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Electroid varieties and the Lagrangian Grassmannian

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Abstract. Recent work of Lam, Bychkov–Gorbounov–Kazakov–Talalaev, and Chepuri–George–Speyer gave a stratification of the totally nonnegative Lagrangian Grassmannian into electroid cells parameterized by cactus networks, paralleling Postnikov's stratification of the totally nonnegative Grassmannian by positroid cells. Electroid varieties arise as an algebro-geometric extension of electroid cells. The combinatorics of these varieties was studied by Lam in 2018. We build on this work and study the geometric properties of electroid varieties. In analogy to results of Knutson, Lam, and Speyer on positroid varieties, we show that electroid varieties are reduced, irreducible, regular in codimension one, compatibly Frobenius split, and form a stratification.

Keywords: Lagrangian Grassmannian, electrical networks, electroid varieties, positroid varieties

1 Introduction

The study of *stratifications* of the Grassmannian dates back to the work of Schubert in the 19th century. His work laid the foundation for the *Bruhat stratification* of the Grassmannian, first introduced by Bruhat for classical groups and later generalized by Chevalley. The Bruhat stratification inspired numerous other stratifications: the *Richardson stratification* [7, 13] and, most recently, *the positroid stratification* [9, 12]. By stratification, we mean a partition of the space into a disjoint union of open strata such that the closure of an open stratum is the union of open strata. The *GGMS decomposition* [5] is a famous failed attempt to stratify the Grassmannian.

Instead of considering the Grassmannian as an algebraic variety, one can instead consider its real points as a smooth real manifold. Inside the real Grassmannian manifold, we define the *totally nonnegative part* as the semi-algebraic subset where all Plücker coordinates are positive. This gives a closed manifold, on which one can also consider *stratifications*, where closures are taken in Euclidean topology. In 2006, Postnikov stratified the totally nonnegative Grassmannian into positroid cells parameterized by plabic

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graphs [12]. Later in 2011, it was extended by Knutson, Lam, and Speyer [9] to a stratification of the Grassmannian variety into positroid varieties. Positroid varieties unite multiple perspectives: they are Zariski closures of positroid cells, projections of Richardson varieties, and intersections of cyclically permuted Schubert varieties [9]. Moreover, they have particularly nice algebro-geometric properties; they are reduced, irreducible, normal, Cohen-Macaulay, and compatibly Frobenius split [9].

It is here that the story of positroid varieties intersects with another: the story of planar electrical networks developed by Curtis–Ingerman–Morrow [4] and Colin de Verdière–Gitler–Vertigan [15]. In 2016, Lam compactified the space of electrical networks, and in-so-doing, formalized its conjectured parallels to the totally nonnegative Grassmannian [10]. Independent work by Chepuri–George–Speyer and Bychkov–Gorbounov–Kazakov–Talalaev later showed that the compactified space is an isotropic Grassmannian abstractly isomorphic to the Lagrangian Grassmannian [2, 3]. Together with work of Lam, this yields a stratification of the totally nonnegative part of the real points of the Lagrangian Grassmannian into electroid cells parameterized by cactus networks. A natural hope is that this stratification of the totally nonnegative part can be extended to a well-behaved stratification of the Lagrangian Grassmannian variety, parallel to the positroid story [9]. In [10], Lam defined electroid varieties as the Lagrangian Grassmannian intersected with certain positroid varieties and understood them combinatorially and set-theoretically. However, people lack fundamental understanding of the algebraic geometry of electroid varieties. For example, it was unknown if electroid varieties are Zariski closures of electroid cells.

We build on the rich story and study the combinatorial algebraic geometry of electroid varieties. In analogy to the work of Knutson, Lam, and Speyer in [9] on positroid varieties, we prove that electroid varieties are reduced, irreducible, regular in codimension one, compatibly Frobenius split, and form a stratification. Our results are stated at the end of Section 2.

2 Background

2.1 Electrical networks and cactus networks

Consider a planar graph Γ embedded in a disc $\mathbb{D}$, with n vertices $v_1, v_2, \dots, v_n$ arranged clockwise on the boundary. We think of each edge as an electrical conductor with some fixed resistance in $\mathbb{R}_{>0}$. Imagine putting voltage V_i at boundary vertex v_i and measuring the resulting current J_i flowing out of vertex v_i. The map from $(V_1, V_2, \dots, V_n)$ to $(J_1, J_2, \cdots J_n)$ is given by an $n \times n$ matrix L called the *response matrix*. The response matrix is symmetric, with rows and columns summing to 0. Two electrical networks are said to be equivalent if their response matrices coincide.

