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A Loehr-Remmel bijection in the $n \times kn$ grid

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Abstract. We extend the pmaj statistic of Loehr and Remmel to labelled Dyck paths in the $n \times kn$ grid, and extend their bijection sending the bistatistic $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$, proving in this way a new combinatorial formula for $\nabla^k e_n$ ($k \geq 1$). At $k = 1$ we recover the original statistic and the original bijection.

Résumé. Nous étendons la statistique pmaj de Loehr et Remmel aux chemins de Dyck étiquetés dans la grille $n \times kn$, et nous étendons leur bijection en transformant la bistatistique $(\mathsf{dinv}, \mathsf{area})$ en $(\mathsf{area}, \mathsf{pmaj})$, démontrant ainsi une nouvelle formule combinatoire pour $\nabla^k e_n$ ($k \geq 1$). Pour $k = 1$, nous retrouvons la statistique et la bijection originales.

Keywords: shuffle theorem, labelled Dyck paths, pmaj, Loehr-Remmel bijection

1 Introduction

In a recent breakthrough [6] Carlsson and Mellit gave a positive solution to the long-standing *shuffle conjecture* [15], which states a combinatorial formula for the Frobenius characteristic of the so-called diagonal harmonics. More precisely, this theorem provides the monomial expansion of the symmetric function ∇e_n, where e_n is the elementary symmetric function of degree n in the variables $x_1, x_2, \dots$, and ∇ is the famous *nabla* operator introduced by Bergeron and Garsia in the 90's. In this formula, to each *labelled Dyck path* in the $n \times n$ grid corresponds a monomial, where the variables $x_1, x_2, \dots$ keep track of the labels, while the variables q and t keep track of the bistatistic (dinv, area).

The bistatistic (dinv, area) has been successfully extended to other combinatorial objects related to Dyck paths, providing an impressive proliferation of theorems and conjectures (see e.g. [7, 5, 4, 19, 3, 18, 20, 21, 10, 12, 9]).

In particular, in [25] Mellit proved a *rational* extension of the *shuffle theorem*, which encompasses a formula for $\nabla^k e_n$ ($k \geq 1$) in terms of labelled Dyck paths in the $n \times kn$ grid.

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The statistic dinv, discovered by Haiman, was originally defined only on "unlabelled" Dyck paths, and together with area provided a combinatorial interpretation of the famous (q,t)-Catalan $\langle \nabla e_n, e_n \rangle$ (here $\langle -, - \rangle$ denotes the Hall scalar product on symmetric functions). At the same time Haglund defined a statistic bounce on Dyck paths, that together with area provided another interpretation of the (q,t)-Catalan. These two interpretations can be proved to be the same by an explicit bijection, usually called ζ, due to Haglund and Loehr [14, 16], sending a Dyck path in the $n \times n$ grid into another one whose bistatistic $(\mathsf{area}, \mathsf{bounce})$ coincides with the bistatistic $(\mathsf{dinv}, \mathsf{area})$ of the original one. Both the statistic bounce and the bijection have been extended first by Loehr [23] to Dyck paths in the $n \times kn$ grid, and then to more general Dyck paths in a rectangular grid, the maps going under the collective name of *sweep maps*, see e.g. [1, 26].

In [24] Loehr and Remmel introduced a new statistic pmaj for labelled Dyck paths in the $n \times n$ grid, as an extension of the bounce to the labelled objects. In this way they provided an alternative combinatorial interpretation of ∇e_n: their formula is in terms of the same objects, but using the bistatistic (area, pmaj). In fact, they proved that the two combinatorial formulas coincide, by defining a remarkable bijection sending the bistatistic $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$. This bijection is an extension of the original ζ map.

Unlike for $(\mathsf{dinv}, \mathsf{area})$, there are not many extensions of the bistatistic $(\mathsf{area}, \mathsf{pmaj})$ to "labelled" objects that are known, even conjecturally (but cf. [11, Sections 2.3 and 2.4]).

In the present article we extend both the statistic pmaj and the bijection of Loehr-Remmel in [24] to labelled Dyck paths in the $n \times kn$ grid ($k \geq 1$). As a corollary, using the rational shuffle theorem of Mellit [25], we get a new formula for the symmetric function $\nabla^k e_n$ in terms of the bistatistic $(\mathsf{area}, \mathsf{pmaj})$. At $k = 1$ we recover the original pmaj and the original bijection in [24].

We like to mention that our extension of the pmaj was actually inspired by the corresponding delay in the *sandpile model* (see [22] for a nice introductory monograph). Indeed, recently, in [8] the authors proved a new combinatorial interpretation of ∇e_n in terms of the sandpile model. Their formula uncovered a surprising bridge between this heavily studied combinatorial dynamical system, first introduced in [2] by Bak, Tang and Wiesenfeld in the context of "self-organized criticality" in statistical mechanics, and the q,t-combinatorics related to Macdonald polynomials, which has the shuffle theorem of Carlsson and Mellit as one of its cornerstones. In [8] the labelled Dyck paths in the $n \times n$ grid are mapped bijectively to *sorted recurrent configurations* of suitable graphs; in this correspondence, the statistic pmaj is sent to a statistic delay on those configurations. Indeed, in a forthcoming article [13] we extend to $\nabla^k e_n$ also the main results of [8], and the pmaj presented here will get sent to the corresponding delay statistic that originally inspired it.

