Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #83, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Generalized orthodontia, punched tableaux, and Lascoux expansion of sheared characters

Avery St. Dizier*[1], Linus Setiabrata†[2], and Tianyi Yu‡[3]

[1] *Michigan State University, East Lansing, MI 48824, USA*
[2] *Massachusetts Institute of Technology, Cambridge, MA 02139, USA*
[3] *Laboratoire d'Algèbre, de Combinatoire et d'Informatique Mathématique, Université du Québec à Montréal, Montreal QC, Canada*

Abstract. We introduce double inhomogeneous analogues $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$ of characters of flagged Schur modules based on Magyar's orthodontia formula for diagrams, giving a new formula for double Grothendieck polynomials. The polynomials $\mathscr{G}_D$ refine *sheared characters*. Using Lascoux's theory of *punched tableaux*, we establish an inhomogeneous extension of Reiner–Shimozono's result that multiplication by $x_1 \cdots x_k$ preserves key positivity, and prove that the reverse-complement $\mathscr{R}_D^{(m,n)}(\mathbf{x})$ of a sheared character expands positively in the basis of Lascoux polynomials.

Keywords: Grothendieck polynomial, flagged Schur module, punched tableaux, Lascoux polynomial, peelable tableaux

1 Introduction

Introduced by Lascoux and Schützenberger in [20], Schubert polynomials serve as distinguished representatives of Schubert varieties in the cohomology of the flag variety of $\mathbb{C}^n$. These polynomials have rich combinatorial structure [3, 4, 8, 13, 15, 17] and play a central role in algebraic combinatorics.

Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$ are indexed by permutations $w \in S_n$, which can be understood graphically through their Rothe diagram

$$D(w) = \{(i, j) \mid 1 \leq i, j \leq n \text{ with } w(i) > j \text{ and } w^{-1}(j) > i\}.$$

A related family of polynomials is that of *key polynomials* κ_α, characters of the type A Demazure modules indexed by weak compositions. Compositions can be viewed graphically through the *skyline diagram* $D(\alpha) = \{(i, j) \mid 1 \leq j \leq \alpha_i\}$. More generally, any diagram $D \subseteq [n] \times [n]$ indexes the corresponding *flagged Schur module* $\mathcal{M}_D$, a representation of the Borel subgroup of GL_n consisting of the upper triangular matrices.

*Email: stdizier@msu.edu.
†Email: setia@mit.edu.
‡Email: yu.tianyi@uqam.ca.

When $D = D(w)$ is a Rothe diagram, the character χ_D of the representation $\mathcal{M}_D$ equals the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ [16]. Similarly any skyline diagram $D = D(\alpha)$ has $\chi_D = \kappa_\alpha(\mathbf{x})$ [25]. The study of characters χ_D of general diagrams has shed light on the combinatorics of Schubert and related polynomials [1, 6, 7, 10, 11].

Generalized orthodontia. When D is *%-avoiding*, Magyar showed that the dual representation of $\mathcal{M}_D$ can be realized geometrically as the space of sections of a certain line bundle on a variety [22]. Through this interpretation, Magyar proved that χ_D is given by a Demazure-like character formula:

$$\chi_D = \omega_1^{k_1} \cdots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1} \pi_{i_2}(\omega_{i_2}^{m_2}(\cdots \pi_{i_\ell}(\omega_{i_\ell}^{m_\ell}) \cdots))). \tag{1.1}$$

Here, $\omega_i = x_1 \cdots x_i$ is a fundamental weight, $\pi_i = \partial_i x_i$ is a Demazure operator, and

$$\mathbf{k}(D) = (k_1, \ldots, k_n), \qquad \mathbf{i}(D) = (i_1, \ldots, i_\ell), \qquad \mathbf{m}(D) = (m_1, \ldots, m_\ell)$$

are combinatorial data comprising the *orthodontic sequence* of D, which builds a %-avoiding diagram from "smaller" %-avoiding diagrams. This formula is a powerful tool, inviting inductive arguments and admitting an explicit tableaux realization ([22, Proposition 16]).

We extend the orthodontia formula to generalizations of Schubert and key polynomials. Among the myriad such generalizations, we focus in two orthogonal directions. Studying K-theory instead of cohomology leads to inhomogeneous analogues called Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x})$ and Lascoux polynomials $\mathfrak{L}_\alpha(\mathbf{x})$. These have the Schubert and key polynomials respectively as their lowest-degree components. Incorporating the torus action on the flag variety leads to double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$; the literature expects but currently lacks a theory of "double Lascoux polynomials". There is considerable interest in extending the current understanding of Schubert polynomials to these generalizations [5, 9, 11, 14, 18, 24, 28].

While there is no known K-theoretic or equivariant analogue of the representation $\mathcal{M}_D$, the combinatorics of orthodontia are a promising avenue for generalization. The inductive structure allows one to proceed without the guiding geometry. Given a %-avoiding diagram D, we describe an algorithm to compute its *double orthodontic sequence*

$$\mathbf{K}(D) = (K_1, \ldots, K_n), \quad \mathbf{i}(D) = (i_1, \ldots, i_\ell), \quad \mathbf{j}(D) = (j_1, \ldots, j_\ell), \quad \mathbf{M}(D) = (M_1, \ldots, M_\ell).$$

Definition 1.1. *For a %-avoiding diagram D with double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$, write*

$$\mathscr{G}_D(\mathbf{x}, \mathbf{y}) := \overline{\omega}_1^{K_1} \overline{\omega}_2^{K_2} \cdots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1, j_1}(\omega_{i_1}^{M_1} \overline{\pi}_{i_2, j_2}(\omega_{i_2}^{M_2} \cdots \overline{\pi}_{i_\ell, j_\ell}(\omega_{i_\ell}^{M_\ell}) \cdots)),$$

and write

$$\mathscr{S}_D(\mathbf{x}, \mathbf{y}) := \omega_1^{K_1} \omega_2^{K_2} \cdots \omega_n^{K_n} \pi_{i_1, j_1}(\omega_{i_1}^{M_1} \pi_{i_2, j_2}(\omega_{i_2}^{M_2} \cdots \pi_{i_\ell, j_\ell}(\omega_{i_\ell}^{M_\ell}) \cdots)) \tag{1.2}$$

for the lowest-degree component of $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$.