Figure 1: The electrical networks with response matrix $L = \begin{bmatrix} -c & c \\ c & -c \end{bmatrix}$ for $c > 0$ and $c = 0$.

Example 1. *For $n = 2$, all response matrices of electrical networks are of the form $L = \begin{bmatrix} -c & c \\ c & -c \end{bmatrix}$. When $c > 0$, L can be realized by the network with a single resistor from v_1 to v_2 of resistance c^{-1}. When $c = 0$, L can be realized by a network with two isolated vertices. See Figure 1.*

From Example 1, we see that the space of possible 2×2 response matrices (or electrical networks up to equivalence) is isomorphic to $\mathbb{R}_{\geq 0}$. Lam [10] found a natural way to compactify the space of response matrices, by introducing cactus networks. Intuitively, a *cactus network* is an electrical network in which we allow subsets of boundary vertices to be “shorted" by wires of infinite conductance. Thus, the space of response matrices for $n = 2$ cactus networks is $\mathbb{R}_{\geq 0} \bigcup \{\infty\}$, homeomorphic to a compact interval. Let E_n denote the space of electrical equivalence classes of (or equivalently, response matrices of) cactus networks with n boundary vertices.

A *pairing* τ on $[2n]$ is a set partition of $[2n]$ into n sets of size 2. Given an electrical network Γ, there is a natural way to associate a pairing $\tau(\Gamma)$. One first constructs the *medial graph* $G(\Gamma)$, which depends only on the underlying unweighted graph of Γ. Place the vertices $1, 2, \ldots 2n$ around the boundary of the disk so that vertex v_i is between $2i - 1$ and $2i$. Then, construct a vertex w_e for each edge e. The vertices w_e and $w_{e'}$ are adjacent if and only if e and e' are incident and share a face. Moreover, if e is incident to some boundary vertex v_i, then w_e is adjacent to both $2i - 1$ and $2i$. Lastly, if v_i is an isolated vertex, we connect $2i - 1$ and $2i$ in the medial graph.

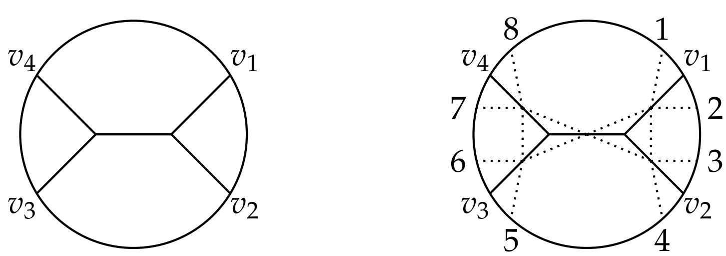

Figure 2: An electrical network (edge weights omitted) and its medial graph. Its medial pairing is $(1, 4), (2, 6), (3, 7), (5, 8)$.

From the medial graph of Γ, we construct the *medial pairing* $\tau(\Gamma)$ on $[2n]$. For each boundary vertex i, a path can be formed by starting at i and proceeding straight through

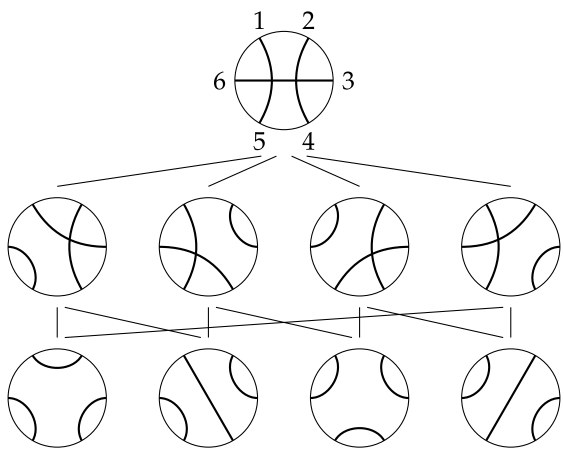

Figure 3: The lower order ideal of pairings on $[6]$ below $\tau = (1,5),(2,4),(3,6)$.

each degree 4 intersection until another boundary vertex j is reached. This path is referred to as a *strand* and is denoted by (i,j) or $\tau(i) = j$. This procedure naturally extends to cactus networks and is invariant under electrical equivalence [4, 10].