In the rest of the present extended abstract we give all the needed definitions and the precise statements of our results.

2 Labelled Dyck paths

Fix $n, k \in \mathbb{N}$ such that $n, k \geq 1$. We will use the common notation $[n] := \{1, 2, \ldots, n\}$.

Definition 1. A *Dyck path* in the $n \times kn$ grid is a lattice path consisting of *north steps* (going from a lattice point (a, b) to the point $(a, b+1)$) and *east steps* (going from a lattice point (a, b) to $(a+1, b)$) going from $(0, 0)$ to (kn, n) that never goes below the *main diagonal* $y = x/k$. We denote by $\mathsf{Dyck}_{n,kn}$ the set of all Dyck paths in the $n \times kn$ grid.

The *diagram* of a Dyck path is its picture in the $n \times kn$ square grid that bounds it.

The diagram of a Dyck path in $\mathsf{Dyck}_{7,21}$ is shown in Figure 1. The squares intersecting the main diagonal are shaded in yellow.

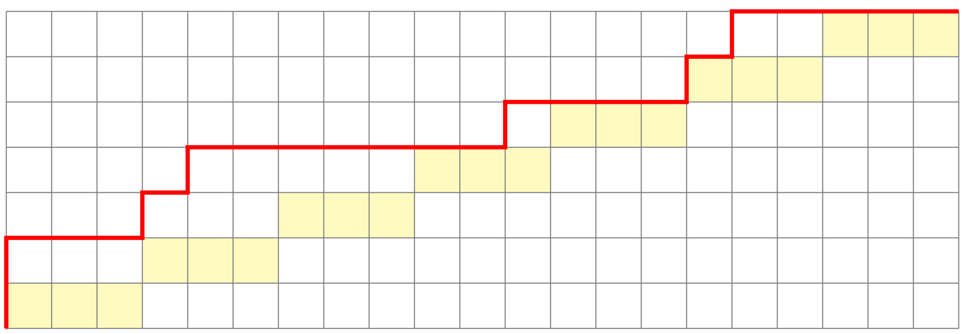

Figure 1: A Dyck path in $\mathsf{Dyck}_{7,21}$.

Given a Dyck path $D \in \mathsf{Dyck}_{n,kn}$, we will refer to the horizontal and vertical sequences of unit squares in its diagram as the *rows* and the *columns* of D, respectively. We count columns increasingly from left to right, and rows from bottom to top.

Definition 2. A *labelled* Dyck path in the $n \times kn$ grid is a Dyck path $D \in \mathsf{Dyck}_{n,kn}$ where each north step is labelled by a positive integer, so that (1) the labels of north steps on the same vertical line have increasing labels when read from bottom to top, and (2) the set of n labels of D is precisely $[n]$. The *diagram* of a labelled Dyck path is the diagram of the associated Dyck path with the label of each vertical step appearing in the unit square to its right. For every $i \in [n]$ we denote by $\ell_i(D)$ the label of the vertical step in row i, and by[1] $f_D(i)$ the column in which the label i appears in the diagram of D. Finally, we denote by $\mathsf{LDyck}_{n,kn}$ the set of all labelled Dyck paths in the $n \times kn$ grid.

The diagram of a labelled Dyck path D in $\mathsf{LDyck}_{7,21}$ is shown in Figure 2; its labels are $\ell_1(D) = 2$, $\ell_2(D) = 4$, $\ell_3(D) = 6$, $\ell_4(D) = 7$, $\ell_5(D) = 1$, $\ell_6(D) = 5$, and $\ell_7(D) = 3$, while $f_D(1) = 12$, $f_D(2) = 1$, $f_D(3) = 17$, $f_D(4) = 1$, $f_D(5) = 16$, $f_D(6) = 4$, and $f_D(7) = 5$.

[1]For $k = 1$ the functions $i \mapsto f_D(i)$ are *parking functions*, and $D \mapsto f_D$ is a bijection between labelled Dyck paths in the $n \times n$ grid and parking functions of size n.

