the Star Decomposition Conjecture for weak k-triangulations on the half-cylinder when $k = 2$ (Theorem 3.5), and we regard this result as the main technical contribution of our work.

The definition of k-triangulations implicitly used by Pilaud is "the projections of a periodic maximal [$(k+1)$-crossing-free sets on the infinite polygon that contain no edge which, when extended periodically, would create a $(k+1)$-self-crossing on the universal cover]" [14]. We further prove that on the half-cylinder, the weak k-triangulations are precisely the k-triangulations they implicitly used when $k = 2$ (Corollary 3.6).

Convention 1.5. In the rest of the paper, k-triangulations refer to weak k-triangulations.

This paper is organized as follows. In Section 2, we define necessary terms including k-triangulations, k-stars, angles, and k-relevant edges. In Section 3, we prove a number of properties about multitriangulations on the half-cylinder. A main result is Theorem 3.5, which serves as a generalization of the Star Decomposition Theorem for n-gons ([16, Theorem 4.1]). In Section 4, we give a bijection between 2-triangulations on the half-cylinder with n marked points and n-periodic k-triangulations on the $2kn$-gon when $k = 2$. In Section 5, we define *chevron pipe dreams*, a symmetric analogue of reduced pipe dreams [1], which captures central symmetries of k-triangulations through mirror symmetry and periodicity conditions. Finally, in Section 6, we prove our main theorem (Theorem 1.3) using the aforementioned results.

This is an extended abstract summarizing our work, and the reader is referred to [17] for a complete version.

2 Background

Generalizing the k-triangulations on a convex n-gon, we define k-triangulations on the half-cylinder with n marked points. Let $\mathcal{C}_n$ denote the annulus with n marked points $\alpha_{[0]}, \ldots, \alpha_{[n-1]}$ on one of its boundary components, where subscripts are congruence classes modulo n. We refer to $\mathcal{C}_n$ as a *half-cylinder* or an $(n, 0)$-*annulus*.

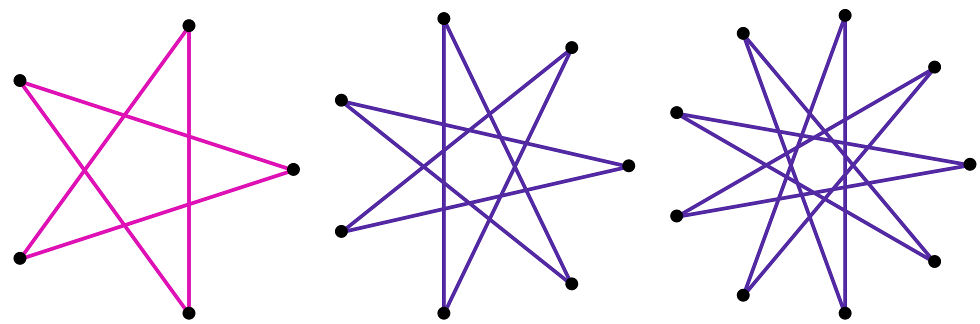


Figure 3: A 2-star in a pentagon, a 3-star in a heptagon, and a 4-star in a nonagon.

The universal cover of $\mathcal{C}_n$, denoted $\overline{\mathcal{C}}_n$, has vertices $\{\alpha_i \mid i \in \mathbb{Z}\}$. Taking any vertex $v = \alpha_{[i]} \in \mathcal{C}_n$, we define $v + j :- \alpha_{[i+j]}$ for any integer j. An *edge* e on $\overline{\mathcal{C}}_n$ is defined by the isotopy class of topological curves connecting u, v and is denoted by $[u, v]$ or $[v, u]$. Similarly, we define $e + n :- [u + n, v + n]$. The covering map $\pi : \overline{\mathcal{C}}_n \to \mathcal{C}_n$, applied to vertices and edges is given by $\pi(\alpha_{[i]}) = \alpha_i$ and $\pi([\alpha_{[i]}, \alpha_{[j]}]) = [\alpha_i, \alpha_j]$. We say an edge e on $\overline{\mathcal{C}}_n$ is a *translation* of an edge f if $e = f + \ell n$ for some $\ell \in \mathbb{Z}$, or equivalently if $\pi(e) = \pi(f)$. Typically, we do not refer to vertices of $\mathcal{C}_n$ or $\overline{\mathcal{C}}_n$ using these subscripts determined by their position.

