Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #17, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Heaps of rhombic dodecahedra, catalan congruences on alternating sign matrices, and bases of the Temperley–Lieb algebra

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Abstract. We prove that the excedance relation on permutations of N. Bergeron and L. Gagnon extends to a congruence of the lattice on alternating sign matrices. We therefore study all congruences of the lattice on ASMs whose quotient is isomorphic to the Stanley lattice on Dyck paths. The maxima of their classes are always covexillary permutations (and all covexillary permutations appear this way), and the minimal permutations in each class are always precisely the 321-avoiding permutations. Finally, any choice of representative permutations in each class yield a basis of the Temperley–Lieb algebra, vastly generalizing the bases arising from the excedance relation.

Keywords: Alternating sign matrices, lattice congruences, Temperley–Lieb algebra

1 Introduction

The quotient $\mathbf{K}[X]/\langle \mathrm{Sym}^+(X) \rangle$ of the polynomial ring on $X = \{x_1, \ldots, x_n\}$ by the ideal generated by symmetric polynomials with no constant terms is of dimension $n!$ and is isomorphic to the regular representation of the symmetric group $\mathfrak{S}_n$. Chevalley–Shephard–Todd's theorem asserts that this statement remains true when the symmetric group is replaced by any real or complex reflection group. In the combinatorics community, a large body of work has been devoted to understanding this representation-theoretic structure in more explicit terms. One notable outcome is A. Lascoux's theory of Schubert polynomials [9, 10], which provides a remarkably elegant basis for this quotient.

*Supported by the French–Austrian project PAGCAP (ANR-21-CE48-0020 & FWF I 5788) and the EU H2020 projet FRESCO (No.101001995).

†Supported by the Spanish project PID2022-137283NB-C21 of MCIN/AEI/10.13039/501100011033 / FEDER, UE, by the Spanish–German project COMPOTE (AEI PCI2024-155081-2 & DFG 541393733), by the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), by the Departament de Recerca i Universitats de la Generalitat de Catalunya (2021 SGR 00697), and by the French–Austrian project PAGCAP (ANR-21-CE48-0020 & FWF I 5788).

There is another closely related case which, however, remains much less understood: the ring of quasi-symmetric polynomials. Recall that these were introduced by I. Gessel [4] as generating series encoding permutations with a prescribed descent set. While many parallels with the theory of symmetric functions have been developed, both from combinatorial and representation-theoretic perspectives [8, 5], the invariant-theoretic viewpoint has long remained elusive. Nevertheless, there have been persistent indications that a structure analogous to the symmetric and coinvariant settings should exist.

On the one hand, the ring of quasi-symmetric polynomials is invariant under a modified action of the symmetric group [5]. In contrast to the classical situation, this action is not faithful and factors through the Temperley–Lieb algebra $\mathrm{TL}(2)$, whose dimension is the n-th Catalan number C_n. On the other hand, J. C. Aval and N. Bergeron [1] noticed that the quotient $\mathbf{K}[X]/\langle \mathrm{QSym}^+(X)\rangle$ is also of dimension C_n. However, the connection between these two occurrences of Catalan numbers is not straightforward. The reason is that the Temperley–Lieb action is not multiplicative, hence the space $\mathbf{K}[X]/\langle \mathrm{QSym}^+(X)\rangle$ does not naturally inherit an action of $\mathrm{TL}(2)$.

In a recent breakthrough [3], N. Bergeron and L. Gagnon constructed a particular set of C_n permutations whose projection in $\mathrm{TL}(2)$ gives a basis, and such that the graded ideal of the vanishing ideal is equal to $\langle \mathrm{QSym}^+(X)\rangle$. They defined the *excedance equivalence* $\equiv$ on permutations of $[n]$, where two permutations are excedance equivalent if they share the same sets of positions and of values of their excedances. They prove that

- (i) the excedance classes are Bruhat intervals, whose minima are the 321-avoiding permutations, and whose maxima are some covexillary permutations,
- (ii) the poset quotient of the Bruhat order on permutations by the excedance relation is isomorphic to the Stanley lattice on Dyck paths,
- (iii) any choice of representatives of the excedance classes yields a basis of $\mathrm{TL}(2)$,
- (iv) the top degree homogeneous components of the vanishing ideal of the set of permutations which are maximal in their excedance classes and the positive degree quasisymmetric polynomials generate the same ideal.

Our work starts from the observation that this excedance relation on the Bruhat order extends to a congruence of the lattice of alternating sign matrices (which is the MacNeille completion of the Bruhat order). Using a combination of lattice and geometric perspectives on alternating sign matrices, we then study all lattice congruences $\equiv$ on alternating sign matrices whose quotient is isomorphic to the Stanley lattice, which we call *catalan congruences*. We encode bijectively these congruences using several combinatorial structures, related to Gelfand–Tsetlin patterns and pipe dreams. Using these descriptions, we finally prove that for any catalan congruence $\equiv$,

- (i) the maxima of classes of $\equiv$ are covexillary permutation matrices,
- (ii) the minimal permutations in the classes of $\equiv$ are the 321-avoiding permutations,
- (iii) any choice of permutations in each class of $\equiv$ yields a basis of $\mathrm{TL}(2)$.