The notation $\overline{\pi}_{i,j}$, $\overline{\omega}^S$, $\pi_{i,j}$, and ω^S are defined in Definition 2.3. See Figure 2 for an example. The following result gives the analogue of Schubert and key polynomials appearing as characters of flagged Schur modules.

Theorem 1.2 ([27, Theorem 1.1])**.** *When $D = D(w)$ is the Rothe diagram of a permutation, $\mathscr{G}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{G}_w(\mathbf{x}, \mathbf{y})$, the double Grothendieck polynomial. Consequently, the lowest degree component is $\mathscr{S}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{S}_w(\mathbf{x}, -\mathbf{y})$, the (positive) double Schubert polynomial.*

Lascoux positivity. A central problem in algebraic combinatorics is the study of basis expansions: given a natural family of polynomials, one seeks to express them in terms of a distinguished basis. In the ideal case, the coefficients are nonnegative integers. Such positivity phenomena can signal hidden combinatorial or geometric structure, with the coefficients encoding enumeration of tableaux, lattice points of polytopes, dimensions of spaces, etc. A classical source of examples is the Schur positivity of symmetric functions, with the celebrated Littlewood–Richardson rule as the bedrock example.

The combinatorics of Schubert positivity and key positivity remains an area of active research [2, 12, 21, 25, 26, 29, 30]. Magyar's orthodontia formula (1.1) can be used to prove key positivity of Schubert polynomials: evaluating (1.1) from right to left,

- Each step $\pi_i(\cdot)$ preserves key positivity of its input since $\pi_i(\kappa_\alpha) \in \{\kappa_\alpha, \kappa_{s_i \cdot \alpha}\}$.
- Each step $\omega_i \times \cdot$ preserves key positivity of its input ([26, Theorem 20]).

For $m, n \in \mathbb{N}$, recall the *reverse complementation involution* $r_{m,n}$ defined on polynomials $f(\mathbf{x}) \in \mathbb{Z}[x_1, \cdots, x_n]$ where each variable has degree at most m by

$$r_{m,n}(f) := (x_1 \cdots x_n)^m f(x_n^{-1}, \cdots, x_1^{-1}).$$

This involution was introduced by the third author in [30] to describe the top homogeneous component of any Lascoux polynomial. This involution intertwines many operators related to the Schubert, key, and Grothendieck polynomials ([30, Lemma 3.4]).

Using the generalized orthodontia formula (Theorem 1.2), we noticed that

$$\begin{aligned} \mathscr{R}_D^{(m,n)} &:= r_{m,n}(\mathscr{S}_D(\mathbf{x}; -\mathbf{1})) \\ &= x_1^m \cdots x_n^m \mathscr{S}_D(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1) \end{aligned}$$

is a well-behaved inhomogeneous analogue of the character χ_D which had not appeared in the literature. For example, the polynomials $\mathscr{R}_D^{(m,n)}$ enjoy a formula similar to Magyar's orthodontia (1.1), obtained by intertwining $r_{m,n}$ past the operators in (1.2). Furthermore, the polynomials $\mathscr{R}_D^{(m,n)}$ appeared to be ([27, Conjecture 1.4]) graded Lascoux positive, i.e. an alternating sum of Lascoux polynomials. We now have two proofs of this Lascoux positivity conjecture.

Theorem 1.3. *For any %-avoiding diagram $D \subseteq [n] \times [m]$, the polynomial $\mathscr{R}_D^{(m,n)}$ is graded Lascoux positive.*

Our first proof of Theorem 1.3 is via reduction to the known special case where $D = D(\alpha)$ is a skyline diagram ([27, Theorem 1.2]).

By taking the lowest-degree part of a Lascoux expansion

$$\mathscr{R}_D^{(m,n)} = \sum_{\alpha} c_{D,\alpha} \mathfrak{L}_\alpha,$$

we recover Reiner–Shimozono's theorem [26, Theorem 20] that χ_D is key positive.

Our second proof of Theorem 1.3 uses the orthodontia-like formula for $\mathscr{R}_D^{(m,n)}$ and parallels the proof of key positivity of χ_D. Resolving [27, Conjecture 1.5], we prove the following inhomogeneous extension of the fact that $\omega_i \times \cdot$ preserves key positivity.

Theorem 1.4. *For any $\alpha \in \mathbb{N}^n$ and $k \in [n]$, the product*

$$x_1 \cdots x_k (1 - x_{k+1}) \cdots (1 - x_n) \mathfrak{L}_\alpha(\mathbf{x})$$

is graded Lascoux positive.

