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The ABCT Variety is a Positive Geometry

Dawei Shen *1 and Emanuele Ventura † 2

[1] *University of Michigan, Department of Mathematics, 530 Church Street, Ann Arbor, USA*
[2] *Politecnico di Torino, Dipartimento di Scienze Matematiche "G. L. Lagrange", Corso Duca degli Abruzzi 24, 10129 Torino, Italy*

Abstract. The *ABCT variety* $V(3,n)$ is the image closure of the rational Veronese map between Grassmannians. It was studied by Arkani-Hamed, Bourjaily, Cachazo, and Trnka in 2011 in the context of MHV and NMHV scattering amplitudes in planar $\mathcal{N}=4$ supersymmetric Yang–Mills theory. From this perspective, $V(3,n)$, along with its totally nonnegative part, is conjectured to be a positive geometry by Lam in 2024.

In this paper, we study the combinatorics and algebraic geometry of $V(3,n)$ and its subvarieties, which arise from iteratively taking the Zariski closures of Euclidean boundaries of the totally nonnegative parts. We draw connections to point configurations on $\mathbb{P}^2$ via the Gelfand–MacPherson correspondence and prove that $V(3,n)$ is a positive geometry, thereby establishing Lam's conjecture.

Keywords: positive geometry, Grassmannian, positroid variety, point configuration

1 Introduction

Scattering amplitudes are central objects in quantum field theory that describe particle collisions. Traditionally, computing amplitudes involves summing a large number of complicated rational functions, but the final answers are always surprisingly simple. *Positive geometries* were introduced by Arkani-Hamed, Bai, and Lam [3] to simplify the computation process. A *positive geometry* is a triple of a projective algebraic variety, a closed semi-algebraic subset of its real points, and a top-degree rational differential form (the *canonical form*), which gives the amplitude (see Definition 2.3). All known examples of positive geometries are proved on a case-by-case basis. Examples include *polytopes* and their blow-ups [5], the totally nonnegative part of *toric varieties* and *Grassmannians* [3], and *Deligne–Mumford compactifications* $\overline{\mathcal{M}}_{0,n}$ [4].

The *ABCT variety* $V(k,n)$ is the subvariety of the Grassmannian $\mathrm{Gr}(k,n)$ defined as the closure of the image of the rational Veronese map from $\mathrm{Gr}(2,n)$ to $\mathrm{Gr}(k,n)$. It was studied by and named after Arkani-Hamed *et al.* [2] in the context of MHV and NMHV

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†emanuele.ventura@polito.it. E.V. is a member of GNSAGA and was partially supported by PRIN 2022 no. 2022NBN7TL.

amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, and by Roiban *et al.* [12] in Witten's twistor string theory. The coefficient in front of a specific Schubert class of the fundamental class of $V(k,n)$ in $H^*(\mathrm{Gr}(k,n))$ is conjectured in [10] and verified for $k = 3$ in [1] to be equal to an Eulerian number and counts solutions to scattering equations [6].

In this paper, we focus on the case $k = 3$ and build on a collaboration involving the first author [1] to study the combinatorial algebraic geometry of $V(3,n)$ and its iterative boundaries induced by the totally nonnegative part $V(3,n)_{\geq 0}$. Our main result proves Lam's conjecture [10] that $V(3,n)$ is a positive geometry and the interior of $\mathrm{Gr}(2,n)_{\geq 0}$ maps holomorphically to the interior of $V(3,n)_{\geq 0}$ such that the canonical forms are related by pushforward. This builds a parallel between $V(3,n)$ and other positive geometries, such as toric varieties, Grassmannians, polytopes, etc. It establishes the foundation for the further conjecture that $V(3,n)_{\geq 0}$ maps diffeomorphically to the interior of the momentum amplituhedron and their canonical forms are related by pushforward [8]. This conjecture can be interpreted as that $V(3,n)$ plays a similar role for the momentum amplituhedron as a toric variety plays for a polytope.