We refer to [6] for many details and all proofs omitted in this extended abstract.

2 Two distributive lattices

2.1 Meet and join representations in distributive lattices

A *lower* (resp. *upper*) *set* of a poset $\mathcal{P}$ is a subset X of $\mathcal{P}$ such that $x \in X$ and $x \geq y$ (resp. $x \leq y$) implies $y \in X$. A *lattice* $\mathbf{L}$ is *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in \mathbf{L}$. An element $x \in \mathbf{L}$ is *join* (resp. *meet*) *irreducible* if it covers (resp. is covered by) a single $x_\star$ (resp. $x^\star$). We denote by $\mathcal{J}(\mathbf{L})$ (resp. $\mathcal{M}(\mathbf{L})$) the subposet of $\mathbf{L}$ induced by its join (resp. meet) irreducibles. The following classical result is known as the fundamental theorem for distributive lattices.

Theorem 2.1 (Birkhoff)**.**
- *The set* $\mathbf{Lo}(\mathcal{P})$ *(resp.* $\mathbf{Up}(\mathcal{P})$*) of lower (resp. upper) sets of any poset* $\mathcal{P}$ *ordered by inclusion forms a distributive lattice.*
- *Conversely, any distributive lattice* $\mathbf{L}$ *is isomorphic to the inclusion poset of lower (resp. upper) sets of its join (resp. meet) irreducible poset* $\mathcal{J}(\mathbf{L})$ *(resp.* $\mathcal{M}(\mathbf{L})$*).*

For $x \in \mathbf{L}$, we denote by $\mathrm{J}(x) := \{j \in \mathcal{J}(\mathbf{L}) \mid j \leq x\}$ the corresponding lower set of $\mathcal{J}(\mathbf{L})$, and by $\mathrm{j}(x) := \max(\mathrm{J}(x))$ the corresponding antichain of $\mathcal{J}(\mathbf{L})$. Define dually $\mathrm{M}(x) := \{m \in \mathcal{M}(\mathbf{L}) \mid x \leq m\}$ and $\mathrm{m}(x) := \min(\mathrm{M}(x))$.

2.2 Dyck paths

A *Dyck path* of semilength n is a path with up steps $(1,1)$ and down steps $(1,-1)$, starting at $(0,0)$, ending at $(2n,0)$, and never passing strictly below the horizontal axis. They define the *Dyck path lattice* $\mathbf{Dyck}_n$ (also known as the *Stanley lattice*) with the relation $P \leq Q$ if P stays below Q. The join (resp. meet) irreducible Dyck paths are those with a single peak (resp. valley), hence the poset $\mathcal{J}(\mathbf{Dyck}_n)$ (resp. $\mathcal{M}(\mathbf{Dyck}_n)$) is isomorphic to the triangular poset $\mathcal{D}_n$ of triples (y_1, y_2, y_3) with $y_1 + y_2 + y_3 = n - 2$, ordered by $(y_1, y_2, y_3) \leq (z_1, z_2, z_3)$ if and only if $y_1 + y_2 \leq z_1 + z_2$ and $y_2 + y_3 \leq z_2 + z_3$.

2.3 Alternating sign matrices

We closely follow the presentation of [11]. Recall that an *alternating sign matrix* (ASM) of order $n \geq 1$ is an $(n \times n)$-matrix $A = (A_{i,j})_{1 \leq i,j \leq n}$ with entries -1, 0 or 1, with row and column sums equal to 1, and such that nonzero entries alternate between 1 and -1 in each row and each column. Its *corner sum matrix* (CSM) and *height function matrix* (HFM) are the $(n+1) \times (n+1)$-matrices $C = (C_{i,j})_{0 \leq i,j \leq n}$ and $H = (H_{i,j})_{0 \leq i,j \leq n}$ defined by $C_{i,j} = \sum_{i' \leq i, j' \leq j} A_{i',j'}$ and $H_{i,j} = i + j - 2C_{i,j}$ for $0 \leq i, j \leq n$. See Figure 1.

The HFMs ordered componentwise define a distributive lattice $\mathbf{HFM}_n$ where the join (resp. meet) of two HFMs H, H' has (i,j)-entry $\max(H_{i,j}, H'_{i,j})$ (resp. $\min(H_{i,j}, H'_{i,j})$).