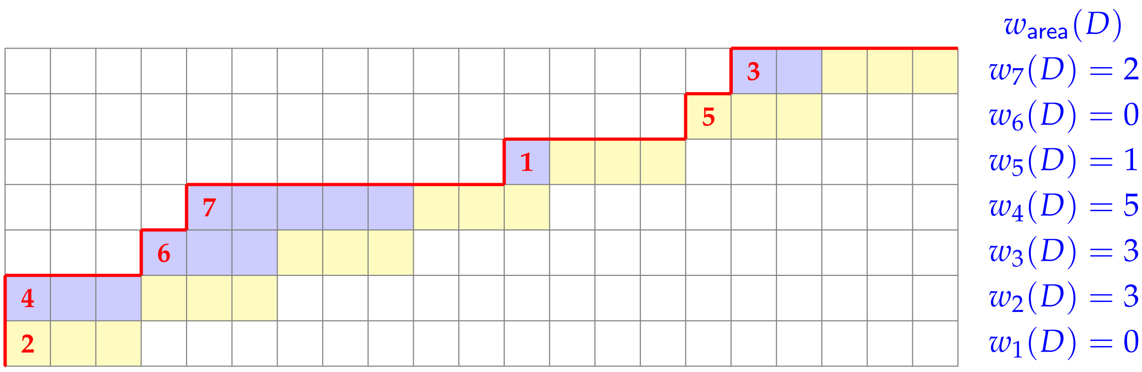

Figure 2: A labelled Dyck path D in $\mathsf{LDyck}_{7,21}$ and its area word. The unit squares contributing to the area are shaded in blue.

2.1 The statistic area

The statistic area is classical.

Definition 3. Given $D \in \mathsf{Dyck}_{n,kn}$, we define its *area word* as the n-tuple

$$w_{\mathsf{area}}(D) := (w_1(D), \dots, w_n(D))$$

where for each $i \in [n]$

$$w_i(D) := \text{\# of integer squares in the } i^{\text{th}}\text{-row between the path and the main diagonal.}$$

Then the *area* of $D \in \mathsf{Dyck}_{n,k}$ is

$$\mathsf{area}(D) := w_1(D) + \cdots + w_n(D).$$

Given $D \in \mathsf{LDyck}_{n,kn}$, its *area word* and its *area* are defined as the ones of the corresponding Dyck path (i.e. ignoring the labels).

Example 1. The area word of the labelled Dyck path D in Figure 2 is shown in the picture, i.e. $w_{\mathsf{area}}(D) = (0, 3, 3, 5, 1, 0, 2)$, hence its area is $\mathsf{area}(D) = 0 + 3 + 3 + 5 + 1 + 0 + 2 = 14$.

2.2 The statistic dinv

For the statistic dinv we will use a reformulation of the geometric description given in [17], which consists of two contributions, tdinv and dinvcorr.

Definition 4. Given a labelled Dyck path $D \in \mathsf{LDyck}_{n,kn}$, for every $i \in [n]$ we set

$$a_{\ell_i(D)}(D) := w_i(D).$$

For example, for D in Figure 2 we have $a_1(D) = w_5(D) = 1$, $a_2(D) = w_1(D) = 0$, $a_3(D) = w_7(D) = 2$, $a_4(D) = w_2(D) = 3$, $a_5(D) = w_6(D) = 0$, $a_6(D) = w_3(D) = 3$, and $a_7(D) = w_4(D) = 5$.

Now we say that a pair $(i, j) \in [n] \times [n]$ of labels is a *temporary diagonal inversion* of D if any of the following two conditions is satisfied:

(A) $f_D(i) < f_D(j)$, $i < j$ and $a_j(D) - k < a_i(D) \leq a_j(D)$.

(B) $f_D(i) < f_D(j)$, $i > j$ and $a_j(D) < a_i(D) \leq a_j(D) + k$.

We denote by $\mathsf{TDinv}(D)$ the multiset[2] of all temporary diagonal inversions of D, and we define the tdinv of D as the cardinality of $\mathsf{TDinv}(D)$, i.e.

$$\mathsf{tdinv}(D) := \#\mathsf{TDinv}(D).$$

This definition is better understood by the next geometric remark.

Definition 5. Given a north step of a Dyck path $D \in \mathsf{Dyck}_{n,kn}$ with endpoints (a, b) and $(a, b+1)$ we define its *main shadow* as the region of $\mathbb{R}^2$ whose points (x, y) satisfy the inequalities

$$\begin{cases} x \leq a \\ \frac{1}{k}(x-a) < y - b \leq \frac{1}{k}(x-a) + 1 \end{cases}, \tag{2.1}$$

while its *upper shadow* is the region of $\mathbb{R}^2$ whose points (x, y) satisfy the inequalities

$$\begin{cases} x \leq a \\ \frac{1}{k}(x-a) < y - b - 1 \leq \frac{1}{k}(x-a) + 1 \end{cases}. \tag{2.2}$$

These shadows are shown in Figure 3; notice that they are parallel to the main diagonal $y = x/k$, and they are half open, as indicated by the dotted borders.

Remark 1. Conditions (A) and (B) correspond to the intersection of the north endpoint of the north step labelled i with the main shadow and the upper shadow of (the north step labelled) j, respectively: see Figure 3.

