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Point counting on intersections of adjoint orbits with ad-nilpotent ideals over $\mathbb{F}_q$

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Abstract. Let $\mathfrak{b}_n(\mathbb{F}_q)$ denote the Lie algebra of upper triangular $n \times n$ matrices with entries in the finite field $\mathbb{F}_q$. For every $\mathfrak{b}_n(\mathbb{F}_q)$-stable ideal $\mathfrak{a}$ of the Lie algebra $\mathfrak{u}_n(\mathbb{F}_q)$ of strictly upper triangular matrices, and every partition μ of n, we obtain an explicit formula for the number of elements of $\mathfrak{a}$ of Jordan type μ. Up to an explicit factor depending on q, our formula is given by the Hall scalar product of a modified Hall–Littlewood function indexed by μ and a chromatic quasisymmetric function associated to $\mathfrak{a}$. In the special case that $\mathfrak{a} = \mathfrak{u}_\Lambda(\mathbb{F}_q)$, where $\mathfrak{u}_\Lambda(\mathbb{F}_q)$ is the nilradical of the standard parabolic subalgebra of $\mathfrak{gl}_n(\mathbb{F}_q)$ associated to a composition Λ of n, our formula reduces to the statement (recently proved by Karp and Thomas) that up to an explicit polynomial factor in q, the number of elements in $\mathfrak{u}_\Lambda(\mathbb{F}_q)$ of Jordan type μ is equal to the coefficient of the monomial $\mathbf{x}^\Lambda$ in the specialization of the dual Macdonald symmetric function $Q_{\mu'}(\mathbf{x}; q^{-1}, t)$ at $t = 0$. We obtain a new and shorter proof of the latter assertion using a parabolic variation of Borodin's division algorithm.

We give four applications of our main result stated above. First, we obtain an analogous formula for the number of points of a nilpotent Hessenberg variety. Second, we demonstrate that a recurrence relation of Kirillov for the number of strictly upper triangular matrices of Jordan type μ is a straightforward consequence of our main result combined with the Cauchy identity for Macdonald polynomials. Third, we give an explicit formula for the number of $X \in \mathfrak{u}_\Lambda(\mathbb{F}_q)$ that satisfy $X^2 = 0$. In the special case $\Lambda = (1^n)$, our formula is different from the one conjectured by Kirillov and Melnikov (and proved by Ekhad and Zeilberger). However, we obtain a new proof of the Kirillov–Melnikov–Ekhad–Zeilberger formula using our latter result and a formula for two-rowed Macdonald polynomials by Jing and Józefiak. Fourth, we give an

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explicit formula for the number of double cosets $\mathrm{U}_1\backslash\mathrm{GL}_n(\mathbb{F}_q)/\mathrm{U}_2$ where U_1 and U_2 denote unipotent subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$ that correspond to two $\mathfrak{b}_n(\mathbb{F}_q)$-stable ideals of $\mathfrak{u}_n(\mathbb{F}_q)$.

Keywords: Finite fields, Jordan form, Macdonald polynomials, q-Hermite polynomials, Hessenberg functions, chromatic quasisymmetric functions.

Detailed proofs of all results in this extended abstract can be found in [2].

1 Ad-nilpotent orbital varieties: main results

Let $\mathsf{U}_n(\mathbb{F}_q)$ be the group of $n \times n$ upper-triangular matrices over the finite field $\mathbb{F}_q$ with all of the diagonal entries equal to 1, and let $\mathfrak{u}_n(\mathbb{F}_q)$ denote the Lie algebra of $n \times n$ upper-triangular nilpotent matrices over $\mathbb{F}_q$. In an influential and elegant paper [18], Kirillov initiated a comprehensive study of the orbit method for $\mathsf{U}_n(\mathbb{F}_q)$, establishing various connections between the adjoint and coadjoint orbit structures of $\mathsf{U}_n(\mathbb{F}_q)$ and $\mathfrak{u}_n(\mathbb{F}_q)$ and representation theory. In connection to these ideas, Kirillov posed the following problem:

Given a partition μ of n, count the number of $X \in \mathfrak{u}_n(\mathbb{F}_q)$ of Jordan type μ.

