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The 1/4-phenomenon of placement probabilities in the Aztec diamond

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Abstract. We consider domino tilings of the Aztec diamond. Using the *Domino Shuffling* algorithm introduced by Elkies, Kuperberg, Larsen, and Propp (1992), we are able to generate domino tilings uniformly at random. In this paper, we investigate the probability of finding a domino at a specific position in such a random tiling. We prove that this placement probability is always equal to 1/4 plus a rational function, whose shape depends on the location of the domino, multiplied by a position-independent factor that involves only the size of the diamond. This result leads to significantly more compact explicit counting formulas compared to previous findings. As a direct application, we derive explicit counting formulas for the domino tilings of Aztec diamonds with 2×2-square holes at arbitrary positions.

Keywords: Aztec diamond, domino tilings, placement probabilities

1 Motivation and main result

The dimer model has risen to a central object in statistical physics. From the point of view of mathematics, it concerns itself with the probabilistic behaviour of random perfect matchings on graphs. A perfect matching of a simple planar graph $G = (V, E)$ is a subset of edges $\mu \subseteq E$ such that any vertex $v \in V$ is contained in exactly one edge of μ (and therefore matched with the other vertex in this edge). If the graphs are subgraphs of the square- or hexagonal lattice, perfect matchings, in the context of statistical physics also called *dimer configurations*, are in bijection with tilings of regions related to the graph by duality. The most prominent examples are rhombus- and domino tilings. For the second, one wants to cover a region consisting of a union of unit squares with domino shaped tiles, i.e. 1×2 rectangles. An example of this idea is given in Figure 1. The connection with perfect matchings is visualised in Figure 2.

Advances in this theory are achieved hand in hand with the development of new enumerative techniques. First breakthroughs were accomplished by Temperley and Fisher [18] and Kasteleyn [9], who simultaneously derived the enumeration formula

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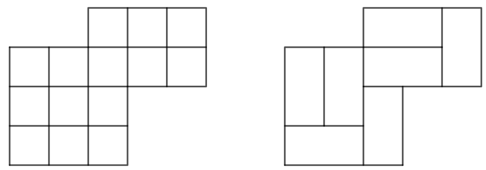

Figure 1: A domino region left and one of its possible domino tilings on the right.

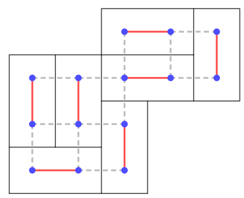

Figure 2: The bijection between domino tilings and perfect matchings of the dual graph.

counting domino tilings of the $n \times m$-rectangle. In the case of rhombus tilings it is MacMahon's Box Formula [15] which counts tilings of semiregular hexagons and connects them with plane partitions. Later on, Kasteleyn [10] developed his method to express the number of perfect matchings of any planar graph as a Pfaffian or determinant, making the dimer model a *solvable* statistical model. Meanwhile, people started to study the correlation functions on this probabilistic model. An early example would be the work of Fisher and Stephenson [6]. Later Kenyon [11] combined the computation of correlations with Kasteleyn's theory. In their simplest form, so-called one-point correlation functions ask for the *placement probability* telling the chance of observing a certain tile at a particular position in a random tiling. Equivalently, they encode the probability that a given edge is contained in a random perfect matching.

For the semiregular hexagon, closed formulas for the placement probabilities were developed by Fischer [5], Gilmore [7] and in even more generality Petrov [17]. However, the complexity of these formulas makes them very cumbersome for many applications. A conjecture by Krattenthaler [13], which is related to our main result, promises vast possible simplifications of the structure of those formulas. In the future, we hope to be able to transfer the ideas of our proof to the setting of the hexagonal lattice to also prove his 20 year old conjecture.

We study the analogous problem of placement probabilities in the Aztec diamond. Here, explicit formulas were found by Helfgott [8] and Cohn, Elkies and Propp [2]. The probabilities were expressed in terms of products of Kravchuk polynomials, whose complexity leads to similar problems as in the case of the semiregular hexagon. In this article we prove a strongly simplifying structural result on the shape of the formulas for those placement probabilities.