Example 2. For the labelled Dyck path D in Figure 2 we have[3] (cf. Figure 4)

$$\mathsf{TDinv}(D) = \left\{\!\!\left\{ \begin{array}{c} (1,3), (2,3), (2,5), (4,1), (4,3), (4,6), \\ (4,7), (6,1), (6,3), (6,5), (6,7), (7,3) \end{array} \right\}\!\!\right\}.$$

[2] Notice that $\mathsf{TDinv}(D)$ will always be a set, but we consider it a multiset since later we will add it to the multiset $\mathsf{Dinvcorr}(D)$.

[3] We denote the multisets with double curly brackets.

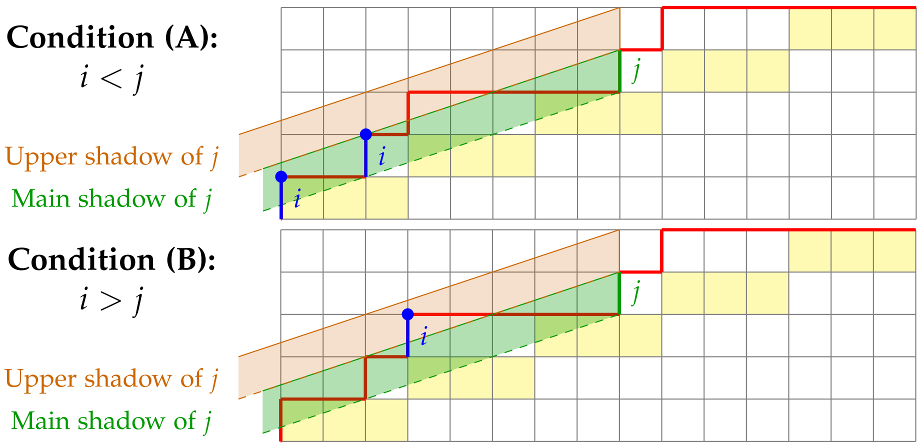

Figure 3: Graphical depiction of conditions (A) and (B) in the definition of $\mathsf{TDinv}(D)$.

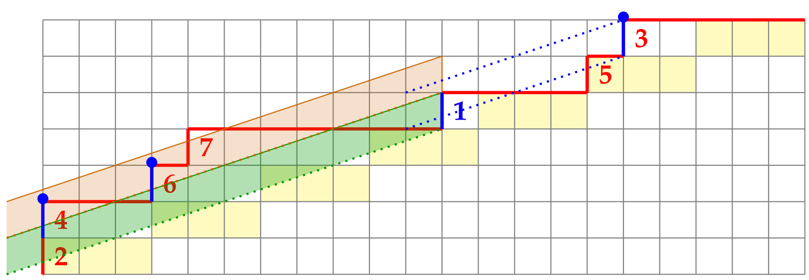

Figure 4: Illustration of the temporary diagonal inversions involving the north step labelled 1.

Definition 6. Given a labelled Dyck path $D \in \mathsf{LDyck}_{n,kn}$, for every east step of its diagram with endpoints (c, d) and $(c+1, d)$ we define its *main shadow* as the region of $\mathbb{R}^2$ whose points (x, y) satisfy the inequalities:

$$\begin{cases} y \leq d \\ \frac{1}{k}(x-c) < y-d < \frac{1}{k}(x-c-1). \end{cases}$$

Notice that the main shadow of every east step of D will intersect at least one north step of D to its west.

The *east step labelling* of D consists of labelling every east step of D with the label of the first north step encountered by its shadow, moving from the east step towards west.

Example 3. In Figure 5 we computed the east step labelling of the labelled Dyck path D in Figure 2.

Definition 7. Given $D \in \mathsf{LDyck}_{n,kn}$, consider its east step labelling. We define the *multiset of diagonal inversion corrections*, denoted $\mathsf{Dinvcorr}(D)$, as the multiset of all pairs (λ, μ)

where λ is the label of an east step of D entirely[4] contained in the main shadow of a north step of D labelled μ. We define $\mathsf{dinvcorr}(D)$ as the cardinality of the multiset $\mathsf{Dinvcorr}(D)$.

Example 4. Consider the labelled Dyck path D of Figure 2, whose east step labelling is shown in Figure 5. We compute[3]

$$\mathsf{Dinvcorr}(D) = \left\{\!\!\left\{ \begin{array}{c} (1,5),(1,5),(2,5),(2,5),(2,1),(6,1),(4,1), \\ (1,3),(6,3),(6,3),(4,3),(4,3),(4,6),(4,6) \end{array} \right\}\!\!\right\},$$

hence $\mathsf{dinvcorr}(D) = 14$.