This problem was investigated by Kirillov himself and revisited later on by Borodin [3], who used a technique called "the division algorithm" to provide a recursive formula for the number of these matrices. Borodin applied this formula to study the asymptotic behavior of the Jordan type of strictly upper triangular $n \times n$ matrices as $n \to \infty$.

One can think of $\mathfrak{u}_n(\mathbb{F}_q)$ as the nilradical of the standard Borel subalgebra of $\mathfrak{gl}_n(\mathbb{F}_q)$. In this setting Kirillov's question is about the number of points of the (not necessarily irreducible) variety obtained as the intersection of $\mathfrak{u}_n(\mathbb{F}_q)$ and a nilpotent orbit of $\mathfrak{gl}_n(\mathbb{F}_q)$. Over $\mathbb{C}$, such varieties are sometimes called *orbital varieties*; see for example [15]. From this viewpoint it is natural to ask Kirillov's question for nilradicals of other parabolic subalgebras, or even more generally, for any ideal of $\mathfrak{u}_n(\mathbb{F}_q)$ that is stable under the action of the standard Borel subalgebra $\mathfrak{b}_n(\mathbb{F}_q)$ of $\mathfrak{gl}_n(\mathbb{F}_q)$. The latter ideals are sometimes simply called *ad-nilpotent ideals* of $\mathfrak{u}_n(\mathbb{F}_q)$. Their connection with nilpotent orbits has attracted attention (for example see [6, 5, 8]). Our primary goal in this paper is to answer this more general form of Kirillov's question (see Theorems 1.6, 1.8 and 1.9). Subsequently, we give various applications of the latter theorems.

Recall that a composition Λ of n is a finite tuple of positive integers whose sum equals n. When the parts of the composition form a weakly decreasing sequence, we call it a partition. The parabolic subgroup of $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$ associated to Λ is denoted by $\mathrm{P}_\Lambda = \mathrm{P}_\Lambda(\mathbb{F}_q)$, and the unipotent radical of P_Λ is denoted by $\mathrm{U}_\Lambda = \mathrm{U}_\Lambda(\mathbb{F}_q)$. The nilradical of the parabolic subalgebra of $\mathfrak{gl}_n(\mathbb{F}_q)$ associated to Λ will be denoted by $\mathfrak{u}_\Lambda = \mathfrak{u}_\Lambda(\mathbb{F}_q)$. For instance when $\Lambda = (2,3,1)$ we have:

$$P_\Lambda = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \quad U_\Lambda = \begin{pmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{u}_\Lambda = \begin{pmatrix} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the $*$'s represent arbitrary elements of $\mathbb{F}_q$. For $\Lambda = (1^n)$, we obtain $\mathfrak{u}_\Lambda = \mathfrak{u}_n(\mathbb{F}_q)$.

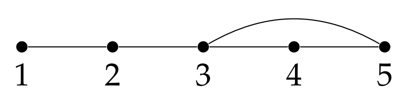
We now give a simple yet quite useful characterization of the ad-nilpotent ideals of $\mathfrak{u}_n(\mathbb{F}_q)$, connecting them to Dyck paths.

Definition 1.1 (Hessenberg function)**.** A Hessenberg function is a non-decreasing function $\mathsf{h}\colon [n] \to [n]$ such that $\mathsf{h}(i) \geq i$ for every $i \in [n]$, where $[n] := \{1, \ldots, n\}$. The set of all Hessenberg functions on $[n]$ will be denoted by $\mathcal{H}_n$. We sometimes identify a Hessenberg function by the n-tuple $(\mathsf{h}(1), \ldots, \mathsf{h}(n))$.