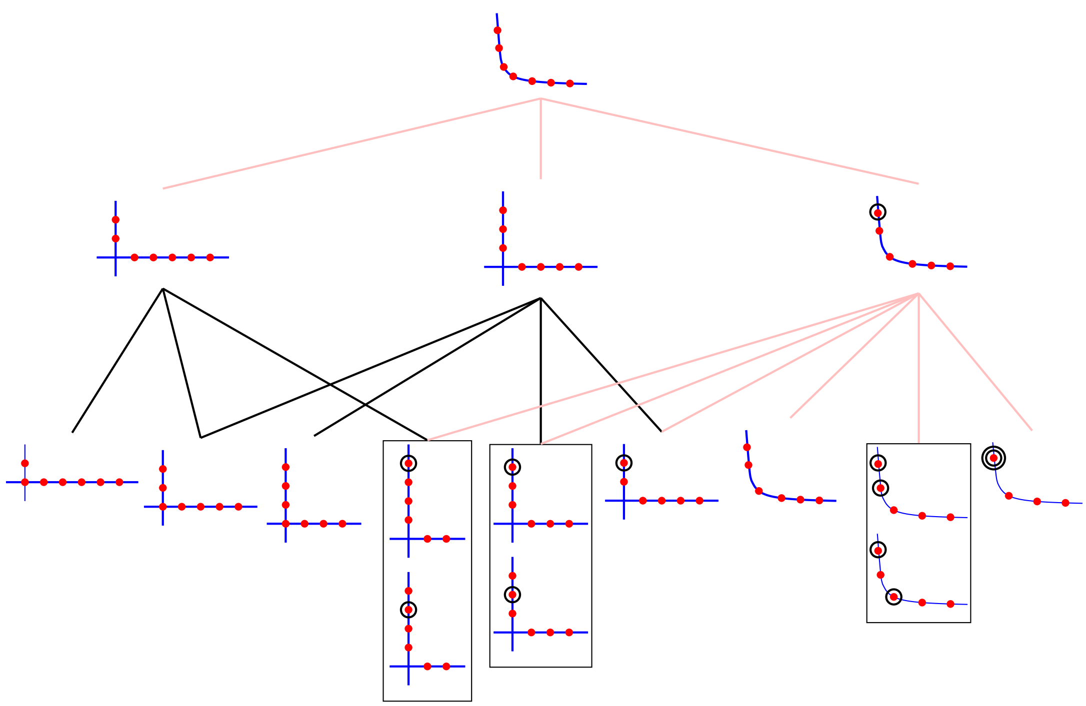
By Definition 2.3, any iterative boundary of a positive geometry X is again a positive geometry. Iterative boundaries naturally form a poset, where $C < C'$ if C is an iterative boundary of C'. For Grassmannians, these are the positroid varieties [3]. Along showing that $V(3,n)$ is a positive geometry, we give an explicit algebro-geometric description of each iterative boundary, describe explicitly the poset structure, and draw connections to *point configurations* on $\mathbb{P}^2$ and the poset of positroids. We emphasize that our results not only contain new combinatorics, but also our proofs of the algebro-geometric side of the results are highly combinatorial (see proof sketches and Example 7.1).

Theorem 1.1. *(1) The ABCT Variety* $(V(3,n), V(3,n)_{\geq 0}, \Omega_{V(3,n)})$ *is a positive geometry. The canonical form* $\Omega_{V(3,n)}$ *equals the pushforward of* $\Omega_{\mathrm{Gr}(2,n)}$ *along the rational Veronese map and has an explicit formula given in Definition 5.1.*

(2) The iterative boundaries of the ABCT variety $V(3,n)$ *are exactly the set-theoretic intersections of* $V(3,n)$ *and certain positroid varieties in* $\mathrm{Gr}(3,n)$*, as in Definition 6.1, and taking totally nonnegative parts. The poset structure of iterative boundaries is the induced subposet structure on these positroids and is graded by the dimension of the iterative boundary.*

(3) The ABCT Variety $V(3,n)$ *and all its iterative algebraic boundaries are reduced, irreducible, rational, normal, Cohen–Macaulay, regular in codimension* 1*, and have expected dimensions from the perspective of point configurations.*

Iterated boundaries of $V(3,7)$ correspond to imposing positroid conditions on point configurations on a common conic in $\mathbb{P}^2$. By "positroid conditions," we mean that columns are set to 0, or give projective points that collide or become collinear. When points are collinear, the conic must be the union of two lines, and thus all other points lie on the other line. The figure below shows the poset of iterated boundaries of codimensions $0, 1, 2$ up to dihedral symmetry: to get the actual poset, we possibly reflect the diagram and then label points in counterclockwise order. A red dot enclosed by 1 or 2 circles denotes 2 or 3 colliding points. Two different configurations are in a box when they are related by a non-dihedral relabeling of points. The seventh codimension 2 boundary is given by setting one column to zero, and thus the configuration only shows six points. Forgetting the conic condition gives a rank 3 matroid on 7 vertices, which is a positroid because the points are labeled counterclockwise in the actual poset. This identifies iterated boundaries with a subset of rank 3 positroids on 7 vertices. Theorem 1.1 states that the poset structure on iterated boundaries is the induced sub-poset. Black covering relations come from covering relations in positroids. Pink covering relations come from non-covering relations in positroids that become covering relations in the induced sub-poset. The smooth conics in row 3 are thin because they impose no conditions, since 5 points determine a conic. The only iterated boundaries of $V(3,7)$ that are not positroid varieties are $V(3,7)$ itself, the codimension 1 boundary given by the right-most configuration on the second row, the codimension 2 boundary isomorphic to $V(3,6)$, and their cyclic rotations. We omit codimension ≥ 3 boundaries in the picture.