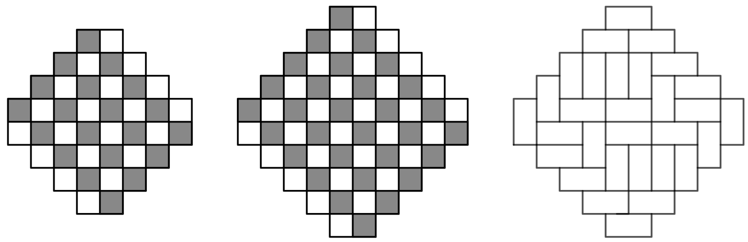

Figure 3: The Aztec diamonds of sizes 4 (left) and 5 (centre) as well as a domino tiling of the Aztec diamond (right).

1.1 Set-up and the main result

The Aztec diamond of size $n \in \mathbb{N}$ is a region in the plane consisting of unit squares aligned as depicted in Figure 3. Throughout this article we assume our diamonds to be coloured black and white in a chessboard pattern such that the left one of the top-most unit squares is black. It was shown by Elkies, Kuperberg, Larsen and Propp [3, 4] that the number of domino tilings of the Aztec diamond of size n is equal to

$$2^{(n+1)n/2}. \tag{1.1}$$

Moreover, also in [4] the researchers developed an algorithm called *Domino Shuffling* which enables us to sample a domino tiling of the Aztec diamond uniformly at random. The implications of this algorithm make it possible to study single placement probabilities. First of all, introduce a coordinate system to our Aztec diamond by assuming that its centre is located at $(0,0)$ and all squares have side length one. We consider the quantity $\mathbb{P}(l,m;n)$ defined to be the probability that in a random tiling of the Aztec diamond of dimension n we observe a horizontal domino tile whose left square is coloured black and such that the centre of its lower side is located at (l,m). The idea is visualised in Figure 4.

The reason why we have to distinguish between horizontal tiles whose left square is black or whose left square is white originates from the Domino Shuffling algorithm, where those two cases lead to different actions in the procedure. Naturally, once we know $\mathbb{P}(l,m;n)$ for all $l,m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we know all possible placement probabilities by just rotating the diamond. Clearly, $\mathbb{P}(l,m;n) = 0$ if $l+m \equiv n \mod 2$, since then the left square at position (l,m) is simply white according to the rules of our colouring.

In [2], Cohn, Elkies and Propp worked out the asymptotic behaviour of $\mathbb{P}(x,y;n)$ for $n \to \infty$ where $(x,y) = (l/n, m/n)$ encodes the relative position inside a *normalised* Aztec diamond. We on the other hand are interested in the exact value of those probabilities for a fixed position in the diamond. We are able to prove the following statement about the shape of an explicit formula.

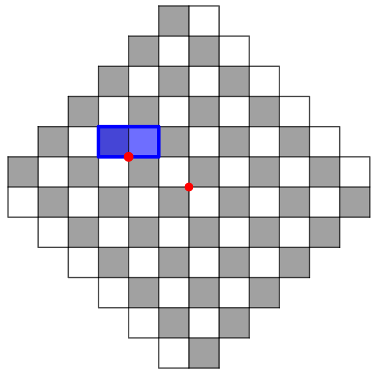

Figure 4: The Aztec diamonds of sizes 6. In blue we marked the space for a horizontal domino tile at position $(-2,1)$. The probability, to observe a horizontal domino at this location in a random tiling is given by $\mathbb{P}(-2,1;6)$.