Our proof of Theorem 1.4 relies on *punched tableaux* [19]. For a subset $S \subseteq [n]$ write

$$\mathbf{x}_S := \prod_{s \in S} x_s \qquad \text{and} \qquad \overline{\mathbf{x}}_S := \left(\prod_{s \in S} x_s \right) \cdot \left(\prod_{s \notin S} (1 - x_s) \right).$$

Punched tableaux were introduced in [19] to study the structure coefficients in the equality of operators

$$\mathbf{x}_{[k]} \pi_w = \sum_{T \in \mathrm{PT}(w)} \pi_{w(T)} \mathbf{x}_{S(T)}. \tag{1.3}$$

We show that (1.3) actually admits an inhomogeneous extension (Proposition 3.9)

$$\overline{\mathbf{x}}_{[k]} \overline{\pi}_w = \sum_{T \in \mathrm{PT}(w)} \overline{\pi}_{w(T)} \overline{\mathbf{x}}_{S(T)}. \tag{1.4}$$

Armed with (1.4), Theorem 1.4 reduces to a much easier problem about Lascoux positivity of certain products of the form $\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda$.

Acknowledgements

We thank David Anderson, Sergey Fomin, Peter Magyar, Karola Mészáros, Mark Shimozono, Vasu Tewari, and Alexander Yong for many helpful conversations and useful comments which improved our exposition.

2 Induction on orthodontic sort order

The main ingredient in our proof of Theorem 1.2 is a nonstandard induction on the set of permutations $w \in S_n$. We first reduce to the special case of *sorted permutations*, whose definition is not important for this exposition. In the sorted case, we observe (see Theorem 2.4) that an "almost-Rothe-diagram" appears after a predictable number of orthodontic steps and show that the corresponding orthodontic polynomials $\mathscr{G}_D(\mathbf{x};\mathbf{y})$ evolve in the same way as do the double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

Definition 2.1. *A diagram D is* %-avoiding *if it does not have a pair of rows $i_1 < i_2$ and a pair of columns $j_1 < j_2$ for which its restriction looks like the configuration in Figure 1.*

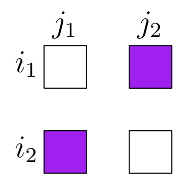


Figure 1: A forbidden configuration in a %-avoiding diagram.

We now define the double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ of a %-avoiding diagram D with columns $D_1, \ldots, D_n \subseteq [n]$. In contrast to Magyar's orthodontia, we track column information as well throughout the process. Begin by setting $K_i := \{j \colon D_j = [i]\}$, and let D_- denote the diagram obtained by replacing any such columns with the empty column. If D_- is empty, then $\mathbf{i} = \mathbf{j} = \mathbf{M}$ is the empty vector and the algorithm terminates.

Definition 2.2. *The diagram D_- above is called the* first orthodontic trimming *of D.*

An *orthodontic step* applied to D_- outputs a diagram $(D_-)''$ which we now describe. A *missing tooth* of a column $C \subseteq [n]$ is an integer $i \in [n]$ so that $i \notin C$ and $i+1 \in C$; any nonempty column of D_- has a missing tooth. Set i_1 to be the smallest missing tooth of the leftmost nonempty column $(D_-)_j$ of D_-. Set j_1 to be $j - \#\{a \leq i_1 \colon a \notin (D_-)_j\}$. Now swap rows i_1 and $i_1 + 1$ to get a diagram $(D_-)'$. Any column of $(D_-)'$ which is a standard interval is necessarily equal to $[i_1]$; set $M_1 = \{j \colon (D_-)'_j = [i_1]\}$. Let $(D_-)''$ denote the diagram obtained from $(D_-)'$ by removing all columns equal to $[i_1]$.

Repeatedly apply orthodontic steps to the resulting diagram $(D_-)''$, keeping track of the data $\mathbf{i}, \mathbf{j}, \mathbf{M}$ at each step, until the diagram is empty. See Figure 2 for an example.

Fix $n \in \mathbb{N}$. The *divided difference operator* $\partial_i \colon \mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]$ is given by

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}, \qquad i \in [n-1].$$

The *Demazure operators* π_i, *isobaric divided difference operators* $\overline{\partial}_i$, and *Demazure–Lascoux operators* $\overline{\pi}_i$ are defined respectively by

$$\pi_i(f) := \partial_i(x_i f), \quad \text{and} \quad \overline{\partial}_i(f) := \partial_i((1 - x_{i+1})f), \quad \overline{\pi}_i(f) := \overline{\partial}_i(x_i f).$$

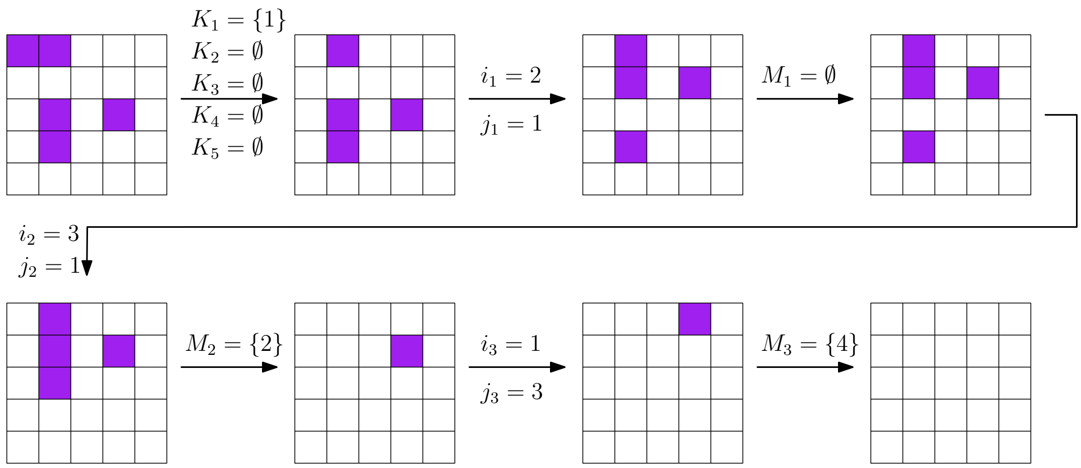


Figure 2: The double orthodontic sequence for $w = 31542$.