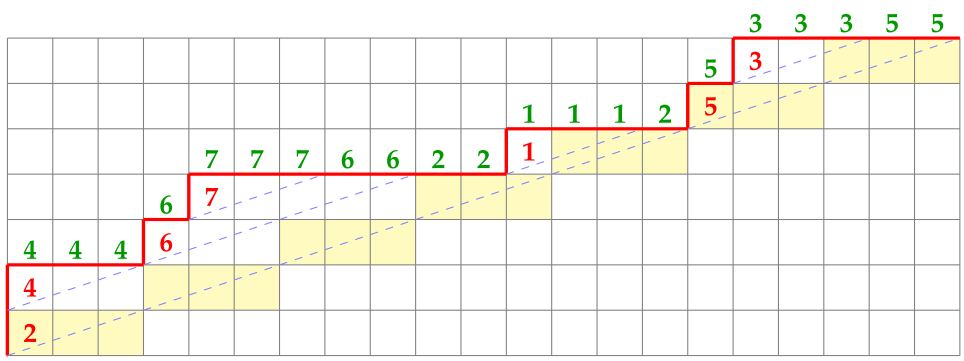

Figure 5: Illustration of the east step labelling of $D \in \mathsf{LDyck}_{7,21}$ from Figure 2.

Definition 8. Given $D \in \mathsf{LDyck}_{n,kn}$, we define the multiset $\mathsf{Dinv}(D)$ of diagonal inversions of D as the sum of the multisets $\mathsf{TDinv}(D)$ and $\mathsf{Dinvcorr}(D)$. We define the dinv of D as the cardinality of $\mathsf{Dinv}(D)$, i.e.

$$\mathsf{dinv}(D) := \mathsf{tdinv}(D) + \mathsf{dinvcorr}(D).$$

Example 5. For the labelled Dyck path D in Figure 2, we have[3]

$$\mathsf{Dinv}(D) = \left\{\!\!\left\{ \begin{array}{c} (2,5),(2,5),(2,5), \\ (1,5),(1,5),(2,1), \\ (1,3),(2,3),(1,3), \\ (4,1),(4,3),(4,1),(4,3),(4,3), \\ (6,1),(6,3),(4,6),(6,5),(6,1),(6,3),(6,3),(4,6),(4,6), \\ (4,7),(6,7),(7,3) \end{array} \right\}\!\!\right\}, \qquad (2.3)$$

so $\mathsf{dinv}(D) = \mathsf{tdinv}(D) + \mathsf{dinvcorr}(D) = 12 + 14 = 26$.

[4]Recall that the main shadow of a north step is half open.

Definition 9. Given $D \in \mathsf{LDyck}_{n,kn}$, we define its *diagonal reading word* $\sigma_{\mathsf{diag}}(D)$ as the word obtained by reading the labels of the north steps of D by sweeping its diagram with diagonals parallel to $y = x/k$, from bottom to top, reading the labels along the same diagonal from left to right. Notice that $\sigma_{\mathsf{diag}}(D)$ is a permutation in $\mathfrak{S}_n$ (in one-line notation).

For example, for the labelled Dyck path D in Figure 2 we have $\sigma_{\mathsf{diag}}(D) = 2513467$.

Recall that given $S \subseteq [n-1]$, the *(Gessel) fundamental quasisymmetric function* $L_{n,S}(x)$ is defined as

$$L_{n,S} = L_{n,S}(x) := \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j = i_{j+1} \implies j \notin S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Finally, the *descent set* of a permutation $\sigma \in \mathfrak{S}_n$ is

$$\mathsf{Des}(\sigma) := \{ i \mid \sigma(i) > \sigma(i+1) \} \subseteq [n-1].$$

We can now state a special case of the so called *rational shuffle theorem* of Mellit.

Theorem 1 ([25]). *We have*

$$\nabla^k e_n = \sum_{D \in \mathsf{LDyck}_{n,kn}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} L_{n, \mathsf{Des}(\sigma_{\mathsf{diag}}(D)^{-1})}.$$

2.3 The statistic pmaj

Our definition of the pmaj statistic for labelled Dyck paths in $\mathsf{LDyck}_{n,kn}$ is new for $k > 1$, while for $k = 1$ we recover the original one defined in [24].

Definition 10. Consider a labelled Dyck path $D \in \mathsf{LDyck}_{n,kn}$. We define a word in the alphabet $[n]$, denoted $\sigma_{\mathsf{pmaj}}(D) := \sigma_1 \sigma_2 \dots \sigma_{kn}$, such that every letter appears k times.
Let $B'_0 := \varnothing$ be the empty multiset[3] and $\sigma_0 := n+1$. Then iterate the following steps for $m = 1, 2, \dots, kn$.

i) Set[5] $B_m := B'_{m-1} + \{\!\{ \lambda^k \mid f_D(\lambda) = m \}\!\} = B'_{m-1} + \{\!\{ \underbrace{\lambda, \lambda, \dots, \lambda}_{k \text{ times}} \mid f_D(\lambda) = m \}\!\}$.

ii) Consider $X_m := \{\!\{ a \in B_m \mid a < \sigma_{m-1} \}\!\}$. There are two cases: if $X_m \neq \varnothing$ then set $\sigma_m := \max(X_m)$, otherwise set $\sigma_m := \max(B_m)$.

iii) Remove an occurrence of σ_m from the multiset B_m to obtain $B'_m := B_m - \{\!\{ \sigma_m \}\!\}$.