Theorem 1. *Consider the Aztec diamond of size $n \in \mathbb{N}$ with $\alpha \in \{0,1,2,3\}$ and $p \in \mathbb{N}$ such that $n = 4p + \alpha$. Dye the Aztec diamond in a chessboard pattern black and white such that the left of the topmost vertices is coloured black. Then, after sampling a tiling uniformly at random, the probability to find a horizontal domino tile at position (l,m) whose left unit square is black equals*

$$\mathbb{P}(l,m;4p+\alpha) = \begin{cases} \frac{1}{4} + 2^{-4p-\alpha}\binom{2p-1}{p}^2 f_{l,m,\alpha}(p) & \text{if } l+m \equiv \alpha+1 \mod 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{l,m,\alpha}(p)$ is a rational function in p.

Remark 1. *By the recursive arguments of the proof of Theorem 1 it is also possible to construct the rational functions $f_{l,m,\alpha}(p)$ explicitly for a given location $(l,m) \in \mathbb{Z}^2$. However, depending on the distance of (l,m) to the origin this procedure may involve many computational steps. Below we give a couple of examples for the function $f_{l,m,\alpha}(p)$.*

$$f_{3,1,3}(p) = \frac{8(1+2p)}{1+p}, \qquad f_{0,-4,1}(p) = -\frac{2(5+6p+3p^2)}{(1+p)^2},$$
$$f_{0,-1,0}(p) = \frac{4(-1+6p+8p^2)}{(1+p)(-1+2p)}, \qquad f_{4,-3,0}(p) = \frac{1-3p-6p^2}{(1+p)(-1+2p)}.$$

In the section below, we present an overview of the proof of Theorem 1. The computational and technical details will be contained in the full version of this extended abstract. The three major steps in the proof are diverted into their own subsections. In the last section we apply the main result to obtain a statement about the structure of enumeration formulas counting domino tilings of Aztec diamonds with 2×2-square holes. Namely, given any position $(l, m) \in \mathbb{Z}^2$ we show the existence of rational functions $g_{l,m,\alpha}(p)$ and $h_{l,m,\alpha}(p)$ such that the number of domino tilings of the Aztec diamond of size $4p + \alpha$ with a 2×2-square hole centred at this location is given by

$$2^{(4p+\alpha+1)(4p+\alpha)/2}\left(\frac{1}{8} + 2^{-4p-\alpha-2}\binom{2p-1}{p}^2 g_{l,m,\alpha}(p) + 2^{-8p-2\alpha}\binom{2p-1}{p}^4 h_{l,m,\alpha}(p)\right).$$

This is used to derive explicit formulas for the number of tilings of diamonds with a central 2×2-square hole. Details are provided in Corollaries 2 and 3.

2 The proof of the main result

The idea of the proof is to reduce the statement by recursive arguments to the single position $(l, m) = (0, 0)$. We will see that the validity of the theorem at the origin already implies its validity at all other positions. At the origin, we show two explicit formulas for $\mathbb{P}(0, 0; 4p + 1)$ and $\mathbb{P}(0, 0; 4p + 3)$ which align with the shape dictated by Theorem 1.

A crucial step in our proof will be the usage of a certain relation between the placement probabilities. The Domino Shuffling algorithm gives rise to a quantity called *net creation rate*.

Definition 1 ([2])**.** *The net creation rate is defined as*

$$\mathrm{Cr}(l, m; n) := 2\big(\mathbb{P}(l, m; n) - \mathbb{P}(l, m-1; n-1)\big). \tag{2.1}$$

Both for $\mathrm{Cr}(l, m; n)$ and $\mathbb{P}(l, m; n)$ there already exist closed formulas [2, 8]. However, they are all stated in terms of Kravchuk polynomials.

Definition 2 (Kravchuk polynomial)**.** *The Kravchuk polynomial* $\mathrm{Kr}(a, b; n)$ *is given as the coefficient of* z^a *in* $(1 + z)^{n-b}(1 - z)^b$*. Thus, we have explicitly*

$$\mathrm{Kr}(a, b; n) = \sum_{j=0}^{a} (-1)^j \binom{b}{j}\binom{n-b}{a-j}.$$

Kravchuk polynomials are a family of discrete orthogonal polynomials. They fulfil a symmetry relation as given in Lemma 1 and a typical three-term recurrence as described in Lemma 2. The results are all taken from [16]. For a general introduction to the theory of hypergeometric orthogonal polynomials see [12].