Definition 2 (Electroid Cell). *The electroid cell E_τ is the set of electrical equivalence classes of cactus networks $[\Gamma] \in E_n$ with the same medial pairing $\tau(\Gamma) = \tau$.*

We say that the medial graph is *critical* if no strand is self-intersecting and no two strands cross more than once. Let the *crossing number* $c(\tau)$ denote the number of strand crossings in a critical medial graph with medial pairing τ. A network Γ is *critical* if its medial graph is critical, or equivalently, if it has $c(\tau(\Gamma))$ edges. Any pair of crossing strands (a,c) and (b,d) with $a < b < c < d$ in cyclic order can be resolved in two ways. Namely, we can have $(a,b),(c,d)$ or $(a,d),(b,c)$. All pairings on $[2n]$ form a natural poset graded by the crossing number $c(\tau)$, where $\tau < \tau'$ if τ can be obtained from τ' by sequentially resolving crossings. Figure 3 shows part of the poset for $n = 3$.

Theorem 3. *[10] The space of electrical equivalence classes of cactus networks is the disjoint union $E_n = \bigsqcup E_\tau$ of electroid cells indexed by pairings on $[2n]$. The Euclidean closure of any electroid cell is the disjoint union $\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$.*

Each electroid cell is an open ball of dimension $c(\tau)$. There exists a critical cactus network Γ for any pairing τ. Letting the resistances vary on a fixed critical network gives a homeomorphism $\mathbb{R}_{>0}^{Edges(\Gamma)} \simeq E_{\tau(\Gamma)}$.

2.2 Embedding E_n into Grassmannian

Let V be a vector space and let $0 \leq k \leq \dim V$. The Grassmannian $Gr(k,V)$ is the space of k-dimensional subspaces of V. If our ground field $\mathbb{F}$ is understood from context, we'll write $Gr(k,m)$ as shorthand for $Gr(k,\mathbb{F}^m)$. We can represent a point of $Gr(k,m)$ as the row span of a full rank $k \times m$ matrix. In this representation, the $\binom{m}{k}$ maximal minors of this matrix form the homogeneous Plücker coordinates on $Gr(k,m)$.

Definition 4 (Grassmann necklace). *A* (k,m)-Grassmann necklace *is an m-tuple* $\mathcal{I} = (I_1, I_2, \ldots, I_m)$ *of subsets* $I_a \in \binom{[m]}{k}$ *such that*

1. $I_{a+1} = I_a$ *if* $a \notin I_a$.
2. $I_{a+1} = I_a \setminus \{a\} \cup \{a'\}$ *if* $a \in I_a$.

For instance, $\mathcal{I} = (124, 234, 134, 124)$ is a $(3,4)$-Grassmann necklace. Let $\leq_a$ be the cyclically shifted order on $[m]$ given by $a < a+1 < \cdots < a-1$ taken modulo m. Then, $\leq_a$ extends naturally to k-subsets of $[m]$ using the lexicographic order with respect to $\leq_a$. Let S denote the k-subsets of $[m]$ which are less than I_a with respect to $\leq_a$ for some a. We define the *open positroid variety* $\mathring{\Pi}_{\mathcal{I}}$ by

$$\mathring{\Pi}_{\mathcal{I}} := \{X \in Gr(k,m) \mid \Delta_I(X) \neq 0 \ \forall I \in \mathcal{I} \text{ and } \Delta_I(X) = 0 \ \forall I \in S\},$$

and the *closed positroid variety* $\Pi_{\mathcal{I}}$ as its Zariski closure. The Grassmannian $Gr(k,m)$ is stratified by the open positroid varieties indexed by (k,m)-Grassmann necklaces [9].

The *totally nonnegative Grassmannian*, $Gr(k,m)_{\geq 0}$ is the locus in $Gr(k,\mathbb{R}^m)$ where all Plücker coordinates are nonnegative. We define a sign flipped version of the totally nonnegative Grassmannian: $Gr(k,2n)^D_{\geq 0}$ is the image of $Gr(k,2n)_{\geq 0}$ under the diagonal matrix $\mathrm{diag}(1,1,-1,-1,1,1,-1,-1,\ldots,(-1)^n,(-1)^n)$. The totally nonnegative Grassmannian is stratified into positroid cells $\mathring{\Pi}^{\geq 0}_{\mathcal{I}}$ [12], we write $\mathring{\Pi}^{D,\geq 0}_{\mathcal{I}}$ for the corresponding sign flipped version.

Lam proves Theorem 3 by embedding E_n into the totally nonnegative Grassmannian

$$i : E_n \hookrightarrow Gr(n+1,2n)_{\geq 0}$$

as a linear slice. Under this embedding, the image of each electroid cell E_τ is the intersection of $i(E_n)$ and a positroid cell $\Pi^{\geq 0}_{\mathcal{I}_\tau}$. We identify the electroid cell with its image and also call the intersection $i(E_n) \bigcap \Pi^{\geq 0}_{\mathcal{I}_\tau}$ an electroid cell. See Example 10 or [10] for a combinatorial description of $\tau \mapsto \mathcal{I}_\tau$.