We now introduce a geometric interpretation of the join irreducible poset following [11, 12]. The join irreducible poset $\mathcal{J}(\mathbf{HFM}_n)$ can be seen geometrically as a poset $\mathcal{T}_n$

$$\begin{pmatrix} \cdot & \cdot & + & \cdot & \cdot \\ \cdot & + & - & + & \cdot \\ \cdot & \cdot & + & - & + \\ + & - & \cdot & + & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 2 & 3 \\ 3 & 4 & 3 & 2 & 3 & 2 \\ 4 & 3 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Figure 1: An ASM, a CSM, and a HFM, all corresponding to each other.

on the integer points inside the $(n-2)$th dilate of a 3-dimensional tetrahedron resting on an edge [11]. See Figure 2. Using barycentric coordinates, this poset is defined on the quadruples $\boldsymbol{x} := (x_1, x_2, x_3, x_4) \in \mathbb{N}^4$ with $x_i \geq 0$ for $i \in [4]$ and $x_1 + x_2 + x_3 + x_4 = n - 2$ ordered by $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $x_1 \geq y_1$, $x_2 \leq y_2$, $x_3 \leq y_3$ and $x_4 \geq y_4$. Its cover relations are the pairs $\boldsymbol{x} \lessdot \boldsymbol{y}$ such that $\boldsymbol{y} - \boldsymbol{x}$ is one of the four vectors $b := (-1, 1, 0, 0)$, $r := (-1, 0, 1, 0)$, $o := (0, 1, 0, -1)$ and $g := (0, 0, 1, -1)$. We can thus see an HFM as a lower set in this 4-colored tetrahedral poset, as illustrated in Figure 2 (left).

Note that via the above-mentioned bijection between ASMs and HFMs, we can transport the lattice $\mathbf{HFM}_n$ on HFMs of order n to a lattice $\mathbf{ASM}_n$ on ASMs of order n. Identifying a permutation σ of $[n]$ with its matrix $P_\sigma := [\delta_{\sigma(i),j}]_{i,j \in [n]}$, this lattice $\mathbf{ASM}_n$ is the MacNeille completion of the Bruhat order on permutations of $[n]$. The join (resp. meet) irreducible ASMs are known to be the matrices of the *bigrassmannian* (resp. *anti-bigrassmannian*) permutations, *i.e.* the permutations σ such that σ and its inverse σ^{-1} have only one descent (resp. ascent). An integer point $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ in the tetrahedron corresponds to the bigrassmannian and anti-bigrassmannian permutations

$$j(\boldsymbol{x}) = \begin{pmatrix} I_{x_1} & 0 & 0 & 0 \\ 0 & 0 & I_{x_2+1} & 0 \\ 0 & I_{x_3+1} & 0 & 0 \\ 0 & 0 & 0 & I_{x_4} \end{pmatrix} \qquad \text{and} \qquad m(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & 0 & \bar{I}_{x_2} \\ 0 & \bar{I}_{x_1+1} & 0 & 0 \\ 0 & 0 & \bar{I}_{x_4+1} & 0 \\ \bar{I}_{x_3} & 0 & 0 & 0 \end{pmatrix}$$

where I_k (resp. $\bar{I}_k$) is the $k \times k$ identity (resp. anti-identity) matrix.

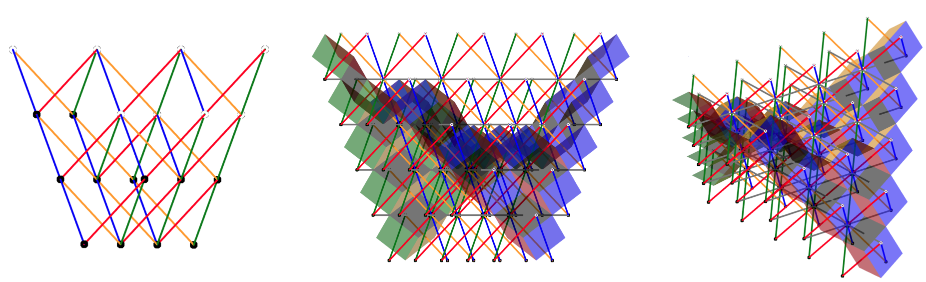


Figure 2: The ASM of Figure 1 seen as a lower set X of the tetrahedron poset $\mathcal{T}_5$ (left) or equivalently as the surface separating X from its complement (middle and right).

It is also interesting to represent an ASM as the surface S separating the lower set X of $\mathcal{T}_n$ and its complement. For this, we include additional edges so that each integer point of the tetrahedron gets 12 incident arrows (one in the direction $\boldsymbol{e}_i - \boldsymbol{e}_j$ for each $i \neq j \in [4]$). The horizontal edges, in direction $\pm(1,0,0,-1)$ and $\pm(0,1,-1,0)$, are colored in gray. The additional endpoints are considered in X (resp. in its complement) if they are incident to an arrow which is also incident to one of the two bottom (resp. top) faces of the tetrahedron. We then add a rhombus orthogonal to each arrow from a point in X to a point in its complement (the color of the rhombus matches the color of the orthogonal arrow). The corresponding surface is illustrated in Figure 2 (middle and right).

As illustrated in Figure 3, looking at this surface S from above (*i.e.* from $(-1,1,1,-1)$) makes transparent some well-known bijections between different models for ASMs:

- Seen from above, the colored rhombi of S forms a *rainbow tiling*. See Figure 3 (a).
- The *ASM* records the saddle points of S. See Figure 3 (b).
- The *six-vertex model* is given by the horizontal diagonals of the rhombi of S, oriented counterclockwise around the surface S. See Figure 3 (c).
- The *HFM* records the height of the surface above integer points. See Figure 3 (d).
- The *fully packed loop* is given by the horizontal diagonals of the red and orange (resp. green and blue) rhombi of S located at odd (resp. even) heights. Figure 3 (e).