2 Positive Geometries

Definition 2.1 (Iterative Boundary). Let X be a d-dimensional irreducible complex projective variety defined over $\mathbb{R}$. Let $X_{\geq 0}$ be a closed semi-algebraic subset of the real points $X(\mathbb{R})$, whose Euclidean interior $\operatorname{int} X_{\geq 0}$ is a d-dimensional orientable manifold and is Euclidean dense in $X_{\geq 0}$. Let C be the divisor on X given by the Zariski closure of the Euclidean boundary $\partial X_{\geq 0}$. Let $C = C_1 \cup C_2 \cup \cdots \cup C_r$ be the irreducible decomposition. Let $C_{i,\geq 0}$ be the Euclidean closure of the intersection $C_i \cap (\operatorname{int} X_{\geq 0})$. Call each C_i (resp., $C_{i,\geq 0}$) an *algebraic (resp., Euclidean) boundary component* of $(X, X_{\geq 0})$. We refer to them as boundaries of X when it is clear from context. Similarly, we may take the boundaries of a boundary pair $(C_i, C_{i,\geq 0})$ and iterate. We call all pairs $(C, C_{\geq 0})$ obtained by iteratively taking boundary components *iterative boundaries* of $(X, X_{\geq 0})$.

Let $X_{\text{reg}} := X \setminus \operatorname{Sing}(X)$ be the regular locus of X. By a rational *differential d-form* on X, we mean a differential d-form on an open subset of the smooth manifold X_{reg}. This is a local section of the canonical bundle on X_{reg}.

Definition 2.2 (Poincaré Residue). Let ω be a rational differential d-form on X with a pole of order one along a prime divisor $Y \subset X$ not contained in the singular locus $\operatorname{Sing}(X)$. A pole of order one is called a *simple* pole. Locally at a smooth point of $Y \setminus \operatorname{Sing}(X)$, we have

$$\omega = \eta \wedge \frac{df}{f} + \eta',$$

where η is a $(d-1)$-form, η' is a d-form, both having no poles along Y, and f is a local coordinate vanishing to order one along Y. The *(Poincaré) residue* of ω along Y is defined as the $(d-1)$-form on Y given by the restriction

$$\operatorname{Res}_Y \omega = \eta_{|_Y},$$

not depending on the choices of η, η', f.

Definition 2.3 (Normal Positive Geometry). [3] Let $(X, X_{\geq 0})$ be as in Definition 2.1. A *normal positive geometry* is a triple $(X, X_{\geq 0}, \Omega_X)$, where X and all of its iterative algebraic boundaries are normal varieties, and Ω_X is the unique rational d-form on X, called the *canonical form*, satisfying the following recursive property.

(i) When $d = 0$, $X = X_{\geq 0}$ is a point and $\Omega_X = \pm 1$, depending on the orientation.

(ii) When $d > 0$, let $(C_i, C_{i,\geq 0})$ for $i = 1, 2, \cdots, r$ be the boundary components of $(X, X_{\geq 0})$ as in Definition 2.1. The form Ω_X has simple poles exactly along the boundary components $\cup C_i$ and nowhere else. Furthermore, each $(C_i, C_{i,\geq 0}, \Omega_{C_i})$ is a positive geometry with the unique canonical form $\Omega_{C_i} = \operatorname{Res}_{C_i} \Omega_X$.

It is remarkable that such a canonical form may exist at all. Even with a candidate canonical form, verifying the definition is still extremely difficult in general, because it requires one to understand and keep track of the complex and real algebraic geometry of not only X itself, but also all iterative boundaries of X. In particular, we need a combinatorial understanding of the partial order on iterative boundaries, where covering relations are induced by taking boundary components. For each covering relation, we need to understand how to take poles and residues of the candidate canonical form.

Remark 2.4. In the spirit of the recursive definition, we do not always need to iterate until we reach points. For example, we can stop the iteration after reaching an iterative boundary that is known to be a positive geometry. Similarly, if we want to inductively show that $V(3, n)$ is a positive geometry, we can stop the iteration after reaching an iterative boundary isomorphic to $V(3, n-1)$. However, $V(3, n)$ has more exotic iterative boundaries, demanding a more involved treatment.

Remark 2.5. We omit the more technical definition of positive geometries without the normality assumption, since we will show that $V(3, n)$ is a normal positive geometry. One benefit of the normality assumption is that we can treat rational d-forms on X as local sections of the canonical bundle on X, rather than X_{reg}. We emphasize that the recursive nature of the definition enforces us to require normality for X itself and all its iterative boundaries. Even without the consideration of canonical forms, identifying all iterative boundaries and proving normality is already nontrivial.