Definition 5 (Electroid space and Electroid varieties). *[10] The electroid space* χ_n *is the Zariski closure of* $i(E_n)$ *in* $Gr(n+1,2n)$.

The open and closed electroid varieties $\mathring{\chi}_\tau$ *and* χ_τ *are defined by*

$$\mathring{\chi}_\tau := \mathring{\Pi}_{\mathcal{I}_\tau} \cap \chi_n, \qquad \chi_\tau := \Pi_{\mathcal{I}_\tau} \cap \chi_n.$$

The electroid cell E_τ is thus the totally nonnegative points of the open electroid variety $\mathring{\chi}_\tau$. In [10], it was shown that χ_n is the disjoint union of the open electroid varieties indexed by the matchings on $[2n]$. However, up until now, little was known about their geometry. It is not even a priori clear that χ_τ is the Zariski closure of $\mathring{\chi}_\tau$.

2.3 The Lagrangian Grassmannian

In 2021, [2] and [3] gave another reason to care about the electroid space, showing that it is abstractly isomorphic to the Lagrangian Grassmannian $LG(n-1, 2n-2)$. If V is a vector space equipped with a skew-symmetric bilinear form Ω, then we define a subspace U of V to be *isotropic* if the restriction of Ω to U is 0. The space of isotropic k-planes in V is denoted $\mathrm{IG}^{\Omega}(k, V)$. We define a skew-symmetric bilinear form on $\mathbb{R}^{2n}$ by

$$\Omega((x_1, \ldots, x_{2n}), (y_1, \ldots y_{2n})) = \sum_{i=1}^{2n} (x_i y_{i+1} - x_{i+1} y_i)$$

with indices modulo $2n$. The bilinear form has a 2-dimensional kernel, and thus its isotropic Grassmannian $\mathrm{IG}^{\Omega}(n+1, \mathbb{R}^{2n})$ is isomorphic to $\mathrm{LG}(n-1, 2n-2)$. We emphasize that the positivity condition is the Plücker positivity on $\mathrm{IG}^{\Omega}(n+1, \mathbb{R}^{2n})$, and thus we mean $Gr(n+1, 2n)^{D}_{\geq 0} \cap \mathrm{IG}^{\Omega}(n+1, \mathbb{R}^{2n})$ when we say "totally nonnegative part of $\mathrm{LG}(n-1, 2n)$." This gives a different totally nonnegative part of $\mathrm{LG}(n-1, 2n)$ than those considered by Lusztig, Rietsch, and Karpman [11, 14, 6].

Theorem 6. *The image $i(E_n)$ is $Gr(n+1, 2n)^{D}_{\geq 0} \cap IG^{\Omega}(n+1, \mathbb{R}^{2n})$ and thus its Zariski closure is $\chi_n = IG^{\Omega}(n+1, 2n) \simeq LG(n-1, 2n-2)$.*

2.4 Main results

Theorem 7. *The Lagrangian Grassmannian $LG(n-1, 2n-2)$ admits a well-behaved stratification by open and closed electroid varieties, extending the stratification of its totally nonnegative part by electroid cells. More precisely, we have the following.*

1. *Open electroid varieties are irreducible, smooth, and have expected dimension.*

2. *Totally nonnegative points of an open electroid variety are Zariski dense in the open electroid variety and are exactly the corresponding electroid cell.*

3. *Any reduced cactus network embeds an open ball $\mathbb{R}^{\ell}_{\geq 0}$ into the totally nonnegative points of an open electroid variety. This can be extended algebraically to embed an algebraic torus $(\mathbb{C}^*)^{\ell}$ as an open dense subscheme of the open electroid variety.*

4. *Closed electroid varieties are reduced, irreducible, regular in codimension one, and have expected dimension.*

5. *There is a Frobenius splitting on $LG(n-1, 2n-2)$ under which the closed electroid varieties are compatibly split.*

6. *Any closed electroid variety χ_τ is the Zariski closure of its open electroid variety $\mathring{\chi}_\tau$ and is a disjoint union $\chi_\tau = \bigsqcup_{\tau' \leq \tau} \mathring{\chi}_{\tau'}$ of open electroid varieties. Thus, they form a stratification.*

7. *There exists an isomorphism between certain closed electroid varieties in $LG(n-1,2n-2)$ and $LG(n-2,2n-4)$, respecting the stratification on both sides by electroid varieties.*

3 Irreducibility

The group GL_m acts on $Gr(k,m)$ by right multiplication. For $a \in \mathbb{C}$ and $1 \leq i \leq m-1$, the element $x_i(a) \in GL_m$ is the elementary matrix differing from the identity matrix by the entry a in the i-th row and $(i+1)$-st column. The generator $x_i(a)$ acts on $Gr(k,m)$ by adding a times column i to column $i+1$. Similarly, $y_i(a) \in GL_m$ is the elementary matrix differing from the identity by the entry a in the $(i+1)$-st row and i-th column. The generator $y_i(a)$ acts by adding a times column $i+1$ to column i. For $a \in \mathbb{C}$, define $u_i(a) := x_i(a)y_{i-1}(a) = y_{i-1}(a)x_i(a)$. We introduce a key embedding relating a pair of open electroid varieties whose pairings are related by crossing adjacent strands.