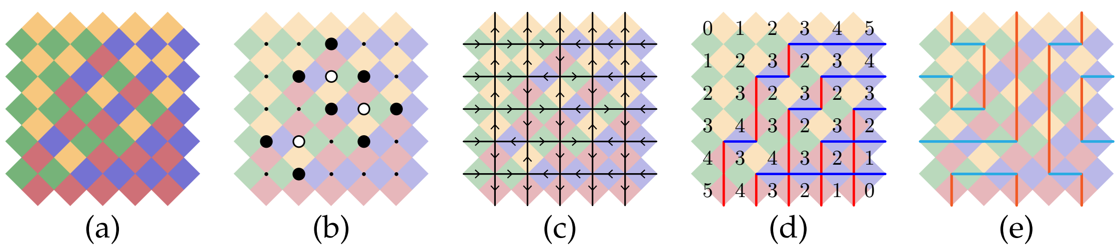


Figure 3: Reading bijections on the surface of the ASM of Figure 1.

Finally, it is also interesting to represent an ASM as a stack of rhombic dodecahedra. The *rhombic dodecahedron* is the convex hull of the 8 vertices $\pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3$ of the standard cube together with the 6 points $\pm 2\boldsymbol{e}_i$ for $i \in [3]$. See Figure 4.

Figure 4: The rhombic dodecahedron (left), and the ASM of Figure 1 seen as a pile of rhombic dodecahedra, as 3d objects in OpenSCAD (middle) and in real life (right).

3 The excedance congruence of $\mathbf{ASM}_n$

This section defines a relevant lattice congruence of $\mathbf{ASM}_n$, extending the excedance relation of the Bruhat order on permutations defined by N. Bergeron and L. Gagnon in [3].

3.1 The excedance relation on permutations

A *noncrossing partition* (NCP) of $[n]$ is a collection $\lambda \subseteq \{(i,j) \mid 1 \le i \le j \le n\}$ such that there is no $(i,j),(k,\ell) \in \lambda$ with $i \le k < j \le \ell$. Equivalently, it is a collection of arcs over the points $1,\dots,n$ on the horizontal line, such that any two arcs do not cross in their interior, nor share their left endpoints nor their right endpoints. We denote by $\lambda^- := \{i \mid (i,j) \in \lambda\}$ the set of left endpoints and $\lambda^+ := \{j \mid (i,j) \in \lambda\}$ the set of of right endpoints of λ. We denote by $\mathbf{NCP}_n$ the transitive closure of the relation $\prec$ on NCPs of $[n]$ defined by $\lambda \prec \mu$ if and only if $\lambda = \mu \smallsetminus \{(i,i+1)\}$ for some $i \in [n-1]$ or $\lambda = \mu \smallsetminus \{(i,k)\} \cup \{(i,j),(j,k)\}$ for some $1 \le i < j < k \le n$. Note that the standard bijection between NCPs of $[n]$ and Dyck paths of semilength n (see Figure 5) sends the poset $\mathbf{NCP}_n$ to the Stanley lattice $\mathbf{Dyck}_n$.

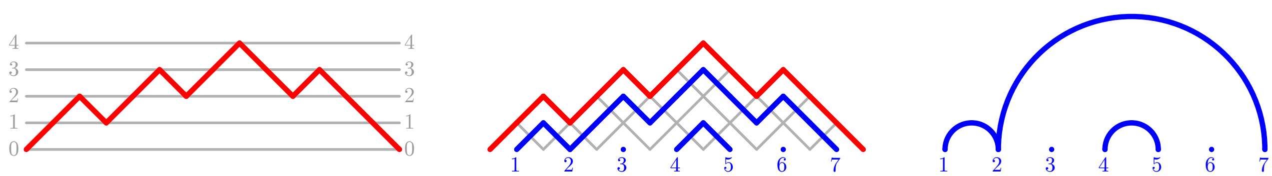


Figure 5: The standard bijection from Dyck paths (left) to NCPs (right).