[5] We will use the exponential notation for the elements of multisets, indicating the multiplicities with the exponents: for example we will write $\{\!\{ 2^3, 4, 5^2 \}\!\}$ for $\{\!\{ 2, 2, 2, 4, 5, 5 \}\!\}$.

It is easy to check that this algorithm is well defined: from the condition on D of being a labelled Dyck path we have that $B_m \neq \varnothing$ for every $m \in [kn]$.
Now for each label $\lambda \in [n]$ define its *pmaj contribution* as

$$p_\lambda(D) := \#\text{ of weak ascents before the leftmost occurrence of } \lambda \text{ in } \sigma_{\mathsf{pmaj}}(D).$$

Finally, we can define the *pmaj statistic* for D by summing these contributions:

$$\mathsf{pmaj}(D) := p_1(D) + p_2(D) + \cdots + p_n(D).$$

Example 6. In Table 1 we record in a table the steps of the algorithm that computes the pmaj of the labelled Dyck path D in Figure 6: we have $\mathsf{pmaj}(D) = 14$ (and $\mathsf{area}(D) = 26$).

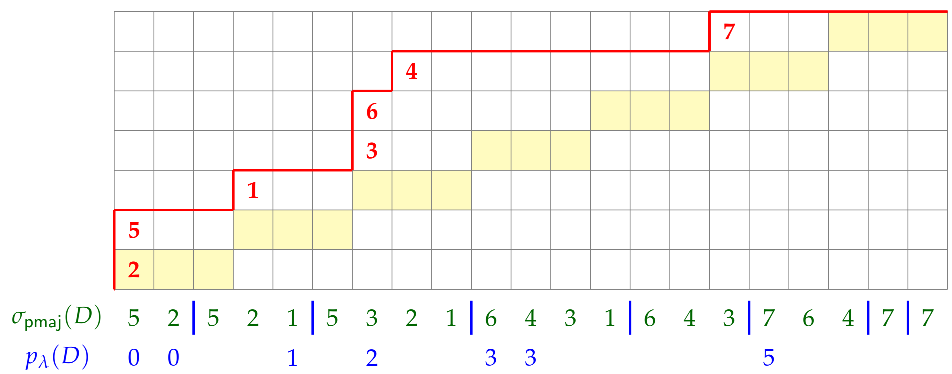

Figure 6: The diagram of $\phi_7^{(3)}(D)$ for D in Figure 2, its $\sigma_{\mathsf{pmaj}}(D)$ word (the bars indicate the weak ascents), and its pmaj contributions.

m	B_m	X_m	σ_m	B'_m
1	$\{\{2^3,5^3\}\}$	$\{\{2^3,5^3\}\}$	5	$\{\{2^3,5^2\}\}$
2	$\{\{2^3,5^2\}\}$	$\{\{2^3\}\}$	2	$\{\{2^2,5^2\}\}$
3	$\{\{2^2,5^2\}\}$	$\varnothing$	5	$\{\{2^2,5\}\}$
4	$\{\{1^3,2^2,5\}\}$	$\{\{1^3,2^2\}\}$	2	$\{\{1^3,2,5\}\}$
5	$\{\{1^3,2,5\}\}$	$\{\{1^3\}\}$	1	$\{\{1^2,2,5\}\}$
6	$\{\{1^2,2,5\}\}$	$\varnothing$	5	$\{\{1^2,2\}\}$
7	$\{\{1^2,2,3^3,6^3\}\}$	$\{\{1^2,2,3^3\}\}$	3	$\{\{1^2,2,3^2,6^3\}\}$
8	$\{\{1^2,2,3^2,4^3,6^3\}\}$	$\{\{1^2,2\}\}$	2	$\{\{1^2,3^2,4^3,6^3\}\}$
9	$\{\{1^2,3^2,4^3,6^3\}\}$	$\{\{1^2\}\}$	1	$\{\{1,3^2,4^3,6^3\}\}$
10	$\{\{1,3^2,4^3,6^3\}\}$	$\varnothing$	6	$\{\{1,3^2,4^3,6^2\}\}$
11	$\{\{1,3^2,4^3,6^2\}\}$	$\{\{1,3^2,4^3\}\}$	4	$\{\{1,3^2,4^2,6^2\}\}$
12	$\{\{1,3^2,4^2,6^2\}\}$	$\{\{1,3^2\}\}$	3	$\{\{1,3,4^2,6^2\}\}$
13	$\{\{1,3,4^2,6^2\}\}$	$\{\{1\}\}$	1	$\{\{3,4^2,6^2\}\}$
14	$\{\{3,4^2,6^2\}\}$	$\varnothing$	6	$\{\{3,4^2,6\}\}$
15	$\{\{3,4^2,6\}\}$	$\{\{3,4^2\}\}$	4	$\{\{3,4,6\}\}$
16	$\{\{3,4,6,7^3\}\}$	$\{\{3\}\}$	3	$\{\{4,6,7^3\}\}$
17	$\{\{4,6,7^3\}\}$	$\varnothing$	7	$\{\{4,6,7^2\}\}$
18	$\{\{4,6,7^2\}\}$	$\{\{4,6\}\}$	6	$\{\{4,7^2\}\}$
19	$\{\{4,7^2\}\}$	$\{\{4\}\}$	4	$\{\{7^2\}\}$
20	$\{\{7^2\}\}$	$\varnothing$	7	$\{\{7\}\}$
21	$\{\{7\}\}$	$\varnothing$	7	$\varnothing$

Table 1: Computation of the pmaj statistic for the labelled Dyck path in Figure 6.