Definition 8 (*i*-**crossing Pair).** *A pair of pairings $\tau \lessdot \tau'$ is called an* i-crossing pair *if τ' is formed from τ by crossing distinct non-intersecting strands i and $i+1$ such that $\tau'(i) = \tau(i+1)$ and $\tau'(i+1) = \tau(i)$.*

Lemma 9. *Let $\tau \lessdot \tau'$ be an i-crossing pair. Then $i+1 \in I_{i+1}(\tau')$, so that $I'_{i+1}(\tau') := I_{i+1} - \{i+1\} \bigcup \{i\}$ is well-defined. We have an open embedding of algebraic varieties*

$$\psi : \mathring{\chi}_\tau \times \mathbb{C} \longrightarrow \mathring{\chi}_\tau \bigsqcup \mathring{\chi}_{\tau'}$$
$$(Y,a) \longmapsto u_i(a).Y$$

where the target is an open subscheme of $\chi_{\tau'}$. The image of ψ contains $\mathring{\chi}_\tau$ and is the dense open subscheme of the target given by the nonvanishing of $\Delta_{I'_{i+1}(\tau')}$. The inverse of ψ can be obtained by $X \mapsto (u_i(-a).X, a)$, where $a = \frac{\Delta_{I_{i+1}(\tau')}}{\Delta_{I'_{i+1}(\tau')}}$. The restriction of ψ to $\mathring{\chi}_\tau \times \mathbb{C}^*$ is an open embedding onto the dense open subscheme of $\mathring{\chi}_{\tau'}$ given by the nonvanishing of $\Delta_{I'_{i+1}(\tau')}$.

Proof Sketch. It was shown in [10] that $u_i(a).Y \in \mathring{\chi}_{\tau'}$ for $a \neq 0, Y \in \mathring{\chi}_\tau$ and $u_i(0).Y = Y$. Thus, we have little difficulty in defining ψ as a morphism with correct source and target. The content of the above theorem is that the image of ψ is an open subscheme of the target on which an inverse morphism can be defined. A key step in the proof is determining how to read the Grassmann necklace directly from τ and using this to translate the i-crossing condition on $\tau \lessdot \tau'$ to conditions relating Grassmann necklaces $\mathcal{I}_\tau$ and $\mathcal{I}_{\tau'}$. With this, we can deduce the vanishing and non-vanishing of certain Plücker coordinates on $\mathring{\chi}_\tau, \mathring{\chi}_{\tau'}$, and Im$\psi$, which allows us to prove that the image of ψ is precisely the open subscheme on which a candidate inverse morphism is defined. □

Figure 4: The pairings $\tau = (1,7),(2,6),(3,4),(5,8)$ and $\tau' = (1,6),(2,7),(3,4),(5,8)$ with subsets $I_7(\tau)+1 = 823$ and $I_7(\tau')+1 = 813$ indicated by the dots.

Example 10. *Instead of embedding χ_n in $Gr(n+1,2n)$, one can opt to embed in $Gr(n-1,2n)$ under the positivity-preserving and positroid-preserving dual isomorphism given by taking orthogonal complement. The former is useful for explicit chart computations, such as in the proof for Frobenius splitting, while the latter makes the combinatorics of translating from pairings to Grassmann necklaces simpler. For combinatorial ease, this example is done using the convention of Gr(n-1,2n); results can then be translated to Gr(n+1,2n).*

Given a pairing τ on $[2n]$, the Grassmann necklace element $I_i(\tau)$ is given as follows. For each strand $(k,\tau(k))$ which is distinct from strand $(i,\tau(i))$, add the endpoint which is smaller with respect to $<_i$ to $I_i(\tau)+1$. Reducing each element by 1 yields $I_i(\tau)$.