Definition 2.3. *For a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$, define the operators*

$$\pi_{i,j}(f) := \partial_i((x_i + y_j)f), \qquad \overline{\pi}_{i,j}(f) := \overline{\partial}_i((x_i + y_j - x_i y_j)f).$$

For $i \in [n]$ and $M \subseteq [n]$ define the quantities

$$\omega_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m), \qquad \overline{\omega}_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m - x_j y_m).$$

In what follows, we need notation whose precise definition is not important for this exposition: each permutation comes with certain numbers denoted i_1 and α.

Theorem 2.4 (cf. [27, Theorem 3.7, Lemma 3.9])**.** *Let w be a nonidentity sorted permutation and write $w' := ws_{i_1} \cdots s_{\alpha+1}$. Then, the first orthodontic trimming $D(w')_-$ of $D(w')$ appears in the orthodontic sequence for $D(w)$. Furthermore,*

$$\mathscr{G}_{D(w)} = \overline{\partial}_{i_1} \cdots \overline{\partial}_{\alpha+1}(\mathscr{G}_{D(w')}).$$

(See Figure 3 for an example.)

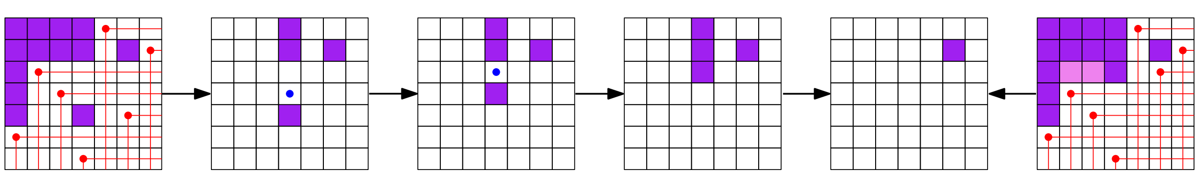


Figure 3: The first few orthodontic steps for $D(w)$, $w = 5723614$, with the first orthodontic trimming $D(w')_-$ for $w' = ws_4s_3s_2 = 5762314$.

Theorem 1.2. *When $D = D(w)$ is the Rothe diagram of a permutation, $\mathscr{G}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{G}_w(\mathbf{x}, \mathbf{y})$, the double Grothendieck polynomial. Consequently, the lowest degree component is $\mathscr{S}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{S}_w(\mathbf{x}, -\mathbf{y})$, the (positive) double Schubert polynomial.*

Proof sketch for Theorem 1.2. We build a partial order $\geq_{\text{os}}$ on S_n in [23, Proposition 5.6] so that the permutation w' appearing in Theorem 2.4 satisfies $w \geq_{\text{os}} w'$. Inducting on this partial order, we may assume that $\mathscr{G}_{D(w')} = \mathfrak{G}_{w'}$. Theorem 2.4 implies that

$$\begin{aligned}
\mathscr{G}_{D(w)} &= \overline{\partial}_{i_1} \cdots \overline{\partial}_{\alpha+1} \left(\mathscr{G}_{D(w')} \right) \\
&= \overline{\partial}_{i_1} \cdots \overline{\partial}_{\alpha+1} \left(\mathfrak{G}_{w'} \right) \\
&= \mathfrak{G}_w.
\end{aligned}$$

□

3 Two proofs of Theorem 1.3

We present two proofs of the graded Lascoux positivity of $\mathscr{R}_D^{(m,n)}$ (Theorem 1.3).

Sheared characters. Our first proof relates $\mathscr{R}_D^{(m,n)}$ to *sheared characters* $\chi_D(\mathbf{x} - \mathbf{1})$ obtained from ordinary characters χ_D by the variable shift $x_i \mapsto x_i - 1$.

Lemma 3.1. *For any %-avoiding diagram D, we have $\mathscr{S}_D(\mathbf{x}; -\mathbf{1}) = \chi_D(\mathbf{x} - \mathbf{1})$.*

Proof sketch. The two polynomials $\mathscr{S}_D(\mathbf{x}; -\mathbf{1})$ and $\chi_D(\mathbf{x} - \mathbf{1})$ satisfy the same orthodontia-like recurrence relations, obtained by specializing (1.2) and (1.1) respectively. □

For $i \in \mathbb{N}$, let φ_i denote the polynomial $\varphi_i := x_1 \cdots x_i(1 - x_{i+1}) \cdots (1 - x_n)$.