Lemma 1 ([16, Thm. 17])**.** *For non-negative integers* a, b *and* n *we have*

$$\mathrm{Kr}(b, a; n) = \frac{a!(n-a)!}{b!(n-b)!}\mathrm{Kr}(a, b; n).$$

Lemma 2 ([16, Thm. 19]). *The Kravchuk polynomials fulfil the following recurrence:*

$$(a+1)\operatorname{Kr}(a+1,b,n) = (n-2b)\operatorname{Kr}(a,b,n) - (n-a+1)\operatorname{Kr}(a-1,b,n).$$

In the language of Kravchuk polynomials we now have the following formula for the creation rates.

Lemma 3 ([2, Prop. 2]). *Let $n > 0$. Suppose l and m are integers with $l+m \equiv n \mod 2$ and $|l|+|m| \leq n$. Set $a := (l+m+n)/2$ and $b := (l-m+n)/2$, then*

$$\operatorname{Cr}(l,m;n+1) = 2^{-n}\operatorname{Kr}(a,b;n)\operatorname{Kr}(b,a;n).$$

For all other integers l and m we have $\operatorname{Cr}(l,m;n+1) = 0$.

This already yields the hint that we actually need to prove a statement about the growth of Kravchuk polynomials. Hence, the idea is to prove that Kravchuk polynomials grow in a way connected to Theorem 1 and deduce from that the correct (homogeneous) growth rate for the creation rates $\operatorname{Cr}(l,m;n)$.

2.1 The Growth Theorem for Kravchuk polynomials

We prove the following statement about Kravchuk polynomials.

Theorem 2 (The Growth Theorem for Kravchuk polynomials). *Let $a,b \in \mathbb{N}$ and $\alpha \in \{0,1,2,3\}$. Then we have*

$$\operatorname{Kr}(a+2p,b+2p;4p+\alpha-1) = (-1)^p\binom{2p-1}{p}g_{a,b,\alpha}(p),$$

where $g_{a,b,\alpha}(p)$ is a rational function in p.

First we notice that it is actually enough to prove Theorem 2 only for the positions $(a,b) \in \{(0,0),(0,1),(1,1)\}$ and all versions of $\alpha \in \{0,1,2,3\}$. That is, since inserting the expression on the right-hand side of Theorem 2 into Lemma 1 we observe that the statement of the theorem holds for (a,b) if and only if it holds for (b,a). Moreover, by the three-term recursion stated in Lemma 2 it follows that if the theorem holds for $(n-1,b)$ and (n,b) it is also true for $(n+1,b)$. Let now $(a,b) \in \mathbb{Z}^2_{\geq 0}$ be arbitrary. Since we assume that the theorem is true for $(0,1)$, it is also true for $(1,0)$ due to Lemma 1. If then the theorem holds for $(0,0)$ and $(1,0)$ it also holds for $(b,0)$ due to the recursion. Therefore it is also true for $(0,b)$. However, since we assume validity for $(0,1)$ and $(1,1)$ the recursion also yields validity for $(b,1)$ and thus also $(1,b)$. Now, since the theorem would apply to $(0,b)$ and $(1,b)$ we derive again from the recursion that the statement holds for (a,b) too. Hence, we deduce that to show Theorem 2 it is actually enough to prove the following proposition.

$\alpha = \dots$	0	1	2	3
$g_{0,0,\alpha}(p)$	1	2	2	$\dfrac{2}{p+1}$
$g_{1,0,\alpha}(p)$	-1	0	2	4
$g_{1,1,\alpha}(p)$	$\dfrac{3-2p}{-1+2p}$	-2	-2	0

Table 1: The rational function $g_{a,b,\alpha}(p)$.