The uniqueness of the canonical form is guaranteed by normality and rationality of X and all iterative boundaries. If two rational top forms both satisfy the recursive definition, their difference is a holomorphic top form on X_{reg}. The rational smooth variety X_{reg} has geometric genus 0, so the difference must be zero [7, Part II, 8.19].

3 Grassmannians and Positroid Varieties

Definition 3.1 (Totally Nonnegative Part). Let $V \subset \mathrm{Gr}(k, n)$ be a closed subvariety. Its *totally nonnegative part* $V_{\geq 0}$ is defined as the semi-algebraic subset of the real points of V such that all Plücker coordinates are nonnegative.

Theorem 3.2. *[3]* $(\mathrm{Gr}(k, n), \mathrm{Gr}_{\geq 0}(k, n), \Omega_{\mathrm{Gr}(k,n)})$ *is a normal positive geometry.*

As in Definition 2.3, contained in the statement of Theorem 3.2 is the statement that all its iterative boundaries are positive geometries. Indeed, iterative boundaries of the Grassmannians are well-understood and are given by positroid varieties Π_f.

Theorem 3.3. *[9, 3] Positroid varieties* Π_f *are reduced, irreducible, normal, Cohen-Macaulay closed subvarieties in* $\mathrm{Gr}(k, n)$*. Furthermore,* $(\Pi_f, (\Pi_f)_{\geq 0})$ *are positive geometries.*

Positroid varieties can be determined by imposing a collection of cyclic rank conditions on the matrix M whose row span represents a point in $\mathrm{Gr}(k, n)$ [9]. The $k = 3$ cyclic rank conditions are given exactly by specifying a cyclic interval $I \subset [n]$ and requiring that the columns of M indexed by I form a submatrix M_I of rank at most $0, 1$, or 2. The rank conditions on M_I can be translated into the vanishing of Plücker coordinates: $\mathrm{rk} M_I \leq r$ if and only if $p_J = 0$ for all subsets J such that $\#(J \cap I) \geq r+1$.

Remark 3.4. The cyclic rank conditions translate to conditions on point configurations on $\mathbb{P}^2$ under the *Gelfand-MacPherson correspondence* from classical invariant theory. Each nonzero column of a $3 \times n$ matrix M gives a point on $\mathbb{P}^2$ after projectivization. Thus, $[M] \in \mathrm{Gr}(3, n)$ gives point configurations on $\mathbb{P}^2$ up to GL_3 action. Now, we have

(1) $\mathrm{rk} M_I \leq 2$ if and only if nonzero columns in I give collinear points on $\mathbb{P}^2$.

(2) $\mathrm{rk} M_I \leq 1$ if and only if nonzero columns in I give the same point on $\mathbb{P}^2$.

(3) $\mathrm{rk} M_I = 0$ if and only if every column in I is zero.

4 The ABCT Variety

Consider the Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$ induced by the map $\mathbb{C}^2 \to S^2\mathbb{C}^2 \cong \mathbb{C}^3$, sending $v \mapsto v \otimes v$. The representation $S^2\mathbb{C}^2$ of GL_2 gives a homomorphism $\mathrm{GL}_2 \to \mathrm{GL}_3$, through which GL_2 acts on $\mathbb{P}^2$. The Veronese map is GL_2-equivariant, so the map $(\mathbb{C}^2)^n \to (\mathbb{C}^3)^n$ descends to the *rational Veronese map* $\vartheta : \mathrm{Gr}(2, n) \dashrightarrow \mathbb{G}(3, n)$ given by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{11}a_{21} & a_{12}a_{22} & \cdots & a_{1n}a_{2n} \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \end{pmatrix}.$$

The map ϑ is only a rational map between the two Grassmannians. For example, it is not defined at $\mathrm{span}\{e_i, e_j\} \in \mathrm{Gr}(2, n)$, because the image matrix is not full rank.

Definition 4.1. The *ABCT variety* $V(3, n)$ is the Zariski closure of the image of ϑ.

Remark 4.2. Although ϑ is a holomorphic map on the interiors, the face structure of $V(k, n)_{\geq 0}$ and $\mathrm{Gr}(2, n)_{\geq 0}$ differ. For instance, $\mathrm{Gr}(2, n)_{\geq 0}$ has 0-dimensional faces that are the $\binom{n}{2}$ torus fixed points $W_{ij} = \langle e_i, e_j \rangle \in \mathrm{Gr}(2, n)$, on which ϑ is not defined.

In this paper, we leverage the following descriptions of $V(3, n)$, proved in a recent collaboration involving the first author [1].