Corollary 3.2. *The polynomial $\mathscr{R}_D^{(m,n)}$ can be obtained from the polynomial $1 \in \mathbb{Z}[\mathbf{x}]$ by repeated application of the operators $\overline{\pi}_i$ and $\cdot \varphi_i$.*

Proof sketch. From the orthodontia formula (1.1):

$$\begin{aligned}
\chi_D(\mathbf{x}) &= \omega_1^{k_1} \cdots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1}(\pi_{i_2}(\omega_{i_2}^{m_2} \cdots \pi_{i_\ell}(\omega_{i_\ell}^{m_\ell}) \cdots))) \\
&= \omega_1^{k_1} \cdots \omega_n^{k_n} \partial_{i_1}(x_{i_1} \omega_{i_1}^{m_1}(\partial_{i_2}(x_{i_2} \omega_{i_2}^{m_2} \cdots \partial_{i_\ell}(x_{i_\ell} \omega_{i_\ell}^{m_\ell}) \cdots))).
\end{aligned}$$

where $\omega_i = x_1 x_2 \cdots x_i$. Let $\omega_{i;1} = (x_1 - 1)(x_2 - 1) \cdots (x_i - 1)$. Observe that

$$\chi_D(\mathbf{x} - \mathbf{1}) = \omega_{1;1}^{k_1} \cdots \omega_{n;1}^{k_n} \partial_{i_1}((x_{i_1} - 1)\omega_{i_1;1}^{m_1}(\partial_{i_2}(x_{i_2} - 1)\omega_{i_2;1}^{m_2}(\cdots \partial_{i_\ell}((x_{i_\ell} - 1)\omega_{i_\ell;1}^{m_\ell}) \cdots))),$$

Recall that $\mathscr{R}_D^{(m,n)} = r_{m,n}(\chi_D(\mathbf{x} - \mathbf{1}))$. The result follows from the identities

$$r_{m,n} \, \partial_i \left((x_i - 1) f \right) = \overline{\pi}_{n-i} \left(r_{m,n}(f) \right) \quad \text{and} \quad r_{m,n} \left(\omega_{i;1} f \right) = \varphi_{n-i} r_{m-1,n}(f).$$

□

Proposition 3.3 ([27, Theorem 1.2])**.** *The polynomial $\mathscr{R}^{(m,n)}_{D(\alpha)}$ is graded Lascoux positive.*

Proof sketch. When $D = D(\alpha)$, we show that the multiplication operators in the orthodontia formula can be commuted past the π operators. Let λ be the partition obtained by sorting the entries of α, and w the minimal length permutation with $\lambda_i = \alpha_{w(i)}$ for each i. Set $c_i = \lambda_i - \lambda_{i+1}$. We show that

$$\mathscr{R}^{(m,n)}_{D(\alpha)} = \overline{\pi}_w(\varphi_1^{c_1} \cdots \varphi_n^{c_n}),$$

Products of φ_i are graded Lascoux positive ([27, Proposition 4.11]), and $\overline{\pi}_w$ preserves Lascoux positivity (since each individual $\overline{\pi}_i$ does). $\square$

First proof of Theorem 1.3. Write $c_{D,\alpha}$ for the structure coefficients in the character-to-key expansion $\chi_D = \sum_\alpha c_{D,\alpha}\kappa_\alpha$. Then

$$\begin{aligned}\mathscr{R}^{(m,n)}_{D} = \quad & r_{m,n}(\chi_D(\mathbf{x}-\mathbf{1})) \quad = r_{m,n}\left(\sum_\alpha c_{D,\alpha}\kappa_\alpha(\mathbf{x}-\mathbf{1})\right) \\ = & \sum_\alpha c_{D,\alpha} r_{m,n}(\kappa_\alpha(\mathbf{x}-\mathbf{1})) = \sum_\alpha c_{D,\alpha}\mathscr{R}^{(m,n)}_{D(\alpha)}\end{aligned}$$

is graded Lascoux positive. $\square$

Punched tableaux. Our second proof relies on the graded Lascoux positivity of $\varphi_i\mathfrak{L}_\alpha$ (Theorem 1.4). We now recall Lascoux's notion of *punched tableaux* and the combinatorics of the expansions (1.3) and (1.4).

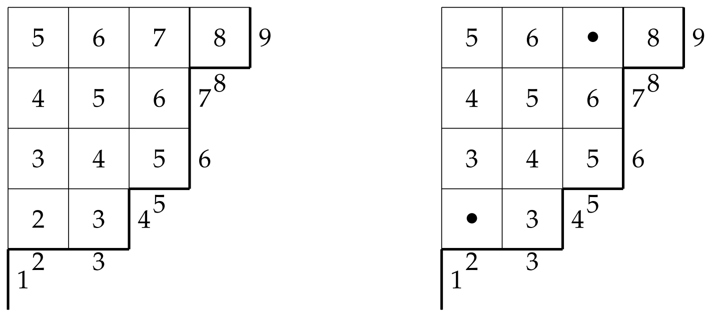
A permutation $\sigma \in S_n$ is called *k-Grassmannian* if $\sigma(i) < \sigma(i+1)$ whenever $i \neq k$. Each k-Grassmannian $\sigma \in S_n$ corresponds to a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ with $\lambda_1 \leq n-k$ via

$$\sigma \mapsto (\sigma(k) - k, \sigma(k-1) - (k-1), \ldots, \sigma(1) - 1).$$

Let $\sigma \in S_n$ be a k-Grassmannian permutation with associated partition λ. Fill the Young diagram $D(\lambda)$ by putting the number $k - i + j$ in $(i, j) \in D(\lambda)$. Label the southeast boundary of $D(\lambda)$ by the numbers 1 through n from bottom left to top right. The resulting diagram together with the filling and labeling is denoted $T(\sigma)$. See Example 3.5.