Proposition 1. *Let $a, b \in \{0,1\}$ but $(a,b) \neq (0,1)$ and $\alpha \in \{0,1,2,3\}$. Then we have*

$$\mathrm{Kr}(a+2p, b+2p, 4p+\alpha-1) = (-1)^p \binom{2p-1}{p} g_{a,b,\alpha}(p),$$

where the rational function $g_{a,b,\alpha}(p)$ can be read off Table 1.

Proof. The identities listed in Table 1 are all proven by comparing the coefficient of z^{a+2p} in the expression

$$(1+z)^{2p+\alpha-1-b}(1-z)^{2p+b},$$

which gives by definition the Kravchuk-polynomial $\mathrm{Kr}(a+2p, b+2p; 4p+\alpha-1)$, and the same coefficient in the generating function when it is reshaped to some expression similar to

$$(1-z^2)^{2p-1}(1+z)^{\alpha-b}(1-z)^{1+b}.$$

□

As a corollary we obtain the suitable growth for the net creation rates.

Corollary 1. *Let $n = 4p+\alpha \geq 1$, suppose l and m are integers with $l+m \equiv n-1 \mod 2$ and $|l|+|m| \leq n-1$. Then*

$$\mathrm{Cr}(l,m;4p+\alpha) = 2^{-4p-\alpha+1}\binom{2p-1}{p}^2 f_{l,m,\alpha}(p),$$

where $f_{l,m,\alpha}(p)$ is a rational function in p.

For the proof one combines Lemma 3 with Theorem 2.

2.2 Reduction to the origin

Having the suitable growth for the net creation rates by Corollary 1 and applying it to Equation (2.1) in the definition of the creation rates, we see inductively that the probability $\mathbb{P}(l,m;4p+\alpha)$ satisfies Theorem 1 *if and only if* $\mathbb{P}(l,0;4p+\alpha)$ does. Hence, it

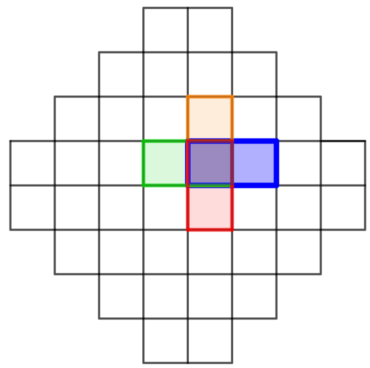

Figure 5: The four possibilities to place a domino tile on the upper-right central unit square. From the point of view of the covered unit square, the covering tile has to be oriented either *east*, *north*, *west* or *south*. By symmetries and our conventions of colouring we have $\mathbb{P}(1,0;4p+\alpha) = \mathbb{P}(\mathit{east}) = \mathbb{P}(\mathit{north})$ and $\mathbb{P}(0,-1;4p+\alpha) = \mathbb{P}(\mathit{west}) = \mathbb{P}(\mathit{south})$.

is enough to show the statement of Theorem 1 for positions along the horizontal axis. Moreover, due to the symmetry of the Aztec diamond we have for all $l, m \in \mathbb{Z}$ and $n \in \mathbb{N}$:

$$\mathbb{P}(-l,m;n) = \mathbb{P}(l,m;n).$$

Therefore, we assume $l \geq 0$. In a next step we show that it is actually enough to prove the statement of Theorem 1 for $\mathbb{P}(0,0;4p+\alpha)$. This is summarised in the next proposition.

Proposition 2. *If we have*

$$\mathbb{P}(0,0;4p+\alpha) = \frac{1}{4} + 2^{-4p-\alpha}\binom{2p-1}{p}^2 f_\alpha(p)$$

for a rational function $f_\alpha(p)$ in p if $\alpha \in \{1,3\}$ and $\mathbb{P}(0,0;4p+\alpha) = 0$ if $\alpha \in \{0,2\}$ then either

$$\mathbb{P}(l,0;4p+\alpha) = \frac{1}{4} + 2^{-4p-\alpha}\binom{2p-1}{p}^2 f_{l,\alpha}(p)$$

with $f_{l,\alpha}(p)$ a rational function in p or $\mathbb{P}(l,0;4p+\alpha) = 0$.