There is a natural order of edges on $\overline{\mathcal{C}}_n$ where $u > v$ if and only if $u = v + \ell$ for some $\ell \in \mathbb{Z}^+$ induced from a counterclockwise orientation on $\overline{\mathcal{C}}_n$. When considering a subset of vertices of $\overline{\mathcal{C}}_n$, we additionally have a *cyclic order* $\prec$ given by the counterclockwise cyclic order on the first polygon, as defined in [11]. On $\overline{\mathcal{C}}_n$, two edges e_1 and e_2 are said to *intersect* if all representatives of the isotopy classes of e_1 and e_2 intersect, or equivalently, $a_1 \prec b_1 \prec a_2 \prec b_2$ for some labeling of vertices $e_1 = [a_1, b_1], e_2 = [a_2, b_2]$. A *k-crossing* on $\overline{\mathcal{C}}_n$ is a set of edges $E = \{e_1, e_2, \dots, e_k\}$ that pairwise intersect.

Take a subset $\{z_0, \dots, z_{2k}\}$ of vertices on $\overline{\mathcal{C}}_n$ ordered such that

$$z_0 \prec z_1 \prec \cdots \prec z_{2k}.$$

A *k-star polygon*, or a *k-star* S on this subset, as defined in [16] and [11], contains all vertices z_i and edges $[z_i, z_{i+k}]$ on $\overline{\mathcal{C}}_n$ with subscripts reduced modulo n. Examples of k-stars are given in Figure 3. We use these definitions of k-crossings and k-stars on $\overline{\mathcal{C}}_n$ to define analogously k-crossings and k-stars on $\mathcal{C}_n$.

As a generalization of the k-stars and the k-triangulations on n-gons, we introduce the following definitions.

Definition 2.1. A *k-star* on $\mathcal{C}_n$ is the projection $\pi(S)$ of a k-star S on $\overline{\mathcal{C}}_n$.

Definition 2.2. A *k-triangulation* T on $\mathcal{C}_n$ is a maximal set of edges such that the lifting $\pi^{-1}(T)$ is $(k+1)$-crossing-free.

We additionally define a *k-triangulation* $\overline{T}$ on $\overline{\mathcal{C}}_n$ to be a maximal set of *n-periodic* edges on $\overline{\mathcal{C}}_n$, that is, $v \in \overline{T} \iff v + n \in \overline{T}$. Equivalently, $\pi(\overline{T})$ is a k-triangulation on $\mathcal{C}_n$.

For vertices $u, v \in \overline{\mathcal{C}}_n$, we say $v - u = j$ if $u + j = v$. The *length* of an edge $[u, v] \in \overline{\mathcal{C}}_n$ is defined by $|v - u|$. We also say $\pi([v, u])$ has *length* $|v - u|$.

An edge $[u, v]$ of $\overline{\mathcal{C}}_n$ can appear in a $(k+1)$-crossing only if $|v - u| > k$. Thus, each k-triangulation T (respectively, $\overline{T}$) contains all edges of length $\leq k$. For simplicity, we adopt the following convention.

Convention 2.3. In this paper, when discussing a k-triangulation T on $\mathcal{C}_n$ (respectively, $\overline{T}$ on $\overline{\mathcal{C}}_n$), we exclude all the edges of length $< k$.

As we will show in Lemma 3.1 later, a k-triangulation T (respectively, $\overline{T}$) contains no edges of length $> kn$. These observations inform us to introduce the following definitions, analogous to those in [11] and [16]. An edge $[u, v] \in \overline{T}$ is a *k-relevant* edge if $k < |v - u| \leq kn$. An edge $[u, v] \in \overline{T}$ is a *k-boundary* edge if $|v - u| = k$. An *angle* of a k-triangulation $\overline{T}$ on $\overline{\mathcal{C}}_n$ is given by a pair of edges $[u, v], [v, w] \in \overline{T}$ such that $u \prec v \prec w$ and there exists no edge $[v, w_0]$ in $\overline{T}$ with $w_0 \prec u \prec v \prec w$. Such an angle is denoted by $\angle(u, v, w)$. An angle $\angle(u, v, w)$ of $\overline{T}$ is called *k-relevant* if at least one of $[u, v]$ and $[v, w]$ is k-relevant with length $< kn$. Given an angle $\angle(u, v, w)$ of $\overline{T}$, an edge $[v, w_0]$ with $w_0 \prec u \prec v \prec w$ is called an *angle bisector* of $\angle(u, v, w)$; or equivalently, we say $[v, w_0]$ *bisects* the angle $\angle(u, v, w)$. An edge $[v, w_0]$ might also be an angle bisector of more than one angle, or an angle bisector of an angle contained in a collection of edges; in these cases we abuse terminology and just call $[v, w_0]$ an angle bisector.