Definition 3.4. *A* punched tableau *for σ is obtained from $T(\sigma)$ by replacing some entries with $\bullet$ such that each column or row contains at most one $\bullet$ and no proper rectangle of at least 2×2 cells contained in T has both the bottom-left and top-right corners equal to $\bullet$.*

Example 3.5. *Consider the 5-Grassmannian permutation $\sigma = 146792358 \in S_9$, associated to the partition $(4, 3, 3, 2, 0)$. The filling $T(\sigma)$ is shown below on the left. Reading $T(\sigma)$ in the* *reading order gives* 5432 6543 765 8*, which is a reduced word of* σ^{-1}*. A punched tableau for* σ *is on the right.*



Definition 3.6. *Let* $\mathrm{PT}(\sigma)$ *denote the set of punched tableaux of* $T(\sigma)$*. For* $T \in \mathrm{PT}(\sigma)$*, the reading word* $\mathrm{word}(T)$ *is obtained by reading its filling in the reading order while ignoring* $\bullet$*. The word* $\mathrm{word}(T)$ *is a reduced word of a permutation which we denote as* $w(T)$*. Define* $S(T) \subseteq [n]$ *as the set of all* i *such that* i *labels a row with no* $\bullet$*, or a column containing a* $\bullet$*.*

In Example 3.5, let $T \in \mathrm{PT}(\sigma)$ be the element on the right. We have

$$\mathrm{word}(T) = 543\,6543\,65\,8 \text{ and } S(T) = \{1, 2, 5, 6, 7\}.$$

Notice that $\mathrm{word}(T)$ is a reduced word of the permutation $w(T)$ where $w(T)^{-1} = 126754398$. In general, the permutation $w(T)^{-1}$ can be read as follows.

Lemma 3.7. *Let* $T \in \mathrm{PT}(\sigma)$*. For* $i \in [k]$*,* $w(T)^{-1}(i)$ *is the label of row* $k+1-i$ *if this row has no* $\bullet$*. Otherwise, it is the label of the column containing the bullet in this row. For* $j \in [n-k]$*,* $w(T)^{-1}(k+j)$ *is the label of column* j *if this column has no* $\bullet$*. Otherwise, it is the label of the row containing the bullet in this column.*

Proof sketch. Follows from induction on the number of cells in $\sigma(T)$. □

Punched tableaux were introduced by Lascoux to describe commutation of certain polynomial operators. Recall that for a subset $S \subseteq [n]$ we write

$$\mathbf{x}_S := \prod_{s \in S} x_s \qquad \text{and} \qquad \overline{\mathbf{x}}_S := \left(\prod_{s \in S} x_s\right) \cdot \left(\prod_{s \notin S} (1 - x_s)\right).$$

In terms of the earlier notation, $\mathbf{x}_{[k]} = \omega_k$ and $\overline{\mathbf{x}}_{[k]} = \varphi_k$.

Theorem 3.8 ([19, Lemma 1.4.2])**.** *Let* σ *be* k*-Grassmannian. Then*

$$\mathbf{x}_{[k]} \pi_{\sigma^{-1}} = \sum_{T \in \mathrm{PT}(\sigma)} \pi_{w(T)} \mathbf{x}_{S(T)}. \tag{1.3}$$

Proposition 3.9. *Let* σ *be* k*-Grassmannian. Then*

$$\overline{\mathbf{x}}_{[k]} \overline{\pi}_{\sigma^{-1}} = \sum_{T \in \mathrm{PT}(\sigma)} \overline{\pi}_{w(T)} \overline{\mathbf{x}}_{S(T)}. \tag{1.4}$$

Proof sketch. By direct computation, we have

$$\mathbf{x}_S\pi_i = \begin{cases} \pi_i\mathbf{x}_{s_iS} + \mathbf{x}_S & \text{if } i \in S,\, i+1 \notin S, \\ \pi_i\mathbf{x}_{s_iS} - \mathbf{x}_{s_iS} & \text{if } i \notin S,\, i+1 \in S, \\ \pi_i\mathbf{x}_S & \text{otherwise,} \end{cases} \qquad \overline{\mathbf{x}}_S\overline{\pi}_i = \begin{cases} \overline{\pi}_i\overline{\mathbf{x}}_{s_iS} + \overline{\mathbf{x}}_S & \text{if } i \in S,\, i+1 \notin S, \\ \overline{\pi}_i\overline{\mathbf{x}}_{s_iS} - \overline{\mathbf{x}}_{s_iS} & \text{if } i \notin S,\, i+1 \in S, \\ \overline{\pi}_i\overline{\mathbf{x}}_S & \text{otherwise.} \end{cases}$$

The equality on the left uniquely determines the expansion of the operator $\mathbf{x}_{[k]}\pi_{\sigma^{-1}}$ into operators of the form $\pi_w\mathbf{x}_S$. The equality on the right uniquely determines the expansion of $\overline{\mathbf{x}}_{[k]}\overline{\pi}_{\sigma^{-1}}$ into operators of the form $\overline{\pi}_w\overline{\mathbf{x}}_S$; it is the same as the expansion of $\mathbf{x}_{[k]}\pi_{\sigma^{-1}}$ into $\pi_w\mathbf{x}_S$. Hence (1.4) follows from (1.3). $\square$

We say a monomial $\mathbf{x}^\alpha$ is *dominant* if $\alpha_1 \geq \alpha_2 \geq \cdots$.