Theorem 4.3 ([1])**.** *The ABCT variety $V(3, n)$ is reduced, irreducible, and Cohen-Macaulay of expected dimension $2n - 4$. Define the map of matrices $\phi : \mathrm{Mat}_{3\times n} \to \mathrm{Mat}_{6\times n}$ by*

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ x_{31} & x_{32} & \cdots & x_{3n} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}^2 & x_{12}^2 & \cdots & x_{1n}^2 \\ x_{21}^2 & x_{22}^2 & \cdots & x_{2n}^2 \\ x_{31}^2 & x_{32}^2 & \cdots & x_{3n}^2 \\ x_{11}x_{21} & x_{12}x_{22} & \cdots & x_{1n}x_{2n} \\ x_{11}x_{31} & x_{12}x_{32} & \cdots & x_{1n}x_{3n} \\ x_{21}x_{31} & x_{22}x_{32} & \cdots & x_{2n}x_{3n} \end{pmatrix}.$$

Then, one has the equality $V(3,n) = \{[M] \in \mathbb{G}(3,n) \mid \operatorname{rk}\phi(M) < 6\}$ as schemes.

Theorem 4.3 is phrased rigorously in [1] using the formalism of degeneracy loci of vector bundle morphisms, which we omit. Concretely, restricted to any chart of $\mathrm{Gr}(3,n)$ by fixing a maximal minor to identity, the rank conditions on the image of the matrix under ϕ are determinantal relations cutting $V(3,n)$ out as a scheme in the chart.

Theorem 4.4 ([1])**.** *Let $n \geq 6$. The ABCT variety $V(3,n)$ is cut out as a scheme by the following quartic relations in the Plücker coordinates on $\mathrm{Gr}(3,n)$.*

$$p_{i_1,i_2,i_3}p_{i_1,i_5,i_6}p_{i_2,i_4,i_6}p_{i_3,i_4,i_5} - p_{i_2,i_3,i_4}p_{i_1,i_2,i_6}p_{i_1,i_3,i_5}p_{i_4,i_5,i_6}$$

where $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ vary through all 6-element subsets of $[n]$.

Remark 4.5. Following Remark 3.4, $V(3,n)$ is exactly the subset of $\mathrm{Gr}(3,n)$ where the corresponding points on $\mathbb{P}^2$ lie on a common conic. Indeed, by Theorem 4.3, if $\phi(M)$ is not full-rank, any nonzero vector in the kernel of $\phi(M)$ is a six-tuple of coefficients for a conic on $\mathbb{P}^2$ containing all points corresponding to nonzero columns of M.

Remark 4.6. Both Theorem 4.3 and 4.4 make it manifest that $V(3,n)$ is invariant under the S_n-action by permuting columns. We emphasize that total nonnegativity is not preserved under S_n-action, but is preserved under cyclically permuting columns and ordering columns reversely.

5 Candidate Canonical Form

It suffices to define the rational form on a dense open chart. Fix coordinates

$$\begin{pmatrix} 1 & 0 & 0 & \alpha_1 & \cdots & \alpha_{n-3} \\ 0 & 0 & 1 & \beta_1 & \cdots & \beta_{n-3} \\ 0 & 1 & 0 & \gamma_1 & \cdots & \gamma_{n-3} \end{pmatrix} \tag{5.1}$$

on a standard affine chart of $\mathrm{Gr}(3,n)$. In fact, we can show that $\Omega_{V(3,n)}$ also equals the pushforward of $\Omega_{\mathrm{Gr}(2,n)}$ via ϑ, details of which we omit.

Definition 5.1 (Canonical form). Define the canonical form $\Omega_{V(3,n)}$ on a chart of $V(3,n)$ by pulling-back the following rational form defined on the chart (5.1)

$$\begin{aligned} &-\frac{1}{f(\prod_{i=1}^{n-3}\beta_i)}\frac{\alpha_{n-3}}{(\alpha_1\gamma_{n-3}-\alpha_{n-3}\gamma_1)\beta_{n-3}\gamma_{n-3}^2}\left(\prod_{i=1}^{n-4}\frac{\gamma_{i+1}^2}{\alpha_i\gamma_{i+1}-\alpha_{i+1}\gamma_i}\right)\cdot \\ &\qquad d\alpha_1 \wedge d\gamma_{n-3} \wedge d\gamma_1 \wedge d\beta_1 \wedge \left(\bigwedge_{i=2}^{n-4} d\left(\frac{\alpha_i}{\gamma_i}\right)\wedge d\beta_i\right)\wedge d\alpha_{n-3}\wedge d\beta_{n-3}, \end{aligned} \tag{5.2}$$

where $f = \frac{\alpha_i\alpha_j}{\beta_i\beta_j}\frac{\beta_i\gamma_j-\beta_j\gamma_i}{\gamma_i\alpha_j-\gamma_j\alpha_i}$ restricted to $V(3,n)$ is independent of the choice of $i \neq j$.