3 Multitriangulations on the half-cylinder

In this section, we prove a series of properties about multitriangulations on the half-cylinder. The main result is Theorem 3.5, which generalizes the Star Decomposition Theorem for n-gons ([16, Theorem 4.1]) to 2-triangulations on the half-cylinder with n marked points.

In the following lemmas, let T denote a k-triangulation on $\mathcal{C}_n$, which lifts to $\pi^{-1}(T) = \overline{T}$ on $\overline{\mathcal{C}}_n$.

Lemma 3.1. A k-triangulation T of $\mathcal{C}_n$ contains no edge of length $\ell > kn$, and exactly one edge of length kn.

In the remainder of this section, we consider only 2-triangulations on $\mathcal{C}_n$.

Lemma 3.2. Let $E = \{[a_1, b_1], [a_2, b_2], [a_3, b_3]\}$ be a 3-crossing labeled such that

$$a_1 < a_2 < a_3 < b_1 < b_2 < b_3.$$

Suppose that for $i = 1$ or $i = 2$, the edge $[a_{i+1}, b_{i+1}]$ is a translation of $[a_i, b_i]$. Then $\pi^{-1}(\pi(E))$ contains a 3-crossing that includes exactly one translation of $[a_i, b_i]$.

In the following lemmas, fix a 2-triangulation T of $\mathcal{C}_n$ which lifts to a 2-triangulation $\pi^{-1}(T) = \overline{T}$ on $\overline{\mathcal{C}}_n$ and a 2-relevant angle $\angle(u,v,w)$ of $\overline{T}$. We say an edge *intersects* $\angle(u,v,w)$ if it intersects both $[u,v]$ and $[v,w]$. The following definitions are consistent with [16] and [11]. We say an edge $e = [a,b]$ is *v-farther* than $f = [c,d]$ if $u \prec a \preceq c \prec v \prec d \preceq b \prec w$ and $e \neq f$. We say e is *v-maximal* if there does not exist any edge intersecting $\angle(u,v,w)$ that is v-farther than e. Let $e = [a,b]$ be the v-maximal edge intersecting $\angle(u,v,w)$ with order $u \prec a \prec v \prec b \prec w$. We work in the next two lemmas towards showing that $[u,b]$ and $[a,w]$ are edges in $\overline{T}$.

Lemma 3.3. The edges $[u,b]$ and $[a,w]$ have length at most $2n$.

Lemma 3.4. Assume that there exists a 2-crossing $F = \{f_1, f_2\} \subset \overline{T} \cup \pi^{-1}(\pi([u,b]))$ such that $F \cup \{[u,b]\}$ is a 3-crossing and F contains a translation of $[u,b]$. Then there exists a 2-crossing $F' \subset \overline{T}$ intersecting $[u,b]$ such that $F' \cup \{[u,b]\}$ is a 3-crossing.

Using the previous lemmas, we could prove the Star Decomposition Theorem for the half-cylinder for $k = 2$.

Theorem 3.5 (Star Decomposition Theorem)**.** Let T be a 2-triangulation on $\mathcal{C}_n$ which lifts to $\overline{T}$ on $\overline{\mathcal{C}}_n$, then any 2-relevant angle $\angle(u,v,w)$ in $\overline{T}$ belongs to a unique 2-star in $\overline{T}$.

We also have the following corollary that relates the k-triangulations on the half-cylinder and the k-triangulations on its universal cover.

Corollary 3.6 (Maximal Lifting Theorem)**.** Given a 2-triangulation T on $\mathcal{C}_n$ which lifts to $\overline{T}$ on $\overline{\mathcal{C}}_n$, let e be an edge on $\overline{\mathcal{C}}_n$ of length $\leq 2n$ that is not contained in $\overline{T}$. Then $\overline{T} \cup \{e\}$ contains a 3-crossing.

We conjecture that Theorem 3.5 and Corollary 3.6 on the half-cylinder can be generalized for arbitrary k [17, Conjectures 3.8, 3.9]. Furthermore, we conjecture that analogous results hold for the k-triangulations on all finite-type ciliated surfaces as well. Following [11, Section 1.2 of Part III], we view any bordered surface X as the quotient of the hyperbolic disc $\mathbb{D}$ by a Fuchsian group Γ, where $\mathbb{D} = \overline{X}$ is the universal cover of X. In this setting, we likewise define the *length* of an edge on $\mathbb{D}$ (note that this length may not always be finite). Consistent with Convention 2.3, we exclude all edges of length $< k$ when discussing a k-triangulation on $\overline{X}$ (or X).