We consider Lemma 9 applied to the $i = 6$-crossing pair $\tau = (1,7),(2,6),(3,4),(5,8)$ and $\tau' = (1,6),(2,7),(3,4),(5,8)$ depicted in Figure 4. Here $I_{i+1}(\tau) = 712$, $I_{i+1}(\tau') = 782$, and $I'_{i+1}(\tau') = 682$. Since $u_6(a)$ acts on $X \in \mathring{\chi}_\tau$ by adding a times column 6 to columns 5 and 7,

$$\Delta_{782}(u_6(a).X) = \Delta_{782}(X) + a\Delta_{682}(X).$$

Since $782 \in \mathcal{I}_{\tau'}$ and $u_6(a).X \in \mathring{\chi}_{\tau'}$, the LHS is nonzero. We have that $782 <_7 712 = I_7(\tau)$ and $X \in \mathring{\chi}_\tau$, so $\Delta_{782}(X) = 0$. Thus, $\Delta_{682}(X)$ is nonzero. Since the action of $u_6(a)$ doesn't change columns $6, 8$, or 2, we have $\Delta_{682}(u_i(a).X) = \Delta_{682}(X) \neq 0$ as desired.

Proof Sketch of Theorem 7, item 1 (Irreducible, Smooth, Dimension Count). We induct on $d = \binom{n}{2} - c(\tau)$. When $d = 0$, $\chi_\tau = \chi_n$ and is abstractly isomorphic to the Lagrangian Grassmannian [3], which is irreducible and smooth of dimension $\binom{n}{2}$. For $\mathring{\chi}_\tau$ with $d > 0$, there exist some strands i and $i+1$ which do not cross in τ. Thus, there exists τ' such that $\tau \lessdot \tau'$ is an i-crossing pair and $c(\tau') = c(\tau)+1$. By the inductive hypothesis, $\mathring{\chi}_{\tau'}$ is irreducible and smooth of dimension $c(\tau')$. By Lemma 9, $\psi_{\mathring{\chi}_\tau \times \mathbb{C}^*} : \mathring{\chi}|_\tau \times \mathbb{C}^* \hookrightarrow \mathring{\chi}_{\tau'}$ is an open embedding onto a dense open subscheme of $\mathring{\chi}_{\tau'}$. Thus, $\mathring{\chi}_\tau$ is irreducible and smooth of dimension $c(\tau)$. □

Proof Sketch of Theorem 7, item 7 implies item 3. By a combinatorial argument, every pairing τ either contains strands i and $i+1$ which cross or contains a strand connecting i to $i+1$. This observation implies that we can either apply Lemma 9 to peel off a torus and thereby decrease the dimension of the open electroid variety or apply Theorem 7 item 7 to pass to a smaller electroid space. Thus, item 3 holds by induction. We independently prove item 7 by constructing morphisms in both directions that are inverses to each other. The isomorphism restricted to totally nonnegative points can be interpreted combinatorially: depending on the parity, it corresponds to removing an isolated boundary vertex or combining two shorted adjacent boundary vertices from the cactus network. □

4 Frobenius Splitting

Let $\mathcal{C}$ be the set of all Grassmann necklaces arising from a medial pairing with exactly $\binom{n}{2} - 1$ crossings. For each $\mathcal{I} \in \mathcal{C}$, $\Pi_{\mathcal{I}}$ is a positroid variety of codimension 2 in $Gr(n+1, 2n)$ given by the vanishing of two Plücker coordinates indexed by the cyclic intervals $[i, i+1, \ldots, i+n]$ and $[i+n, i+n+1, \ldots, i+2n]$. On χ_n, these Plücker coordinates are equal [3], so each such $\chi_{\mathcal{I}}$ is cut out by the vanishing of one interval Plücker inside χ_n and is a divisor on χ_n. That is, $\chi_{\mathcal{I}}$ is the zero locus of a global section of $\mathcal{O}_{\chi_n}(1)$. Since there are n such pairs of Plücker coordinates, there are n distinct electroid divisors.

We fix a monomial term order $\prec$ on the polynomial ring $k[L_{ij}]$. This term order is obtained from the following weighting on the variables L_{ij}. Position $1, 2, \ldots, n$ uniformly on a unit disk and declare the weight of the variable L_{ij} to be the negative log of the Euclidean distance between vertices i and j. Ptolemy's Theorem guarantees that under this weighting, $L_{pr}L_{qs} \prec L_{pq}L_{rs}$ and $L_{pr}L_{qs} \prec L_{ps}L_{qr}$ for any (p, q, r, s) arranged cyclically.

Proof Sketch of Theorem 7, item 5 *(Frobenius Splitting).* First, we show that all codimension 1 electroid varieties are compatibly split. We compute the anticanonical bundle on χ_n to be $\mathcal{O}(n)$, so that the union of all electroid divisors form an anticanonical divisor, giving a global section of $\omega^*_{\chi_n}$. On an affine chart of χ_n given in [3], we explicitly expand this section in local coordinates. Finally, under the monomial ordering given above, we compute the leading term of the local expansion and verify a criterion for Frobenius Splitting that the leading term is the product of all variables [8].