Proposition 5.2. *In the coordinates of the chart* (5.1), *let* $\varphi_4 : V(3,n) \dashrightarrow V(3,n-1)$ *be the rational map forgetting the fourth column. Let* p_{145} *be the Plücker coordinate. Then*

$$\Omega_{V(3,n)} = -\frac{\beta_1\gamma_2}{p_{145}}\varphi_4^*\Omega_{V(3,n-1)} \wedge d\log\beta_1 \wedge d\log\gamma_1.$$

6 Iterative Boundaries

We define a family of subvarieties in $V(3,n)$ by intersecting $V(3,n)$ with certain positroid varieties and possibly taking the reduced subscheme structure on the intersection. They turn out to be exactly the iterative boundaries of $V(3,n)$. Fix a partition $A_0 \bigsqcup(\bigsqcup_{i=1}^r I_i)$ of $[n]$, where I_i's are cyclic intervals in $[n]$. Fix a partition $[r] = A \bigsqcup B$ where A, B are cyclic intervals in $[r]$. Define two families of positroid varieties in $\mathrm{Gr}(3,n)$ as follows.

$\Pi(A_0; I_1, \cdots, I_r) := \{[M] \in \mathrm{Gr}(3,n) : \mathrm{rk}M_{A_0} = 0,\ \mathrm{rk}M_{I_i} \leq 1 \text{ for all } i\}$, and
$\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B) :=$
$\{[M] \in \mathrm{Gr}(3,n) : \mathrm{rk}M_{A_0} = 0,\ \mathrm{rk}M_{I_i} \leq 1 \text{ for all } i,\ \mathrm{rk}M_{\bigsqcup_{a\in A} I_a} \leq 2,\ \mathrm{rk}M_{\bigsqcup_{b\in B} I_b} \leq 2\}$.

In fact, $\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B)$ is always contained in $V(3,n)$. However, $\Pi(A_0; I_1, \cdots, I_r)$ is contained in $V(3,n)$ if and only if $\#(\bigsqcup_{i=1}^r I_i) \leq 5$.

Definition 6.1. We call $\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B)$ *boundaries of collinear configurations*. Let $V(A_0; I_1, \cdots, I_r) := \Pi(A_0; I_1, \cdots, I_r) \cap V(3,n)$ be the scheme-theoretic intersection. We call its reduced subscheme a *boundary of colliding configurations*.

Remark 6.2. Following Remark 3.4, $\Pi(A_0; I_1, \cdots, I_r)$ is the subset of $\mathrm{Gr}(3,n)$ such that columns in A_0 are zero and nonzero columns in each I_i give the same point on $\mathbb{P}^2$. Intersecting with $V(3,n)$ means requiring that these points furthermore lie on a conic.

$\Pi(A_0; I_a \text{ for } a \in A | I_b \text{ for } b \in B)$ is the subset such that the above colliding conditions hold and furthermore, nonzero columns from $\bigsqcup_{a\in A} I_a$ and from $\bigsqcup_{b\in B} I_b$ respectively give collinear points on $\mathbb{P}^2$. Thus, all points lie on the union of two lines, which is a conic. Indeed, this positroid variety is contained in $V(3,n)$.

We now give the key construction of incidence schemes for studying the boundaries of colliding configurations. If $\#I_i = 1$ for all i, $V(A_0; I_1, \cdots, I_r)_{\mathrm{red}} = V(3, n - \#A_0)$ by forgetting zero columns in A_0. The geometry of $V(3,n)$ can be understood separately. We hereby assume that $\#I_1 \geq 2$ by symmetry as in Remark 4.6. Fix $s, t \in I_1$.

Definition 6.3. Define the incidence closed subscheme of $V(A_0; I_1, \cdots, I_r)_{\mathrm{red}} \times \mathbb{P}^1$ by

$$\mathcal{I}_{st}(A_0; I_1, \cdots, I_r) = \{([M], [x:y]) : y \cdot M_{\{s\}} = x \cdot M_{\{t\}}\},$$

where $M_{\{s\}}$ denotes the s-th column of the matrix M. The incidence condition can be rephrased in Plücker coordinates as $p_{s,j,k}(M)y = p_{t,j,k}(M)x$ for all j,k.