3 The map $\phi_n^{(k)}$

To define our bijection, we need one more definition.

Definition 11. Consider $D \in \mathsf{LDyck}_{n,kn}$ and let $\sigma_{\mathsf{diag}}(D) = \sigma_1\sigma_2\cdots\sigma_n$ be its diagonal reading word. We define the *dinv contribution* of a label $\lambda \in [n]$ by

$$d_\lambda(D) := \#\{\{(\sigma_i, \sigma_j) \in \mathsf{Dinv}(D) \mid \sigma_{\max(i,j)} = \lambda\}\}.$$

Clearly we have $\sum_{\lambda\in[n]} d_\lambda(D) = \mathsf{dinv}(D)$.

For example, for the labelled Dyck path D in Figure 2, $\sigma_{\mathsf{diag}}(D) = 2513467$, and we compute $d_2(D) = 0$, $d_5(D) = 3$, $d_1(D) = 3$, $d_3(D) = 3$, $d_4(D) = 5$, $d_6(D) = 9$, and $d_7(D) = 3$: cf. Equation (2.3), where each row shows the pairs contributing to a $\lambda \in [7] \setminus \{2\}$ (2 is the first letter of $\sigma_{\mathsf{diag}}(D)$ hence necessarily $d_2(D) = 0$).

Definition 12. Given $D \in \mathsf{LDyck}_{n,kn}$, define $\phi_n^{(k)}(D)$ as the unique labelled Dyck path in $\mathsf{LDyck}_{n,kn}$ such that

$$f_{\phi_n^{(k)}(D)}(\sigma_i) = k(i-1) - d_{\sigma_i}(D) + 1\,,$$

where $\sigma_{\mathsf{diag}}(D) = \sigma_1\sigma_2\cdots\sigma_n$.

This definition is better understood by an example.

Example 7. Consider the labelled Dyck path D in Figure 2 (here $n = 7$ and $k = 3$). We already computed $\sigma_{\mathsf{diag}}(D) = 2513467$, and its dinv contributions $d_2(D) = 0$, $d_5(D) = 3$, $d_1(D) = 3$, $d_3(D) = 3$, $d_4(D) = 5$, $d_6(D) = 9$, and $d_7(D) = 3$. The definition is simply saying that label 2 has to occur in $\phi_7^{(3)}(D)$ in column $3(1-1) - d_2(D) + 1 = 1$, label 5 in column $3(2-1) - d_5(D) + 1 = 1$, label 1 in column $3(3-1) - d_1(D) + 1 = 4$, label 3 in column $3(4-1) - d_3(D) + 1 = 7$, label 4 in column $3(5-1) - d_4(D) + 1 = 8$, label 6 in column $3(6-1) - d_6(D) + 1 = 7$, and label 7 in column $3(7-1) - d_7(D) + 1 = 16$. The diagram of $\phi_7^{(3)}(D)$ is shown in Figure 6.

The main result of this article is the following theorem.

Theorem 2. *The correspondence $D \mapsto \phi_n^{(k)}(D)$ is a well-defined bijection of $\mathsf{LDyck}_{n,kn}$ into itself such that*

$$\mathsf{area}(\phi_n^{(k)}(D)) = \mathsf{dinv}(D) \quad \text{and} \quad \mathsf{pmaj}(\phi_n^{(k)}(D)) = \mathsf{area}(D). \tag{3.1}$$

For example, Equation (3.1) can be checked for D in Figure 2 (cf. Examples 1, 5 and 6).
To state our main corollary, we need our last definition.

Definition 13. Given $D \in \mathsf{LDyck}_{n,kn}$, we define its *row word* as the word $\sigma_{\mathsf{row}}(D)$ obtained by reading the labels in the diagram of D by rows, from bottom to top.

For example, the labelled Dyck path D in Figure 6 has $\sigma_{\mathsf{row}}(D) = 2513647$.

The following version of the shuffle theorem for $\nabla^k e_n$ can be deduced from Theorem 2 and Theorem 1 with little more work.

Corollary 1. *We have*

$$\nabla^k e_n = \sum_{D \in \mathsf{LDyck}_{n,kn}} q^{\mathsf{area}(D)} t^{\mathsf{pmaj}(D)} L_{n, \mathsf{Des}(\sigma_{\mathsf{row}}(D)^{-1})}.$$

We leave as an open problem to find an extension of the pmaj statistic to labelled Dyck paths in the $n \times m$ grid that recovers the formulas in [25].