Conjecture 3.7 (Star Decomposition Conjecture)**.** Given a finite-type ciliated surface X and its universal cover $\pi : \overline{X} \to X$. Let T be a k-triangulation on X which lifts to $\overline{T}$ on $\overline{X}$. Let $\angle(u,v,w)$ be an angle in $\overline{T}$. Suppose that $[u,v]$ and $[v,w]$ are not the preimage of the same edge $e \in T$ under π, then $\angle(u,v,w)$ belongs to a unique k-star contained in $\overline{T}$.

Conjecture 3.8 (Maximal Lifting Conjecture)**.** [cf. [11, Remark 12 of Part III] and Remark 1.4] Given a finite-type ciliated surface X and its universal cover $\pi : \overline{X} \to X$. Let T be a k-triangulation on X which lifts to $\overline{T}$ on $\overline{X}$. Let e be an edge on $\overline{X}$ such that e is not contained in $\overline{T}$ and $\pi^{-1}(\pi(e))$ does not contain a $(k+1)$-crossing. Then, $\overline{T} \cup \{e\}$ contains a $(k+1)$-crossing.

4 Bijection: 2-triangulations on the half-cylinder and n-periodic 2-triangulations on the $4n$-gon

Definition 4.1. An *n-periodic* k-triangulation on a $2kn$-gon is a k-triangulation on the $2kn$-gon that is invariant under rotation by $\frac{2\pi}{2k} = \frac{\pi}{k}$ radians.

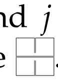
An example of a 3-periodic 2-triangulation on the 12-gon was given in Figure 1.

Theorem 4.2. There is a canonical bijection between 2-triangulations on $\mathcal{C}_n$ and n-periodic 2-triangulations on the $4n$-gon.

As shown in [16], every k-triangulation on the $2kn$-gon has exactly $2kn - 2k$ k-stars, $k(2kn - 2k - 1)$ k-relevant edges, and $k(4kn - 2k - 1)$ edges. The following counts then follow almost directly from the bijection in Theorem 4.2.

Corollary 4.3. Any 2-triangulation on $\mathcal{C}_n$ contains exactly $n - 1$ 2-stars, $2(n - 1)$ 2-relevant edges, and $2(2n - 1)$ total edges.

5 Chevron pipe dreams

In the 2010s, Vincent Pilaud and Christian Stump independently established a connection between pipe dreams and multitriangulations [13, Section 4.1.4], [18, Theorem 2.1] [15, Section 7]. Starting from a k-triangulation on the n-gon, the following is the construction of the pipe dream in the $(n-1) \times (n-1)$ staircase polyomino shape given by them: in the NW staircase of an $(n-1) \times (n-1)$ grid as follows: Label the rows $n, n-1, \dots, 2$ from top to bottom and the columns $1, 2, \dots, n-1$ from left to right. In each box (i, j) where there is an edge between vertices i and j in the k-triangulation, place a bump tile  . In all other boxes, place a crossing tile . This process yields a natural bijection between k-triangulations on the n-gon and reduced pipe dreams for the permutation $\pi_{n,k} := [1, \dots, k, n-k, n-k-1, \dots, k+1] \in \mathfrak{S}_{n-k}$ [18]. For clarity, we call them *staircase pipe dreams*. Further, reduced staircase pipe dreams for the permutation $\pi_{n,k}$ are in natural bijection with subsets P of the word $Q = c^k \mathrm{w}_\circ(c)$ such that $Q \setminus P$ is a reduced expression for w_0 [3].

In [3], an analogous result is shown in type B. In particular, k-triangulations of the $2m$-gon invariant under rotation by 180° are in bijection with subsets P of the word $Q = c^m$ such that $Q \setminus P$ is a reduced expression for $w_0 \in B_{m-k.}$ In [17], we construct *chevron pipe dreams* which arise naturally from the bijection between type B k-triangulations and these subword complexes. In particular, chevron pipe dreams more naturally capture the possible symmetries of k-triangulations.

Lemma 5.1. The chevron pipe dream $\mathrm{Chev}_{2m,k}$ corresponding to a k-triangulation of the $2m$-gon is reduced, and moreover, each pair of pipes crosses exactly once.