Proposition 6.4. *One has a surjective morphism obtained by projecting to the first factor*

$$\pi_1 : \mathcal{I}_{st}(A_0; I_1, \cdots, I_r) \subset V(A_0; I_1 \cdots, I_r)_{\text{red}} \times \mathbb{P}^1 \to V(A_0; I_1 \cdots, I_r)_{\text{red}}$$

π_1 is birational and induces an isomorphism of sheaves $\mathcal{O}_{V(A_0;I_1,\cdots,I_r)_{\text{red}}} \cong (\pi_1)_* \mathcal{O}_{\mathcal{I}_{st}(A_0;I_1,\cdots,I_r)}$.

Proposition 6.5. *The morphism obtained by projecting to the second factor*

$$\pi_2 : \mathcal{I}_{st}(A_0; I_1, \cdots, I_r) \subset V(A_0; I_1 \cdots, I_r)_{\text{red}} \times \mathbb{P}^1 \to \mathbb{P}^1$$

is trivialized over standard charts of $\mathbb{P}^1 = \{[x/y : 1]\} \cup \{[1 : y/x]\}$. More precisely, let U_1 and U_2 be the preimages of the charts $\{[x/y : 1]\}$ and $\{[1 : y/x]\}$, respectively. Restrictions of π_2 to each U_i factors through an isomorphism to some product space and the projection map on the product space to the second factor

$$U_1 \simeq V(A_0; I_1', \cdots, I_r)_{\text{red}} \times \{[x/y : 1]\} \to \{[x/y : 1]\}$$

and

$$U_2 \simeq V(A_0; I_1', \cdots, I_r)_{\text{red}} \times \{[1 : y/x]\} \to \{[1 : y/x]\},$$

where $V(A_0; I_1', \cdots, I_r)_{\text{red}}$ is the boundary of colliding configurations inside $V(3, n-1)$ obtained by identifying $s, t \in I_1$ as a single index to form I_1' of size $\#I_1 - 1$.

We have but omit the ideal description of $V(A_0; I_1, \cdots, I_r)_{\text{red}}$ as a subscheme of $V(3, n)$ in terms of both matrix coordinates and Plücker coordinates, generalizing Theorem 4.3 and Theorem 4.4. Surprisingly, the extra relations needed to cut out the reduced subscheme structure also have a combinatorial point configuration interpretation visible on the totally nonnegative part, details of which we omit.

Example 6.6. Here is an example that makes the combinatorial nature of the above theorems manifest. Let $A_0 = \emptyset, n = 7$. Consider the boundary divisor $V(12, 3, 4, 5, 6, 7)_{\text{red}}$ in $V(3, 7)$, given by the condition that columns 1 and 2 combined have rank at most 1. If column 1 or 2 is nonzero, they give parallel columns and the incidence condition means that their ratio determines a unique point in $\mathbb{P}^1$. If both columns are zero, the incidence condition poses no constraints on $\mathbb{P}^1$. Thus, π_1 is an isomorphism on the open subscheme where column 1 or 2 is nonzero. If both columns are zero, we have a $\mathbb{P}^1$ worth of fiber. We see that π_1 is surjective, birational, and behaves like a blow-up. Now, fixing a point in $\mathbb{P}^1$ fixes the ratio between column 1 or 2. So the fiber over each point in $\mathbb{P}^1$ is isomorphic to $V(3, 6)$ and we can recover all preimages in $\mathcal{I}_{12}(12, 3, 4, 5, 6, 7)$ by doubling a parallel column according to the prescribed ratio. We see that π_2 is a locally trivial $V(3, 6)$-family over $\mathbb{P}^1$. We comment that the incidence scheme construction easily relates point counting of these boundaries to point counting of $V(3, n)$ over finite fields.

7 The ABCT Variety $V(3,n)$ is a positive geometry

Proof Sketch of Theorem 1.1. To prove statement (3), notice that the boundaries of collinear configurations are positroid varieties, so these properties are known [9, 11]. As for the boundaries of colliding configurations, notice that $V(A_0; I_1, \cdots, I_r)_{\mathrm{red}} \subset V(3,n)$ behaves at least as well as $\mathcal{I}_{st}(A_0; I_1, \cdots, I_r)$ by Proposition 6.4, which behaves as well as the boundary $V(A_0; I_1', \cdots, I_r)_{\mathrm{red}} \subset V(3, n-1)$ by Proposition 6.5. In this way, we decrease both n and the size of I_1 as long as some $\#I_i \geq 2$. The base case of the induction is when each I_i has size 1 so that the boundary component is isomorphic to $V(3,m)$ by forgetting zero columns. A nontrivial argument shows that (3) holds in the base case. Statement (2) is proved by the same spirit. Leveraging the incidence scheme $\mathcal{I}_{st}$, we reduce to the case of $V(3,n)$, which itself requires a nontrivial argument.

To check statement (1), the uniqueness part of the definition is taken care of by statement (3), following Remark 2.5. A priori, we need to verify the recursive property as in Definition 2.3 for each covering relation $C \lessdot C'$ in the poset of iterative boundaries. Similar to the above and by Remark 2.4, we can reduce to the case $C' = V(3,n)$ by a nontrivial argument leveraging the incidence scheme.