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References

[1] D. Armstrong, N. A. Loehr, and G. S. Warrington. "Sweep maps: a continuous family of sorting algorithms". *Adv. Math.* **284** (2015), pp. 159–185. DOI.

[2] P. Bak, C. Tang, and K. Wiesenfeld. "Self-organized criticality: An explanation of the 1/f noise". *Phys. Rev. Lett.* **59** (1987), pp. 381–384. DOI.

[3] F. Bergeron, J. Haglund, A. Iraci, and M. Romero. "The super nabla operator". Preprint, arXiv:2303.00560 [math.CO] (2023). 2023.

[4] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "Dens, nests and the Loehr-Warrington conjecture". *J. Amer. Math. Soc.* **38**.4 (2025), pp. 1049–1106. DOI.

[5] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "A shuffle theorem for paths under any line". *Forum Math. Pi* **11** (2023), Paper No. e5, 38. DOI.

[6] E. Carlsson and A. Mellit. "A proof of the shuffle conjecture". *J. Amer. Math. Soc.* **31**.3 (2018), pp. 661–697. DOI.

[7] E. Carlsson and A. Mellit. "A combinatorial formula for the nabla operator". *Compos. Math.* **161**.4 (2025), pp. 800–830. DOI.

[8] M. D'Adderio, M. Dukes, A. Iraci, A. Lazar, Y. Le Borgne, and A. Vanden Wyngaerd. "Shuffle theorems and sandpiles". *Comm. Math. Phys.* **406**.4 (2025), Paper No. 83, 23. DOI.

[9] M. D'Adderio, A. Iraci, Y. Le Borgne, M. Romero, and A. Vanden Wyngaerd. "Tiered trees and Theta operators". *Int. Math. Res. Not. IMRN* 24 (2023), pp. 20748–20783. DOI.

[10] M. D'Adderio, A. Iraci, and A. Vanden Wyngaerd. "Theta operators, refined delta conjectures, and coinvariants". *Adv. Math.* **376** (2021), Paper No. 107447, 59. DOI.

[11] M. D'Adderio, A. Iraci, and A. Vanden Wyngaerd. "Decorated Dyck paths, polyominoes, and the delta conjecture". *Mem. Amer. Math. Soc.* **278**.1370 (2022), pp. vii+119. DOI.

[12] M. D'Adderio and A. Mellit. "A proof of the compositional delta conjecture". *Adv. Math.* **402** (2022), Paper No. 108342, 17. DOI.

[13] M. D'Adderio and A. Sgubin. "Sorted sandpiles and Dyck paths in the $n \times kn$ grid". *to appear* (2026+).

[14] J. Haglund. "Conjectured statistics for the q, t-Catalan numbers". *Adv. Math.* **175**.2 (2003), pp. 319–334. DOI.

[15] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. "A combinatorial formula for the character of the diagonal coinvariants". *Duke Math. J.* **126**.2 (2005), pp. 195–232. DOI.

[16] J. Haglund and N. Loehr. "A conjectured combinatorial formula for the Hilbert series for diagonal harmonics". *Discrete Math.* **298**.1-3 (2005), pp. 189–204. DOI.

[17] A. Hicks and E. Leven. "A simpler formula for the number of diagonal inversions of an (m, n)-parking function and a returning fermionic formula". *Discrete Math.* **338**.3 (2015), pp. 48–65. DOI.

[18] A. Iraci, R. Pagaria, and G. Paolini. "Falling stars: a fall-decorated rational shuffle theorem". Preprint, arXiv:2508.20935 [math.CO] (2025). 2025.

[19] A. Iraci and M. Romero. "Delta and theta operator expansions". *Forum Math. Sigma* **12** (2024), Paper No. e30, 37. DOI.

[20] D. Kim, S. J. Lee, and J. Oh. "Shuffle formula in science fiction for Macdonald polynomials". *Compos. Math.* **161**.5 (2025), pp. 1075–1127. DOI.

[21] D. Kim and J. Oh. "Extending the science fiction and the Loehr–Warrington formula". Preprint, arXiv:2409.01041 [math.CO] (2024). 2024.

[22] C. J. Klivans. *The mathematics of chip-firing*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2019, pp. xii+295.

[23] N. A. Loehr. "Conjectured statistics for the higher q, t-Catalan sequences". *Electron. J. Combin.* **12** (2005), Research Paper 9, 54. DOI.

[24] N. A. Loehr and J. B. Remmel. "Conjectured combinatorial models for the Hilbert series of generalized diagonal harmonics modules". *Electron. J. Combin.* **11**.1 (2004), Research Paper 68, 64. DOI.

[25] A. Mellit. "Toric braids and (m, n)-parking functions". *Duke Math. J.* **170**.18 (2021), pp. 4123–4169. DOI.

[26] H. Thomas and N. Williams. "Sweeping up zeta". *Selecta Math. (N.S.)* **24**.3 (2018), pp. 2003–2034. DOI.