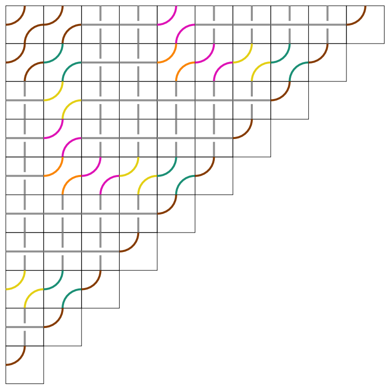


Figure 4: The staircase pipe dream corresponding to the 3-periodic 2-triangulation on a 12-gon in Figure 1.

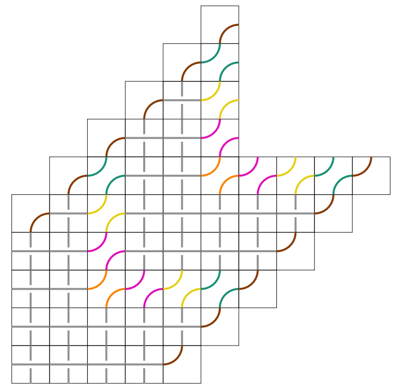


Figure 5: The chevron pipe dream which results from applying the cutting and gluing operations as outlined in [17] to the staircase pipe dream in Section 5.

With Theorem 4.2 as an intermediate step, we use chevron pipe dreams to show that every edge of a 2-triangulation on $\mathcal{C}_n$ can be flipped uniquely, leading to results which are discussed further in Section 6.

6 The flip graph of rotationally symmetric 2-triangulations

Let $\tilde{E}_{2kn,n}$ denote the set of equivalence classes of edges of the $2kn$-gon of the convex n-gon after identification via rotation by $\frac{\pi}{k}$. Each element of $\tilde{E}_{2kn,n}$ is an n-periodic edge, namely, a set $\tilde{e}$ containing an edge e and its rotational copies by integer multiples of $\frac{\pi}{k}$. We identify an n-periodic k-triangulation on the $2kn$-gon T with a subset $\tilde{T} \subset \tilde{E}_{2kn,n}$. When $k = 2$, elements of $\tilde{E}_{4n,n}$ correspond to edges of $\mathcal{C}_n$ as described in Theorem 4.2, and $\tilde{T}$ corresponds to a 2-triangulation on $\mathcal{C}_n$.

In the following, let T be an n-periodic 2-triangulation on the $4n$-gon, and let $\tilde{e} \in \tilde{T}$ be a set of edges containing a 2-relevant edge e and its rotational copies, which we call a 2-*relevant n-periodic edge*.

Proposition 6.1. For each 2-relevant n-periodic edge $\tilde{e} \in \tilde{T}$, there exists a unique 2-relevant n-periodic edge $\tilde{f} \notin \tilde{T}$ such that replacing $\tilde{e}$ with $\tilde{f}$ yields another n-periodic 2-triangulation $(\tilde{T} \setminus \tilde{e}) \cup \tilde{f}$.

We say that we obtain the n-periodic 2-triangulation on the $4n$-gon $(T \setminus \tilde{e}) \cup \tilde{f}$ from T by flipping the n-periodic edge e. An example of a 3-periodic 2-triangulation of the 12-gon T and the result of flipping a 3-periodic edge $\tilde{e}$ are illustrated in Figure 6. Let

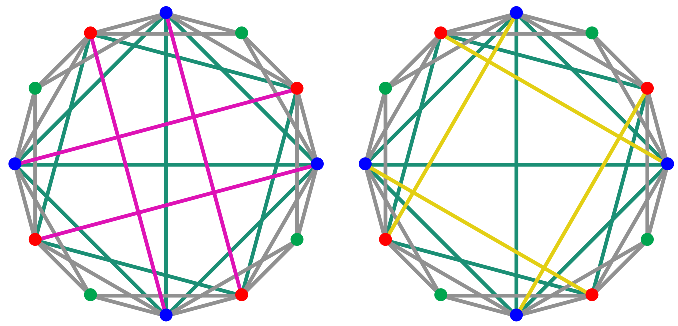


Figure 6: (Left) The 3-periodic 2-triangulation T of the 12-gon. (Right) The 2-triangulation $(T \setminus \tilde{e}) \cup \tilde{f}$.

$G^n_{4n,2}$ denote the *flip graph* of n-periodic 2-triangulations on the $4n$-gon, the graph whose vertices are n-periodic 2-triangulations on the $4n$-gon and whose edges are pairs related by a flip. Equivalently, $G^n_{4n,2}$ is the graph of flips of 2-triangulations on the half-cylinder $\mathcal{C}_n$.