One can naturally associate a graph to h which is the graph with vertex set $[n]$ and set of edges $E = \{\{i,j\} \mid i < j \leq \mathsf{h}(i)\}$. This graph, known as the *indifference graph* of h, will be denoted by $\mathsf{G}(\mathsf{h})$. A Hessenberg function $\mathsf{h} \in \mathcal{H}_n$ also produces a partial order on $[n]$ by declaring $i \prec_{\mathsf{h}} j$ if and only if $\mathsf{h}(i) < j$. We denote the poset $([n], \preceq_{\mathsf{h}})$ by $\mathcal{P}_{\mathsf{h}}$. The graph $\mathsf{G}(\mathsf{h})$ is the *incomparability graph* of the poset $\mathcal{P}_{\mathsf{h}}$. Associated to h, define a Lie subalgebra of $\mathfrak{u}_n(\mathbb{F}_q)$ by $\mathfrak{u}_{\mathsf{h}} = \mathfrak{u}_{\mathsf{h}}(\mathbb{F}_q) \overset{\text{def}}{=} \{(a_{ij}) \in \mathfrak{u}_n(\mathbb{F}_q) : a_{ij} = 0 \text{ for } j \leq \mathsf{h}(i)\}$.

Example 1.2. For $\mathsf{h} \in \mathcal{H}_n$ where $\mathsf{h} = (2,3,5,5,5)$ we have

$$\mathfrak{u}_{\mathsf{h}} = \left\{ \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : * \in \mathbb{F}_q \right\},$$



$\mathsf{G}(\mathsf{h})$

Proposition 1.3. *The assignment $\mathsf{h} \mapsto \mathfrak{u}_{\mathsf{h}}$ is a bijective correspondence between $\mathcal{H}_n$ and ad-nilpotent ideals of $\mathfrak{u}_n(\mathbb{F}_q)$.*

Throughout this extended abstract, we denote $\sum_{i=1}^n \mathsf{h}(i)$ by $|\mathsf{h}|$. The (non-commutative) product of $\mathsf{h}_1 \in \mathcal{H}_n$ and $\mathsf{h}_2 \in \mathcal{H}_m$ is defined by

$$\mathsf{h} = \mathsf{h}_1 \sqcup \mathsf{h}_2 \in \mathcal{H}_{n+m}; \qquad \mathsf{h}(i) = \begin{cases} \mathsf{h}_1(i) & 1 \leq i \leq n; \\ n + \mathsf{h}_2(i-n) & n+1 \leq i \leq n+m. \end{cases}$$

The Hessenberg function $k_n = (n, \ldots, n) \in \mathcal{H}_n$ is called the complete Hessenberg function of size n. To each composition $\Lambda = (\Lambda_1, \ldots, \Lambda_s)$ of n we assign the Hessenberg function $k_\Lambda \overset{\text{def}}{=} k_{\Lambda_1} \sqcup k_{\Lambda_2} \sqcup \cdots \sqcup k_{\Lambda_s}$, and we note that $\mathfrak{u}_{k_\Lambda} = \mathfrak{u}_\Lambda$.

For any $k \geq 1$ let J_k be the $k \times k$ nilpotent (upper triangular) Jordan block. Given a partition $\mu = (\mu_1, \ldots, \mu_k)$ of n (written $\mu \vdash n$), we use J_μ to denote the square block-diagonal matrix of size $|\mu| = \mu_1 + \cdots + \mu_k$ with diagonal blocks $J_{\mu_1}, \ldots, J_{\mu_k}$. We say that the Jordan type of a matrix is μ when its Jordan form is J_μ.

Definition 1.4 (Jordan type counting function)**.** Let μ be a partition of n. For a given composition Λ of n we set $\mathrm{F}_{\mu\Lambda}(q) \stackrel{\text{def}}{=} \#\left(\mathcal{C}_\mu \cap \mathfrak{u}_\Lambda(\mathbb{F}_q)\right)$, where $\mathcal{C}_\mu = \mathcal{C}_\mu(\mathbb{F}_q)$ is the $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy class of J_μ. For a Hessenberg function $h \in \mathcal{H}_n$ set $\mathrm{F}_{\mu\mathsf{h}}(q) \stackrel{\text{def}}{=} \#\left(\mathcal{C}_\mu \cap \mathfrak{u}_\mathsf{h}(\mathbb{F}_q)\right)$.