For $V(3,n)$, in particular, we need to verify that our form $\Omega_{V(3,n)}$ has no poles outside of boundary components of $V(3,n)$ established in statement (2). We show this by a highly combinatorial argument of intersecting divisors on $\Omega_{V(3,n)}$ obtained by taking preimage of poles of $\Omega_{V(3,n-1)}$. We then go through a combinatorially nontrivial residue computation along the boundary components and verify that they always give the correct canonical form. $\square$

Example 7.1. We first show how the perspective of point configurations gives expected dimensions. The expected dimension can be seen to coincide with the correct dimension by the incidence scheme construction and standard positroid variety theory.

Fixing a standard affine chart of $\mathrm{Gr}(3,7)$ fixes a 3×3 submatrix to identity and thus fixes 3 points on $\mathbb{P}^2$. We freely choose 2 more columns with 6 degrees of freedom, giving a total of 5 points on $\mathbb{P}^2$. These 5 points determine a conic, leaving 1 degree of freedom for each of the remaining 2 points and 1 degree of freedom each for rescaling the columns. Thus, $\dim V(3,7) = 6 + 2 + 2 = 10$. Indeed, $\dim V(3,7) = \dim \mathrm{Gr}(2,7) = 10$.

Codimension 1 boundary components of $V(3,7)$, up to cyclic rotations, are given by

$$\Pi(1,2|3,4,5,6,7), \quad \Pi(1,2,3|4,5,6,7), \quad V(12,3,4,5,6,7)_{\mathrm{red}}$$

The two positroid boundaries give point configurations where a subset of points lie on some line and the complement lie on another line. The 3 points fixed by the chart cannot lie on the same line. Thus, the chart fixes 2 points on one line and 1 point on another line. We choose 1 point freely on $\mathbb{P}^2$ to determine the second line. The remaining 3 points must lie on the union of two fixed lines. The point configuration space is

$2+3\times 1 = 5$-dimensional. Thus, these positroid varieties have dimension $5+4=9$. As for $V(12,3,4,5,6,7)_{\mathrm{red}}$, we fix $M_{\{234\}}$ to identity matrix, which fixes points from columns $2,3,4$. By definition, column 1 gives the same point as column 2 and is also fixed. Among the remaining 3 points, 2 moves freely to determine the conic, on which the third point moves with 1 degree of freedom. The point configuration space has dimension $2\times 2+1=5$, which again gives total dimension $5+4=9$ for the boundary.

We then illustrate the combinatorics behind why the above three are all the codimension 1 boundary components. To pass to a boundary, we take the vanishing of one Plücker coordinate. Due to the *submodular* property of Plücker coordinates on $\mathrm{Gr}_{\geq 0}(3,n)$, we can assume that it is an interval Plücker. Consider $V(3,7)\cap\{p_{123}=0\}$. By Theorem 4.4, $V(3,7)$ is cut out by 7 quartics, one of which is $p_{123}p_{156}p_{246}p_{345} - p_{234}p_{126}p_{135}p_{456}$. Requiring $p_{123}=0$ forces the second monomial to vanish. Set-theoretically, this means the vanishing of some Plücker coordinate among $p_{234}, p_{126}, p_{135}, p_{456}$. Each possibility, combined with Plücker relations, imply more vanishing Plückers from the theory of positroid varieties. For example, the vanishing of p_{123} and p_{234} cuts the union of two positroid varieties, respectively defined by $\mathrm{rk}M_{23}\leq 1$ and $\mathrm{rk}M_{1234}\leq 2$, resulting in two possible sets of more Plückers to vanish. In this way, the vanishing of one Plücker, combined with any of the 7 quartics, results in lots of possible outcomes. In each outcome, we have different sets of vanishing Plückers. The vanishing of these Plückers can then be combined with any of the other 6 quartics to have more possibilities of sets of vanishing Plückers. We have a combinatorial way to keep track of the chain reaction and explosion of possibilities, in order to verify the correct boundaries in the Theorem.

Finally, we illustrate the combinatorics behind showing that $\Omega_{V(3,n)}$ does not have poles outside of the boundary. Proposition 5.2 implies that the poles of $\Omega_{V(3,7)}$ are contained in the union of the preimage of the poles of $\Omega_{V(3,6)}$, under the map of forgetting some column, and the locus defined by the vanishing of a Plücker. Doing this for every column gives that the poles are contained in the intersection of large unions of divisors on $V(3,7)$. The problem has the same flavor as the above, where we keep track of the nontrivial combinatorics of the chain reactions of the vanishing of Plücker coordinates.

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