Definition 3.1 ([3])**.** *A weak excedance of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i,σ_i) such that $i \le \sigma_i$. Define the excedance positions $\mathrm{E}_{\mathrm{pos}}(\sigma) := \{i \mid i \le \sigma_i\}$ and values $\mathrm{E}_{\mathrm{val}}(\sigma) := \{\sigma_i \mid i \le \sigma_i\}$. The excedance relation on $\mathfrak{S}_n$ is defined by $\sigma \equiv \tau$ if $\mathrm{E}_{\mathrm{pos}}(\sigma) = \mathrm{E}_{\mathrm{pos}}(\tau)$ and $\mathrm{E}_{\mathrm{val}}(\sigma) = \mathrm{E}_{\mathrm{val}}(\tau)$. We denote by $\mathfrak{S}_n/\!\equiv$ the poset quotient on equivalence classes of $\equiv$, defined by $X \le Y$ if and only if there are $x \in X$ and $y \in Y$ such that $x \le y$ in Bruhat order.*

Definition 3.2 ([3])**.** *For a NCP λ of $[n]$, define the set of permutations*

$$C_\lambda := \left\{\sigma \in \mathfrak{S}_n \;\middle|\; \mathrm{E}_{\mathrm{pos}}(\sigma) = [n] \smallsetminus \lambda^- \text{ and } \mathrm{E}_{\mathrm{val}}(\sigma) = [n] \smallsetminus \lambda^+\right\}.$$

Theorem 3.3 ([3])**.** *For any $n \in \mathbb{N}$,*

1. *For any $\lambda \in \mathbf{NCP}_n$, the set C_λ is an interval $[\sigma_\lambda, \tau_\lambda]$ of the Bruhat order on $\mathfrak{S}_n$, where the permutation σ_λ (resp. τ_σ) avoids the pattern 321 (resp. 3412).*
2. *The map $\lambda \mapsto C_\lambda$ is a bijection from the NCPs of $[n]$ to the excedance classes of $\mathfrak{S}_n$.*
3. *The quotient $\mathfrak{S}_n/\!\equiv$ is isomorphic to the Stanley lattice: $\lambda \le \mu \iff C_\lambda \le C_\mu$.*
4. *Any choice of representatives of the excedance classes yields a basis of the Temperley–Lieb algebra $\mathrm{TL}_n(2)$ (see Section 5.3 for details).*

3.2 The excedance congruence on ASM

We now extend the excedance relation on permutations to a lattice congruence on $\mathbf{ASM}_n$. For this, we consider the diagonal and the superdiagonal of the corresponding HFMs.

Definition 3.4. *The excedance congruence of* $\mathbf{HFM}_n$ *is the equivalence relation* $\equiv$ *defined by* $H \equiv H'$ *if and only if* $H_{i,j} = H'_{i,j}$ *for all* $0 \le i, j \le n$ *with* $j - i \in \{0, 1\}$*. The excedance congruence of* $\mathbf{ASM}_n$ *is the corresponding equivalence relation* $\equiv$ *on* $\mathbf{ASM}_n$*.*

Proposition 3.5. *The excedance congruence on* $\mathbf{ASM}_n$

- *is a lattice congruence (i.e.* $A \equiv A'$ *and* $B \equiv B' \Rightarrow A \vee B \equiv A' \vee B'$ *and* $A \wedge B \equiv A' \wedge B'$*),*
- *restricts to the excedance relation on* $\mathfrak{S}_n$ *(i.e.* $\sigma \equiv \tau \iff P_\sigma \equiv P_\tau$ *for any* $\sigma, \tau \in \mathfrak{S}_n$*).*

We will now describe the classes of the excedance congruence. For this, we first associate to an ASM A the NCP $\boldsymbol{\lambda}(A)$ corresponding to the Dyck path whose height sequence is given by the diagonal and superdiagonal of the HFM corresponding to A. See Figure 6 for an illustration.

$$\begin{pmatrix} \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot \\ \cdot & \cdot & + & \cdot & \cdot & - & + \\ \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\ + & \cdot & - & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot \\ \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \mathbf{0} & \mathbf{1} & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & \mathbf{2} & \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & \mathbf{2} & \mathbf{3} & 4 & 5 & 4 & 5 \\ 3 & 4 & 3 & \mathbf{2} & \mathbf{3} & 4 & 5 & 4 \\ 4 & 5 & 4 & 3 & \mathbf{4} & \mathbf{3} & 4 & 3 \\ 5 & 4 & 3 & 4 & 3 & \mathbf{2} & \mathbf{3} & 2 \\ 6 & 5 & 4 & 5 & 4 & 3 & \mathbf{2} & \mathbf{1} \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & \mathbf{0} \end{pmatrix}$$

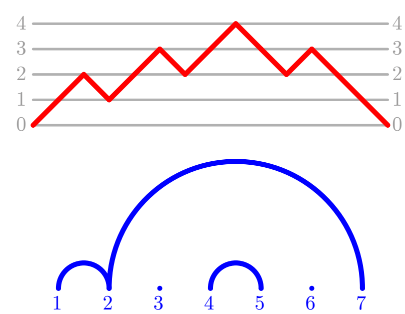


Figure 6: An ASM (left), its HFM (middle), its Dyck path (top right), and its NCP (bottom right).

The following are the analogues of Definition 3.2 and Theorem 3.3.

Definition 3.6. *For* $\lambda \in \mathbf{NCP}_n$*, we define* $D_\lambda := \{A \in \mathbf{ASM}_n \mid \boldsymbol{\lambda}(A) = \lambda\}$*.*

Theorem 3.7.