Theorem 6.2. The graph $G^n_{4n,2}$ is regular of degree $2(n-1)$.

Proof. The regularity of $G^n_{4n,2}$ follows from the count of 2-relevant edges in Corollary 4.3, in addition to the fact that every 2-relevant n-periodic edge can be flipped and this flip is unique, as shown in Proposition 6.1. □

We additionally define a simplicial complex $\Delta_{\mathcal{C}_n,k}$ whose vertices are k-relevant edges of $\mathcal{C}_n$ and whose facets correspond to k-triangulations on $\mathcal{C}_n$. Corollary 3.6 shows that the faces of $\Delta_{\mathcal{C}_n,2}$ are sets of k-relevant edges of $\mathcal{C}_n$ such that the lifting to $\overline{\mathcal{C}}_n$ is $(k+1)$-crossing-free. The results of this paper additionally imply the following:

Corollary 6.3. The simplicial complex $\Delta_{\mathcal{C}_n,2}$ is pure of dimension $2(n-1)$; moreover, it is a weak pseudomanifold (without boundary).

Proof. Purity of dimension $2(n-1)$ follows from the count of 2-relevant edges in Corollary 4.3. As every k-relevant edge can be uniquely flipped as shown in Proposition 6.1, each codimension 1 face lies in exactly 2 distinct facets, and thus $\Delta_{\mathcal{C}_n,2}$ is a weak pseudomanifold without boundary. □

Proof of Theorem 1.3. This follows directly from Theorem 6.2 and Corollary 6.3. □

In [18, Theorem 2.1], the simplicial complex $\Delta_{n,k}$, whose vertices are diagonals of the n-gon of length greater than k and whose facets correspond to k-triangulations, is shown to be naturally isomorphic to a subword complex in type A_{n-k-1}.

In [3, Theorem 2.10], an analogous result is shown in type B. In particular, the B_{m-k} subword complex considered has vertices corresponding to a diagonal and its image under rotation by 180°, and facets corresponding to k-triangulations of the $2m$-gon invariant under rotation by 180°.

We give an analogous result for the simplicial complex $\Delta_{\mathcal{C}_n,2}$ below. Let Q be a word in the generators S of a finite Coxeter group W. For an element $\pi \in W$, the subword complex $\mathcal{SC}(Q, \pi)$, which was introduced in [10], has faces given by subwords P of Q for which the complement $Q \setminus P$ contains a reduced expression of π.

Proposition 6.4. The simplicial complex $\Delta_{\mathcal{C}_n,2}$ is isomorphic to a union of subword complexes $\bigcup_{w^2=w_0} \mathcal{SC}(c^n, w)$, where w_0 is the longest element of the Coxeter group B_{2n-2} and c is the word $s_1, s_2 \cdots, s_{2n-2}$.

Conjecture 6.5. The simplicial complex $\Delta_{\mathcal{C}_n,k}$ is isomorphic to a union of subword complexes $\bigcup_{w\in\mathcal{A}_k} \mathcal{SC}(c^n, w)$, where w_0 is the longest element of the Coxeter group B_{kn-k}, c is the word $s_1, s_2, \ldots, s_{nk-k}$, and $\mathcal{A}_k := \{w \mid w^k = w_0, \ell(w) = \ell(w_0)/k\} \subset B_{kn-k}$.

Note that when $k \geq 3$, for w satisfying $w^k = w_0$, it is not automatically true that $\ell(w) = \ell(w_0)/k$.

Conjecture 6.6. The simplicial complex $\Delta_{\mathcal{C}_n,k}$ has $\binom{2(n-1)}{n-1}^k$ facets.

In [8, Chapter 6], a specific class of unions of subword complexes, called *glued subword complexes*, are shown to be vertex-decomposable spheres. As we have shown that $\Delta_{\mathcal{C}_n,2}$ is a weak pseudomanifold – analogous to the result for n-gons [9, Theorem 1], we make the following conjecture.

Conjecture 6.7. The simplicial complex $\Delta_{\mathcal{C}_n,k}$ is a piecewise linear sphere.

By [7, Proposition 2.3], the only ciliated surfaces whose multitriangulation complexes are finite (that is, contain finitely many simplices) are polygons and half-cylinders. Hence, an affirmative answer to Conjecture 6.7, together with [9, Theorem 1], would imply that all finite multitriangulation complexes are piecewise linear spheres.

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