Theorem 1.4. *For any $\alpha \in \mathbb{N}^n$ and $k \in [n]$, the product $\overline{\mathbf{x}}_{[k]}\mathfrak{L}_\alpha(\mathbf{x})$ is graded Lascoux positive.*

Proof sketch of Theorem 1.4. If $\alpha_i < \alpha_{i+1}$ for some $i \in [n-1] \setminus \{k\}$, then $\overline{\mathbf{x}}_{[k]}\mathfrak{L}_\alpha(\mathbf{x}) = \overline{\pi}_i(\overline{\mathbf{x}}_{[k]}\mathfrak{L}_{s_i\alpha}(\mathbf{x}))$; thus graded Lascoux positivity of $\overline{\mathbf{x}}_{[k]}\mathfrak{L}_\alpha(\mathbf{x})$ would follow from that of $\overline{\mathbf{x}}_{[k]}\mathfrak{L}_{s_i\alpha}(\mathbf{x})$. Hence we reduce to the case where $\alpha_i \geq \alpha_{i+1}$ for all $i \in [n-1] \setminus \{k\}$.

Write λ for the partition obtained by reordering α and σ for the permutation so that $\alpha_i = \lambda_{\sigma(i)}$ for each i. Observe that σ^{-1} is k-Grassmannian and that

$$\overline{\mathbf{x}}_{[k]}\mathfrak{L}_\alpha(\mathbf{x}) = \overline{\mathbf{x}}_{[k]}\overline{\pi}_{\sigma^{-1}}(\mathbf{x}^\lambda) = \sum_{T \in \mathrm{PT}(\sigma)} \overline{\pi}_{w(T)}(\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda). \tag{3.1}$$

It remains to show each $\overline{\pi}_{w(T)}(\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda)$ is graded Lascoux positive by considering two cases. If $\mathbf{x}_{S(T)} \cdot \mathbf{x}^\lambda$ is a dominant monomial, then $\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda$ is graded Lascoux positive by Proposition 3.10 below. Otherwise, there exists i such that $i \notin S(T)$, $i+1 \in S(T)$, and $\lambda_i = \lambda_{i+1}$. In this case, $\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda$ can be written as $f \cdot (1-x_i)x_{i+1}$ for some f that is symmetric in x_i, x_{i+1}; Lemma 3.7 implies that $(w(T))(i) > (w(T))(i+1)$, so $w(T)$ has a reduced word ending with i, and $\overline{\pi}_{w(T)}(\overline{\mathbf{x}}_{S(T)} \cdot \mathbf{x}^\lambda) = 0$. $\square$

We now prove the result used in the proof above.

Proposition 3.10. *Let λ be a partition and $S \subseteq [n]$ be a set. If $\mathbf{x}_S \cdot \mathbf{x}^\lambda$ is a dominant monomial, then $\overline{\mathbf{x}}_S \cdot \mathbf{x}^\lambda$ is graded Lascoux positive.*

Proof sketch. For any $a \leq b$, let us write $\mathbf{x}_{a,b} := x_1 \cdots x_a(1-x_{a+1}) \cdots (1-x_b)$. We show (see Example 3.11) that the product $\overline{\mathbf{x}}_S \cdot \mathbf{x}^\lambda$ can be written as a product

$$\mathbf{x}_{j_1,m_1}\mathbf{x}_{j_2,m_2} \cdots \mathbf{x}_{j_\ell,m_\ell} \quad \text{with} \quad j_1 \leq m_1 \leq j_2 \leq m_2 \leq \cdots \leq j_\ell \leq m_\ell. \tag{3.2}$$

A direct computation shows that if $\alpha \in \mathbb{N}^m$ has m parts (so that $\mathfrak{L}_\alpha$ depends only on the variables $x_1, \ldots, x_m$), the product $\mathbf{x}_{j,k}\mathfrak{L}_\alpha$ is graded Lascoux positive whenever $m \leq j$. It follows that any prefix product in (3.2) is graded Lascoux positive. $\square$

Example 3.11. *Take $n = 6$, $\lambda = (3, 3, 2, 1, 1)$, and $S = \{3, 4\}$. Equation (3.2) expands the product $\overline{\mathbf{x}}_{\{3,4\}} \cdot x_1^3 x_2^3 x_3^2 x_4 x_5 = (1 - x_1)(1 - x_2)x_3 x_4(1 - x_5)(1 - x_6) \cdot x_1^3 x_2^3 x_3^2 x_4 x_5$ as the product*

$$\Big((1 - x_1)(1 - x_2)\Big)\Big(x_1 x_2 x_3\Big)\Big(x_1 x_2 x_3 x_4(1 - x_5)\Big)\Big(x_1 x_2 x_3 x_4 x_5(1 - x_6)\Big) = \mathbf{x}_{0,2}\mathbf{x}_{3,3}\mathbf{x}_{4,5}\mathbf{x}_{5,6}.$$

Second proof of Theorem 1.3. By Corollary 3.2, the polynomial $\mathscr{R}_D^{(m,n)}$ can be obtained from the polynomial $1 \in \mathbb{Z}[\mathbf{x}]$ by repeated application of $\overline{\pi}_i$ and $\cdot\varphi_i$. The operator $\overline{\pi}_i$ preserves Lascoux positivity because $\overline{\pi}_i(\mathfrak{L}_\alpha) \in \{\mathfrak{L}_\alpha, \mathfrak{L}_{s_i\alpha}\}$, and the operator $\cdot\varphi_i$ preserves Lascoux positivity by Theorem 1.4. □

References

[1] D. Anderson. “Filtrations and recursions for Schubert modules”. 2024. arXiv:2408.16694.

[2] S. Assaf and A. Schilling. “A Demazure crystal construction for Schubert polynomials”. *Algebr. Comb.* **1**.2 (2018), pp. 225–247. DOI.