1. *For any* $\lambda \in \mathbf{NCP}_n$*, the set* D_λ *is an interval of* $\mathbf{ASM}_n$*, whose maximum is the permutation matrix* P_{τ_λ}*, and we have* $C_\lambda = \{\sigma \in \mathfrak{S}_n \mid P_\sigma \in D_\lambda\}$*.*
2. *The map* $\lambda \mapsto D_\lambda$ *is a bijection from the NCPs of* $[n]$ *to the excedance classes of* $\mathbf{ASM}_n$*.*
3. *The lattice quotient* $\mathbf{ASM}_n/\equiv$ *is isomorphic to the Stanley lattice:* $\lambda \le \mu \iff D_\lambda \le D_\mu$*.*

The goal of this paper is to show that all properties of Theorem 3.3 actually hold for any lattice quotient of $\mathbf{ASM}_n$ isomorphic to $\mathbf{Dyck}_n$.

4 Catalan congruences of $\mathbf{ASM}_n$

This section focuses on the following generalization of Section 3.2.

Definition 4.1. *A catalan congruence is a congruence of the lattice $\mathbf{ASM}_n$ on alternating sign matrices whose quotient is isomorphic to the Stanley lattice $\mathbf{Dyck}_n$ on Dyck paths.*

In this section, we describe all catalan congruences and encode them bijectively using several combinatorial structures, in particular Gelfand–Tsetlin patterns and pipe dreams.

4.1 Quotients of distributive lattices

Quotients of distributive lattices are distributive (by the definition of $\vee$ and $\wedge$ on $\mathbf{L}/\equiv$), and behave properly with respect to Birkhoff's fundamental theorem of distributive lattices (Theorem 2.1). Recall that we denote by $j_\star$ the single element covered by a join irreducible j.

Theorem 4.2. *For a congruence $\equiv$ on a distributive lattice $\mathbf{L}$, denote by $\mathcal{J}(\equiv)$ the subposet of $\mathcal{J}(\mathbf{L})$ induced by the join irreducibles j such that $j \not\equiv j_\star$. Then*

- *an element $x \in \mathbf{L}$ is minimal in its congruence class if and only if the corresponding antichain $\mathrm{j}(x)$ is contained in $\mathcal{J}(\equiv)$,*
- *the quotient $\mathbf{L}/\equiv$ is isomorphic to the distributive lattice of lower sets of $\mathcal{J}(\equiv)$.*

Dual statements hold for meet irreducibles.

Theorem 4.3 ([13])**.** *For two finite distributive lattices $\mathbf{L}$ and $\mathbf{K}$, the lattice congruences of $\mathbf{L}$ whose quotients are isomorphic to $\mathbf{K}$ are in bijection with the subposets of $\mathcal{J}(\mathbf{L})$ isomorphic to $\mathcal{J}(\mathbf{K})$.*

4.2 Walls

We now specialize Theorem 4.3 to understand all catalan congruences of $\mathbf{ASM}_n$. We have seen in Section 2 that $\mathcal{J}(\mathbf{ASM}_n)$ is isomorphic to the tetrahedron poset $\mathcal{T}_n$, and $\mathcal{J}(\mathbf{Dyck}_n)$ is isomorphic to the triangular poset $\mathcal{D}_n$. Hence quotients of $\mathbf{ASM}_n$ isomorphic to $\mathbf{Dyck}_n$ are in bijection with subposets of $\mathcal{T}_n$ isomorphic to $\mathcal{D}_n$. We now show that this isomorphism always factorizes through the projection $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + x_3, x_4)$ of $\mathcal{T}_n$.

Proposition 4.4. *Let $\mathcal{P}$ be a subposet of the tetrahedron poset $\mathcal{T}_n$ isomorphic to the triangular poset $\mathcal{D}_n$. For any $(y_1, y_2, y_3) \in \mathcal{D}_n$, the subposet $\mathcal{P}$ contains exactly one element (x_1, x_2, x_3, x_4) such that $x_1 = y_1$ and $x_4 = y_3$ (and thus $x_2 + x_3 = y_2$).*

In other words, Proposition 4.4 states that the subposets of $\mathcal{T}_n$ isomorphic to $\mathcal{D}_n$ can be seen as vertical (possibly undulating) sections of $\mathcal{T}_n$, which motivates the following.

Definition 4.5. *A wall of order n is a subposet of $\mathcal{T}_n$ isomorphic to $\mathcal{D}_n$.*

Example 4.6. *The excedance quotient defined in Section 3.2 is given by the undulating wall* $\{(x_1, x_2, x_3, x_4) \in \mathcal{T}_n \mid x_3 - x_2 \in \{0, -1\}\}$.

4.3 Depth triangles, catalan triangles, and bicolored pipe dreams

For any wall $\mathcal{P} \subset \mathcal{T}_n$, the projection $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + x_3, x_4)$ is a bijection from $\mathcal{P}$ to $\mathcal{D}_n$ by Proposition 4.4. We can therefore encode the wall $\mathcal{P}$ by recording the depth $x_3 - x_2$ of the point (x_1, x_2, x_3, x_4) in the direction of the projection. See Figure 10. We now characterize the triangular arrays obtained this way, transform them to certain Gelfand–Tsetlin patterns by an appropriate shift, and connect them to pipe dreams.