Proof. We use induction on the horizontal coordinate $l \geq 0$. We already assume the statement for the case $l = 0$. By the observations at the beginning of this subsection we may therefore assume that Theorem 1 holds for $\mathbb{P}(0,m;4p+\alpha)$ for all $m \in \mathbb{Z}$. For $l = 1$ we use that we are working with probabilities in random tilings and thus every unit square must be covered by some domino tile which can be oriented in only four possible ways. Comparing with Figure 5 and since probabilities add up to one, we deduce that

$$\mathbb{P}(1,0;4p+\alpha) = \frac{1}{2}\Big(1 - 2\mathbb{P}(0,-1;4p+\alpha)\Big).$$

Now, since we assumed that Theorem 1 holds for $\mathbb{P}(0,-1;4p+\alpha)$ we observe that it is also true for $\mathbb{P}(1,0;4p+\alpha)$.

Now, for arbitrary $l > 0$ we apply the same trick as described in Figure 5. Using the same notions and notations we obtain

$$\mathbb{P}(l,0;4p+\alpha) = \mathbb{P}(east) = 1 - \mathbb{P}(north) - \mathbb{P}(west) - \mathbb{P}(south),$$

and all the probabilities on the right-hand side correspond to quantities $\mathbb{P}(l', m'; 4p+\alpha)$ with $l' < l$. Thus our induction hypothesis applies and also $\mathbb{P}(l,0;4p+\alpha)$ is expressible as

$$\frac{1}{4} + 2^{-4p-\alpha}\binom{2p-1}{p}^2 f_{l,\alpha}(p)$$

for some rational function $f_{l,\alpha}(p)$. □

2.3 The case of the origin

The proof of Theorem 1 is finished once we show its statement for position $(0,0)$. To do so, we prove the following.

Proposition 3. *We have*

$$\mathbb{P}(0,0;4p+1) = \frac{1}{4} + 2^{-4p}\binom{2p-1}{p}^2, \tag{2.2}$$

$$\mathbb{P}(0,0;4p+3) = \frac{1}{4} \tag{2.3}$$

and $\mathbb{P}(0,0;4p) = \mathbb{P}(0,0;4p+2) = 0$.

Proof. First of all, the equation $\mathbb{P}(0,0;4p) = \mathbb{P}(0,0;4p+2) = 0$ just follows from the convention of our colouring since a horizontal domino at position $(0,0)$ will always have a white square on its left if the size of the diamond is even.

For the more interesting equations we apply a powerful tool from general perfect matching enumeration called Kuo condensation [14]. It tells that for a planar graph $G = (V,E)$ with distinct vertices $a,b,c,d \in V$, which are located in the same face of G and appear in cyclic order, we have for the numbers of perfect matchings

$$\begin{aligned} M(G)M(G\backslash\{a,b,c,d\}) &= M(G\backslash\{a,b\})M(G\backslash\{c,d\}) - M(G\backslash\{a,c\})M(G\backslash\{b,d\}) \\ &\quad + M(G\backslash\{a,d\})M(G\backslash\{b,c\}), \end{aligned}$$

where $M(H)$ denotes the number of perfect matchings of a subgraph H of G. Now, in order to prove Equation (2.3), the simpler of the two identities, we proceed as follows.