To extend to all electroid varieties, it suffices to show that we can reach all electroid varieties from electroid hypersurfaces by iteratively taking intersections and components [1]. This boils down to a combinatorial study of the poset of medial pairings, showing that if a pairing does not correspond to an electrical divisor, there exist two distinct pairings covering it. □

Example 11. *When $n = 5$, the five electroid divisors are given by the vanishing of Plücker coordinates $\Delta_{123456}, \Delta_{234567}, \ldots, \Delta_{56789,10,11}$. A local chart of $IG^{\Omega}(n+1, 2n)$ is given in [3] such that the Plücker coordinate $\Delta_{i,i+1,\ldots,i+5}$ is the minor of the response matrix L in rows $i+1$, $i+2$ and columns $i+4$, $i+5 \bmod 5$, with leading term $L_{(i+1)(i+4)}L_{(i+2)(i+3)}$. For example,*

$\Delta_{123456} = L_{24}L_{35} - L_{25}L_{34}$ *has leading term* $L_{25}L_{34}$. *Thus, the local expansion for these five electroid divisors has leading term* $\prod_{i=1}^{5}\left(L_{(i+1)(i+4)}L_{(i+2)(i+3)}\right) = \prod_{1\le i<j\le 5} L_{ij}$.

Proof of Theorem 7, item 2, item 6, and item 4 except R1. Reducedness is guaranteed by compatibly split [1]. The fact that closed electroid varieties are the disjoint union of open electroid varieties is proved in [10]. Thus, $\chi_\tau = \bigsqcup_{\mu\le\tau} \mathring{\chi}_\mu$ is a Zariski closed set containing $(\mathring{\chi}_\tau)_{\ge 0}$. We induct on $c(\tau)$ to show that the Zariski closure of $(\mathring{\chi}_\tau)_{\ge 0}$ contains χ_τ.

When $c(\tau) = 0$, $(\mathring{\chi}_\tau)_{\ge 0}$ is a point. By irreducibility and reducedness of the open electroid varieties and the dimension count, $\mathring{\chi}_\tau = \chi_\tau$ is a reduced point.

Suppose the result holds for all $c(\tau) < c$. Take a pairing τ with $c(\tau) = c$. Recall from Theorem 3 that the Euclidean closure of $(\mathring{\chi}_\tau)_{\ge 0}$ equals $\bigsqcup_{\mu\le\tau}(\mathring{\chi}_\mu)_{\ge 0}$. Since the Zariski topology is coarser than the Euclidean topology, the Zariski closure of $(\mathring{\chi}_\tau)_{\ge 0}$ contains the Zariski closure of $\bigsqcup_{\mu\le\tau}(\mathring{\chi}_\mu)_{\ge 0}$. In particular, this contains the Zariski closure of $(\mathring{\chi}_\mu)_{\ge 0}$ for each $\mu < \tau$, which equals χ_μ by the inductive hypothesis. On the other hand, we know from item 1 that $\mathring{\chi}_\tau$ is irreducible with Krull dimension $c(\tau)$. By Theorem 3, $(\mathring{\chi}_\tau)_{\ge 0}$ is a ball of the same dimension, $c(\tau)$, contained in $\mathring{\chi}_\tau$. So, $(\mathring{\chi}_\tau)_{\ge 0}$ is dense in $\mathring{\chi}_\tau$.

Combining the above, the Zariski closure of $(\mathring{\chi}_\tau)_{\ge 0}$ contains $\mathring{\chi}_\tau \bigcup(\bigcup_{\mu<\tau}\chi_\mu) = \chi_\tau$ as a set. Since the right hand side is reduced, we have containment as a scheme. Thus, the electroid cell is dense in the open electroid variety, which in turn is dense in the closed electroid variety. Irreducibility and dimension count for closed electroid varieties follow from those for electroid cells and open electroid varieties. □

So far, we have addressed every assertion in Theorem 7 except that closed electroid varieties are regular in codimension 1.

5 Regular in Codimension One

Given a covering relation $\tau \lessdot \tau'$, call $\tau \lessdot \tau'$ a *regular pair* if the closed electroid variety $\chi_{\tau'}$ is regular along the generic point of the open electroid variety $\mathring{\chi}_\tau$. From Theorem 7 Item 1, open electroid varieties are smooth. Thus, regularity in codimension 1 of closed electroid varieties is guaranteed by asserting that all covering relations are regular pairs.

Now that open electroids are smooth, Lemma 9 shows that i-crossing pairs are regular. However, most covering relations are obtained by uncrossing non-adjacent strands, and thus are not i-crossing pairs.