Definition 4.7. *A depth triangle of order n is a map $\delta : \mathcal{D}_n \to \mathbb{Z}$ such that*

- *the bottom row is $00\cdots0$, that is, $\delta(i, 0, n-2-i) = 0$ for all $0 \le i \le n-2$,*
- $|\delta(y_1, y_2, y_3) - \delta(y_1, y_2 - 1, y_3 + 1)| = 1 = |\delta(y_1, y_2, y_3) - \delta(y_1 + 1, y_2 - 1, y_3)|$.

Note that a depth triangle of order n has $n-1$ rows (or diagonals). See Figure 7.

$$
\begin{array}{ccccc} & & -2 & & \\ & -1 & & -1 & \\ 0 & & 0 & & 0 \end{array}
\qquad
\begin{array}{ccccc} & & 0 & & \\ & -1 & & -1 & \\ 0 & & 0 & & 0 \end{array}
\qquad
\begin{array}{ccccc} & & 0 & & \\ & 1 & & -1 & \\ 0 & & 0 & & 0 \end{array}
\qquad
\begin{array}{ccccc} & & 0 & & \\ & -1 & & 1 & \\ 0 & & 0 & & 0 \end{array}
\qquad
\begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 0 & & 0 & & 0 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 1 & & 1 & \\ 0 & & 0 & & 0 \end{array}
$$

Figure 7: The 6 depth triangles of order 4.

Definition 4.8. *A Gelfand–Tsetlin pattern of order n is a map $\gamma : \mathcal{D}_n \to \mathbb{N}$ such that $\gamma(y_1, y_2 - 1, y_3 + 1) \le \gamma(y_1, y_2, y_3) \le \gamma(y_1 + 1, y_2 - 1, y_3)$. We say that γ is doubly gapless if $\gamma(y_1, y_2, y_3) - \gamma(y_1, y_2 - 1, y_3 + 1) \le 1$ and $\gamma(y_1 + 1, y_2 - 1, y_3) - \gamma(y_1, y_2, y_3) \le 1$. A catalan triangle of order n is a doubly gapless Gelfand–Tsetlin pattern with bottom row $12\cdots(n-1)$. Again, note that a catalan triangle of order n has $n-1$ rows (or diagonals). See Figure 8.*

$$
\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}
$$

Figure 8: The 6 catalan triangles of order 4.

Definition 4.9. *A pipe dream P is a filling of a triangular shape with crossings $+$ and contacts so that all pipes entering on the left side exit on the top side [2, 7]. A bicolored pipe dream is a pipe dream whose pipes are colored red or blue so that each contact is bichromatic. Note that there is no color restriction on the crossings: they are allowed to be both monocolor or bicolor. See Figure 9.*

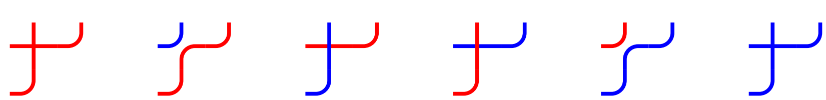
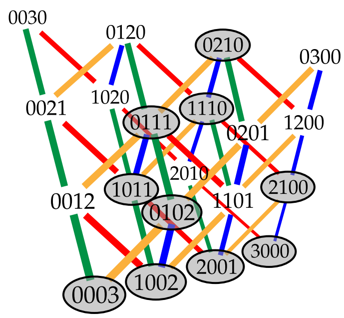


Figure 9: The 6 bicolored pipe dreams with 2 pipes.

The following statement is illustrated in Figure 10.



$$\begin{array}{ccccccc} & & & -1 & & & \\ & & 0 & & 0 & & \\ & -1 & & 1 & & -1 & \\ 0 & & 0 & & 0 & & 0 \end{array} \qquad \begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 3 & & \\ & 1 & & 3 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

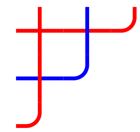


Figure 10: A subposet of $\mathcal{T}_5$ isomorphic to $\mathcal{D}_5$ (left) and its corresponding depth triangle (middle left), catalan triangle (middle right), and bicolored pipe dream (right).

Theorem 4.10. *The catalan congruences of* $\mathbf{ASM}_n$ *are in bijection with walls of order* n*, depth triangles of order* n*, catalan triangles of order* n*, and bicolored pipe dreams with* $n-2$ *pipes.*

Example 4.11. *Following Example 4.6, for the excedance quotient of Section 3.2,*

- *the depth triangle is given by* $\delta(y_1, y_2, y_3) = 2\lfloor y_2/2 \rfloor - y_2 = -(y_2 \mod 2)$ *(in other words, odd rows are* 0 *while even rows are* -1*)*
- *the catalan triangle is given by* $\gamma(y_1, y_2, y_3) = y_1 + \lfloor y_2/2 \rfloor + 1$.
- *the bicolored pipe dream is the pipe dream with no crossing (hence waving pipes alternating colors), whose longest pipe is blue.*

4.4 The lattice of catalan triangles

Finally, we note that the set of catalan triangles carries itself a natural distributive lattice structure.