Since the tilings of the Aztec diamond are in bijection with perfect matchings of its dual graph, we apply this identity with a, b, c, d being the central four unit squares of the Aztec diamond. After straightforward computations which make use of the symmetry of the Aztec diamond we arrive at the equation

$$\#\text{Tilings of } \mathcal{A}\backslash\{a, b, c, d\} = 2^{(4p+4)(4p+3)/2} \cdot 2\mathbb{P}(0, 0; 4p+3)^2, \tag{2.4}$$

where $\mathcal{A}\backslash\{a, b, c, d\}$ in this case just denotes the Aztec diamond of size $4p+3$ which has a central 2×2-square hole. However, by a theorem of Ciucu [1] it is known, that for holey Aztec diamonds of size $4p+3$ we have

$$\#\text{Tilings of } \mathcal{A}\backslash\{a, b, c, d\} = 2^{8p^2+14p+3}.$$

Inserting this into Equation (2.4) this proves the identity of Equation (2.3).

On the other hand, to show Equation (2.2) we apply Kuo condensation to a region as it is shown in Figure 6. After removing all forced tiles in the subregions appearing in the Kuo condensation formula we arrive at a certain quadratic equation for $\mathbb{P}(0, 0; 4p+1)$. For this, one shows that

$$\frac{1}{4} + 2^{-4p}\binom{2p-1}{p}^2$$

is a solution and the second solution does not give correct values for the probabilities. This completes the proof of Proposition 3 and therefore also the proof of Theorem 1.

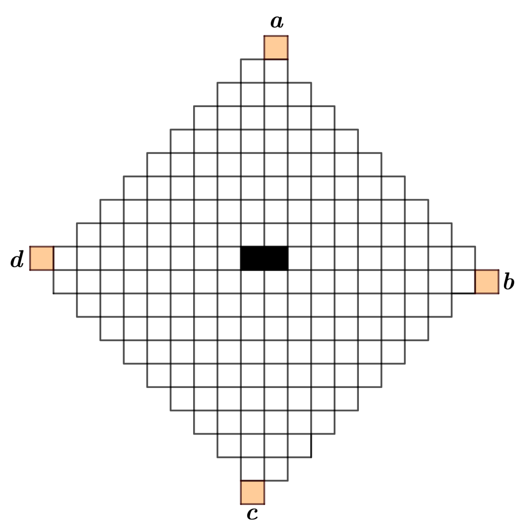

Figure 6: The domino domain $\mathcal{A}^+$: an Aztec diamond of size $4p+1$ (here $p = 2$) with a horizontal domino shaped hole at position $(0, 0)$ and additional squares a, b, c, d in particular positions of the corners. (Always added at second position at each tip, when read in clockwise order.)

□

3 Application: tiling enumeration of Aztec diamonds with 2×2-square holes.

In this last section we apply Theorem 1 to count domino tilings in an Aztec diamond which has a 2×2-square hole whose centre is located at $(l, m) \in \mathbb{Z}^2$.

Corollary 2. *The number of domino tilings of an Aztec diamond of size* $4p + \alpha$*,* $\alpha \in \{0, 1, 2, 3\}$ *with a* 2×2*-square hole centred at position* $(l, m) \in \mathbb{Z}^2$ *is equal to*

$$2^{(4p+\alpha+1)(4p+\alpha)/2}\left(\frac{1}{8} + 2^{-4p-\alpha-2}\binom{2p-1}{p}^2 g_{l,m,\alpha}(p) + 2^{-8p-2\alpha}\binom{2p-1}{p}^4 h_{l,m,\alpha}(p)\right),$$

where $g_{l,m,\alpha}(p)$ *and* $h_{l,m,\alpha}(p)$ *are two rational functions in* p*.*

Proof. The formula in Corollary 2 is again obtained by applying Kuo condensation [14] to the four unit squares forming the square hole and inserting the expressions provided by Theorem 1. □

In particular, we obtain explicit enumeration formulas for the number of domino tilings of Aztec diamonds with a central 2×2-square hole. Closed expressions were already derived by Mihai Ciucu in [1] for the numbers of tilings of diamonds of size $4p + 2$ and $4p + 3$. By Corollary 2 we are now able to state two compact formulas counting the tilings of holey Aztec diamonds of the other two cases, i.e. diamonds of size $4p$ and $4p + 1$.