[3] N. Bergeron and S. Billey. “RC-graphs and Schubert polynomials”. *Experiment. Math.* **2**.4 (1993), pp. 257–269.

[4] S. Billey, W. Jockusch, and R. P. Stanley. “Some Combinatorial Properties of Schubert Polynomials”. *J. Algebraic Combin.* **2**.4 (1993), pp. 345–374. DOI.

[5] F. Castillo, Y. Cid-Ruiz, F. Mohammadi, and J. Montaño. “Double Schubert polynomials do have saturated Newton polytopes”. *Forum Math. Sigma* **11** (2023), Paper No. e100, 9. DOI.

[6] N. J. Fan and P. L. Guo. “Upper bounds of Schubert polynomials”. *Sci. China Math.* **65**.6 (2022), pp. 1319–1330. DOI.

[7] A. Fink, K. Mészáros, and A. St. Dizier. “Schubert polynomials as integer point transforms of generalized permutahedra”. *Adv. Math.* **332** (2018), pp. 465–475. DOI.

[8] S. Fomin and A. N. Kirillov. “The Yang-Baxter equation, symmetric functions, and Schubert polynomials”. *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).* Vol. 153. 1-3. 1996, pp. 123–143. DOI.

[9] S. Fomin and A. N. Kirillov. “Grothendieck polynomials and the Yang-Baxter equation”. *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*. DIMACS, Piscataway, NJ, sd, pp. 183–189.

[10] P. L. Guo and Z. Lin. “Schubert polynomials and patterns in permutations”. *Math. Z.* **313**.2 (2026), Paper No. 16. DOI.

[11] E. S. Hafner, K. Mészáros, L. Setiabrata, and A. St. Dizier. “M-convexity of vexillary Grothendieck polynomials via bubbling”. *SIAM J. Discrete Math.* **38**.3 (2024), pp. 2194–2225. DOI.

[12] J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg. “Refinements of the Littlewood-Richardson rule”. *Trans. Amer. Math. Soc.* **363**.3 (2011), pp. 1665–1686. DOI.

[13] J. Huh, J. P. Matherne, K. Mészáros, and A. St. Dizier. "Logarithmic concavity of Schur and related polynomials". *Trans. Amer. Math. Soc.* **375**.6 (2022), pp. 4411–4427. DOI.

[14] A. Knutson and E. Miller. "Subword complexes in Coxeter groups". *Adv. Math.* **184**.1 (2004), pp. 161–176. DOI.

[15] A. Knutson and E. Miller. "Gröbner geometry of Schubert polynomials". *Ann. of Math. (2)* **161**.3 (2005), pp. 1245–1318. DOI.

[16] W. Kraśkiewicz and P. Pragacz. "Schubert functors and Schubert polynomials". *European J. Combin.* **25**.8 (2004), pp. 1327–1344. DOI.

[17] T. Lam, S. J. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compos. Math.* **157**.5 (2021), pp. 883–962. DOI.

[18] T. Lam, S. J. Lee, and M. Shimozono. "Back stable K-theory Schubert calculus". *Int. Math. Res. Not. IMRN* 24 (2023), pp. 21381–21466. DOI.

[19] A. Lascoux. "Polynomials". Preprint, August 28, 2012. Available at `https://web.archive.org/web/20170312164301/http://phalanstere.univ-mlv.fr/%7Eal//ARTICLES/CoursYGKM.pdf`. 2013.

[20] A. Lascoux and M.-P. Schützenberger. "Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux". *C. R. Acad. Sci. Paris Sér. I Math.* **295**.11 (1982), pp. 629–633.

[21] C. Lenart. "Noncommutative Schubert calculus and Grothendieck polynomials". *Adv. Math.* **143**.1 (1999), pp. 159–183. DOI.

[22] P. Magyar. "Schubert polynomials and Bott-Samelson varieties". *Comment. Math. Helv.* **73**.4 (1998), pp. 603–636. DOI.

[23] K. Mészáros, L. Setiabrata, and A. St. Dizier. "An orthodontia formula for Grothendieck polynomials". *Trans. Amer. Math. Soc.* **375**.2 (2022), pp. 1281–1303. DOI.

[24] O. Pechenik, D. E. Speyer, and A. Weigandt. "Castelnuovo-Mumford regularity of matrix Schubert varieties". *Selecta Math. (N.S.)* **30**.4 (2024), Paper No. 66, 44. DOI.

[25] V. Reiner and M. Shimozono. "Key polynomials and a flagged Littlewood–Richardson rule". *J. Combin. Theory Ser. A* **70**.1 (1995), pp. 107–143. DOI.

[26] V. Reiner and M. Shimozono. "Percentage-avoiding, northwest shapes and peelable tableaux". *J. Combin. Theory Ser. A* **82**.1 (1998), pp. 1–73. DOI.

[27] L. Setiabrata and A. St. Dizier. "Double orthodontia formulas and Lascoux positivity". 2024. arXiv:2410.08038.

[28] A. Weigandt. "Bumpless pipe dreams and alternating sign matrices". *J. Combin. Theory Ser. A* **182** (2021), Paper No. 105470, 52. DOI.

[29] A. Weigandt. "Changing Bases with Pipe Dream Combinatorics". 2025. arXiv:2506.07306.

[30] T. Yu. "Connection between Schubert polynomials and top Lascoux polynomials". *Algebr. Comb.* **8**.4 (2025), pp. 955–970. DOI.