Proposition 4.12. *The set of catalan triangles of order* n *ordered by coordinatewise comparison forms a distributive lattice* $\mathbf{Cat}_n$*, whose join-irreducibles poset is isomorphic to the poset of quadruples* $(x_1, x_2, x_3, x_4) \in \mathbb{N}^4$ *with* $x_1 + x_2 + x_3 + x_4 = n - 2$*, ordered by* $\mathbf{x} \le \mathbf{y}$ *if and only if* $x_2 + x_3 + x_4 \le y_2 + y_3 + y_4$, $x_2 + x_4 \le y_2 + y_4$, $x_3 + x_4 \le y_3 + y_4$ *and* $x_4 \le y_4$.

Remark 4.13. *By Theorem 4.10, Proposition 4.12 also translates to the depth triangles of Definition 4.7 and the bicolored pipe dreams of Definition 4.9.*

Remark 4.14. *The lattice* $\mathbf{Cat}_n$ *has an automorphism corresponding to the diagonal reflection of pipe dreams and an anti-automorphism corresponding to color exchange of pipe dreams.*

Example 4.15. *The minimum (resp. maximum) depth triangles, catalan triangles, and bicolored pipe dreams are given by:*

- $\delta_{\min}(y_1, y_2, y_3) = -y_2$ *(resp.* $\delta_{\max}(y_1, y_2, y_3) = y_2$*),*
- $\gamma_{\min}(y_1, y_2, y_3) = y_1 + 1$ *(resp.* $\gamma_{\max}(y_1, y_2, y_3) = y_1 + y_2 + 1 = n - 1 - y_3$*),*
- *the completely red (resp. blue) pipe dreams with only crossings.*

5 Properties of catalan congruences

5.1 Maxima of classes and covexillary permutations

We now describe the maxima of the classes of catalan congruences of $\mathbf{ASM}_n$.

Theorem 5.1. *The following assertions are equivalent for an ASM A of order n:*

(i) A is a covexillary permutation matrix (i.e. avoiding the pattern 3412),

(ii) there exists a wall of order n containing all meet irreducibles of $\mathrm{m}(A)$,

(iii) there exists a catalan congruence of $\mathbf{ASM}_n$ such that A is maximal in its class.

Example 5.2. *Following Example 4.15, the maxima of the classes of the congruence corresponding to the minimal (resp. maximum) catalan triangle are the 231 (resp. 312) avoiding permutations.*

5.2 Minimal permutations in classes and 321 avoidance

We now describe the minimal permutations in classes of catalan congruences of $\mathbf{ASM}_n$. In contrast to Section 5.1, the minimum of the classes are not necessarily permutations. However, it turns out that each class contains a minimal permutation.

Proposition 5.3. *Let $\equiv$ be a catalan congruence on $\mathbf{ASM}_n$. Let $\sigma \in \mathfrak{S}_n$ containing the pattern 321, and let $i < j < k$ be such that $\sigma(i) > \sigma(j) > \sigma(k)$. Then at least one of the permutations $\sigma \circ (i, j)$ and $\sigma \circ (j, k)$ is congruent to σ. Moreover, if both are, then $\sigma \circ (i, k)$ is also congruent to σ.*

Theorem 5.4. *For a catalan congruence on $\mathbf{ASM}_n$,*

- *each class contains a unique minimal permutation,*
- *the minimal permutations of the classes are precisely the 321-avoiding permutations.*

5.3 Bases of the Temperley–Lieb algebra

Extending the work of N. Bergeron and L. Gagnon [3], we finally construct various bases of the Temperley–Lieb algebra $\mathrm{TL}_n(2)$ using the catalan congruences studied in Section 4. Recall that, for $q \in \mathbb{C}$, the *Temperley–Lieb algebra* $\mathrm{TL}_n(q)$ is the $\mathbb{C}$-algebra generated by $e_1, \ldots, e_{n-1}$ subject to the *Jones relations*

$$\begin{aligned} e_i^2 &= qe_i \quad \text{for } i \in [n-1], & e_ie_{i+1}e_i &= e_i \quad \text{for } i \in [n-2], \\ e_{i+1}e_ie_{i+1} &= e_{i+1} \quad \text{for } i \in [n-2], & e_ie_j &= e_je_i \quad \text{for } i, j \in [n-1] \text{ with } |i-j| \neq 1. \end{aligned}$$

It is well known that the set T_n of 321-avoiding permutations of $[n]$ forms a basis of $\mathrm{TL}_n(2)$. This was extended to Theorem 3.3 (4) in [3]. We extend their result to our catalan congruences of $\mathbf{ASM}_n$.

Theorem 5.5. *Any choice of representative permutations in the congruence classes of any catalan congruence on $\mathbf{ASM}_n$ forms a basis of $\mathrm{TL}_n(2)$.*

DOI
— V.

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