Corollary 3. *The Aztec diamond with a central* 2×2*-square hole has*

$$2^{2p(4p+1)}\left(\frac{1}{8} + 2^{-4p}\binom{2p-1}{p}^2 + 2^{-8p+1}\binom{2p-1}{p}^4\right),$$

tilings if the diamond is of size $4p$ *and*

$$2^{(2p+1)(4p+1)}\left(\frac{1}{8} + 2^{-4p}\binom{2p-1}{p}^2 + 2^{-8p+1}\binom{2p-1}{p}^4\right),$$

tilings if the diamond is of size $4p + 1$*.*

References

[1] M. Ciucu. "Enumeration of Perfect Matchings in Graphs with Reflective Symmetry". *Journal of combinatorial theory, Series A 77, 67-97* (1997).

[2] H. Cohn, N. Elkies, and J. Propp. "Local statistics for random domino tilings of the Aztec diamond". *Duke Math. J.* **85**.1 (1996), pp. 117–166. DOI.

[3] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. "Alternating-sign matrices and domino tilings. I". *J. Algebraic Combin.* **1**.2 (1992), pp. 111–132. DOI.

[4] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. "Alternating-sign matrices and domino tilings. II". *J. Algebraic Combin.* **1**.3 (1992), pp. 219–234. DOI.

[5] I. Fischer. "Enumeration of rhombus tilings of a hexagon which contain a fixed rhombus in the centre". *J. Combin. Theory Ser. A* **96**.1 (2001), pp. 31–88. DOI.

[6] M. E. Fisher and J. Stephenson. "Statistical mechanics of dimers on a plane lattice. II. Dimer correlations and monomers". *Phys. Rev. (2)* **132** (1963), pp. 1411–1431.

[7] T. Gilmore. "Inverting the Kasteleyn matrix for holey hexagons". 2017. arXiv:1701.07092. Link.

[8] H. Helfgott. "Edge Effects on Local Statistics in Lattice Dimers: A Study of the Aztec Diamond (Finite Case)". 2000. arXiv:math/0007136. Link.

[9] P. W. Kasteleyn. "The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice". *Physica* (1961), pp. 1209–1225.

[10] P. W. Kasteleyn. "Dimer Statistics and Phase Transitions". *Journal of Mathematical Physics* **4**.2 (Feb. 1963), pp. 287–293. DOI.

[11] R. Kenyon. "Local statistics of lattice dimers". *Ann. Inst. H. Poincaré Probab. Statist.* **33**.5 (1997), pp. 591–618. DOI.

[12] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics. With a foreword by T. H. Koornwinder. Springer-Verlag, Berlin, 2010, pp. xx+578. DOI.

[13] C. Krattenthaler. "A (conjectural) 1/3-phenomenon for the number of rhombus tilings of a hexagon which contain a fixed rhombus". 2001. arXiv:math/0101009. Link.

[14] E. Kuo. "Graphical Condensation Generalizations Involving Pfaffians and Determinants". 2006. arXiv:math/0605154. Link.

[15] P. A. MacMahon. *Combinatory analysis. Vol. I, II (bound in one volume)*. Dover Phoenix Editions. Reprint of *An introduction to combinatory analysis* (1920) and *Combinatory analysis. Vol. I, II* (1915, 1916). Dover Publications Inc., Mineola, NY, 2004, pp. ii+761.

[16] F. J. MacWilliams and N. Sloan. *The theory of error correcting codes*. North-Holland Pub. Co., 1977.

[17] L. Petrov. "Asymptotics of random lozenge tilings via Gelfand-Tsetlin schemes". *Probab. Theory Related Fields* **160**.3-4 (2014), pp. 429–487. DOI.

[18] H. N. V. Temperley and M. E. Fisher. "Dimer problem in statistical mechanics—an exact result". *Philos. Mag. (8)* **6** (1961), pp. 1061–1063.