Lemma 12. *Suppose* $\tau \lessdot \tau'$ *and* τ *is obtained from* τ' *by uncrossing non-adjacent strands* b *and* c. *Let* $a < b < c < d$ *be non-adjacent numbers in cyclic order such that* $\tau'(b) = d, \tau'(c) = a$ *and* $\tau(b) = a, \tau(c) = d$. *Suppose there exists* i *such that* $\tau(i) \ne i+1$ *and strands* $i, i+1$ *cross in* τ'. *We can construct* σ *and* σ' *such that* $\sigma \lessdot \tau$ *and* $\sigma' \lessdot \tau'$ *are* i*-crossing pairs. If either of the following holds,*

1. $a \leq i \leq b-1$,
2. $d < i \leq a-1$, $a < \tau(i) < b$, and $c \leq \tau(i+1) < d$,

then $\sigma \lessdot \sigma'$ and $\sigma \lessdot \sigma'$ is a regular pair if and only if $\tau \lessdot \tau'$ is a regular pair.

Proof Sketch. We construct a morphism "gluing" the ψ morphisms associated to the two i-crossing pairs. Notice that since both are i-crossing pairs, the formulae for ψ are both given by $Y \mapsto u_i(a).Y$. Thus, it is relatively easy to define the glued morphism with correct source and target. It remains to be proven that inverses glue. This involves showing that a certain pair of rational functions are well-defined and agree on open subschemes that are *dense*. Our approach is similar to the proof of Lemma 9 but is more involved. □

Proof Sketch of Theorem 7, item 4, R1 part. To show that $\tau \lessdot \tau'$ is a regular pair, it suffices to show that moves from Lemma 12 connect any covering relation to an i-crossing pair. This is done by a combinatorial algorithm which explicitly constructs such a path. □

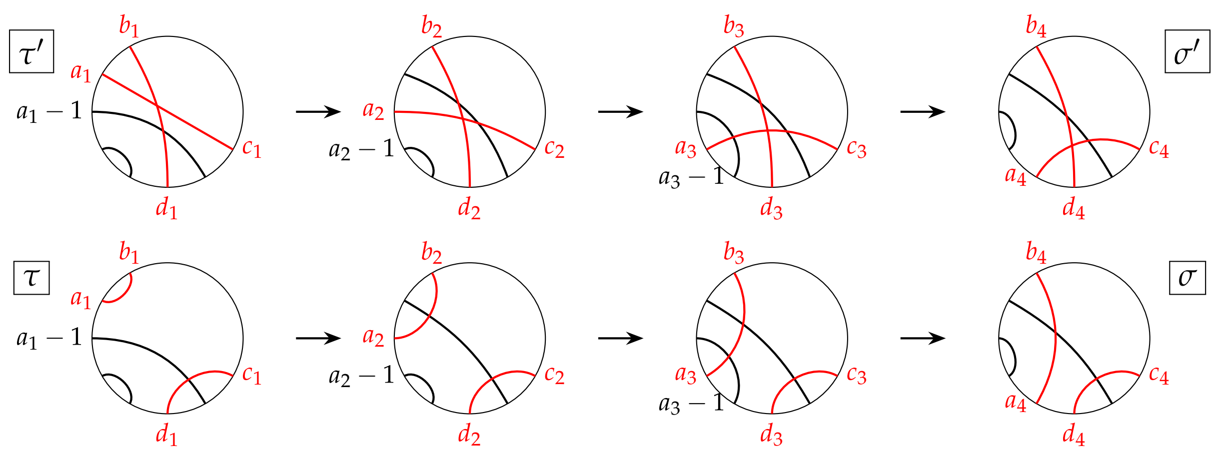

Figure 5: Under the algorithm, $\tau \lessdot \tau'$ is transformed into an i-crossing pair $\sigma \lessdot \sigma'$.

Example 13. *We use the pair $\tau \lessdot \tau'$ depicted in Figure 5 to illustrate the algorithm transforming $\tau \lessdot \tau'$ into an i-crossing pair. The strategy of the algorithm is to apply a "regular pair"-preserving move from Lemma 12 to iteratively decrease the distance between strands a and d until the strands are adjacent. At each iteration, $\tau'(a-1)$ lies in one of three intervals, dictating a different move.*

- *Move 1: If $\tau'(a-1) \in (c,d)$, then cross strands $a, a-1$.*
- *Move 2: If $\tau'(a-1) \in (d,a)$, then cross strands $a, a-1$.*
- *Move 3: If $\tau'(a-1) \in (a,b)$, then uncross strands $a, a-1$.*

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