Recall that the length of a partition μ, denoted by $\ell(\mu)$, is the number of its parts. It is convenient to define $\mu_i = 0$ for all $i > \ell(\mu)$. The conjugate of μ, denoted by $\mu' = (\mu'_1, \ldots, \mu'_l)$, is defined by $\mu'_j = \#\{1 \leq i \leq k : \mu_i \geq j\}$. In particular we have $\mu'_1 = \ell(\mu)$ and $l = \ell(\mu') = \mu_1$. For instance the conjugate of $\mu = (3, 2, 2, 1)$ is $\mu' = (4, 3, 1)$. Given partitions $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_s)$ of n, we say that ν "dominates" μ, and we write $\mu \trianglelefteq \nu$, if and only if $\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \nu_j$ for $i \geq 1$. Note that conjugation reverses the dominance order, i.e., $\nu' \trianglelefteq \mu'$. We also remark that $\ell(\nu) \leq \ell(\mu)$ when $\mu \trianglelefteq \nu$.

Assigned to each partition μ we define $\mathsf{n}(\mu) \stackrel{\text{def}}{=} \sum_{i \geq 1}(i-1)\mu_i = \sum_{i \geq 1}\binom{\mu'_i}{2}$. As usual, to any $\mu = (\mu_1, \ldots, \mu_k)$ we associate a Young diagram of shape μ: a left-justified shape of k rows of boxes, where the i-th row from the top has μ_i boxes for $1 \leq i \leq k$. A standard Young tableau is a filling of the Young diagram of μ by $\{1, \ldots, |\mu|\}$ with strictly increasing rows and columns. A semi-standard Young tableau of shape μ is a filling of this Young diagram by positive integers with weakly increasing rows and strictly increasing columns. A semi-standard Young tableau with shape μ has content Λ if there are Λ_i occurrences of i for every $i \geq 1$. The number of semi-standard Young tableaux with shape μ and content Λ is called a Kostka number and it is denoted by $\mathrm{K}_{\mu\Lambda}$.

For partitions ν and μ, we write $\nu \subseteq \mu$ to mean that the Young diagram of μ contains the Young diagram of ν, i.e., $\nu_i \leq \mu_i$ for all $i \geq 1$. The set-theoretic difference of the Young diagrams of μ and ν, denoted by μ/ν, is called a skew diagram. A horizontal k-strip is a skew diagram with k boxes with no two squares in the same column. For instance the skew diagram $(5, 4, 4, 1)/(4, 4, 2)$ is a horizontal 4-strip. For a composition Λ of n we use $\mathrm{sort}(\Lambda)$ to denote the partition obtained by sorting the parts of Λ in decreasing order. For a non-negative integer n we adopt the following conventions for q-integers, q-factorials, and q-binomial coefficients:

$$[0]_q \stackrel{\text{def}}{=} 0, \quad [n]_q \stackrel{\text{def}}{=} \frac{q^n - 1}{q - 1}, \quad [0]_q! \stackrel{\text{def}}{=} 1, \quad [n]_q! \stackrel{\text{def}}{=} [1]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{\text{def}}{=} \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Definition 1.5. Let λ and μ be partitions of n with $\ell(\lambda) = s$. We define

$$b_{\mu\lambda}(q) \stackrel{\text{def}}{=} \sum_{\mathcal{F}} \prod_{j=1}^{s} \theta_{\mu^j/\mu^{j-1}}(q),$$

with

$$\theta_{\eta/\rho}(q) \overset{\text{def}}{=} \frac{[|\eta| - |\rho|]_q!}{[\eta_1 - \rho_1]_q!} \prod_{i \geq 1} \begin{bmatrix} \rho_i - \rho_{i+1} \\ \eta_{i+1} - \rho_{i+1} \end{bmatrix}_q, \tag{1.1}$$

where η/ρ is a horizontal strip and $\mathcal{F} := \mathcal{F}(\mu, \lambda)$ is the set of all chains of partitions $\emptyset = \mu^0 \subseteq \mu^1 \subseteq \cdots \subseteq \mu^s = \mu$, such that μ^j/μ^{j-1} is a horizontal λ_j-strip. The set $\mathcal{F}(\mu, \lambda)$ is naturally in bijection with semistandard tableaux of shape μ and content λ. By convention, $b_{\mu\lambda}(q) = 0$ when $\mathcal{F} = \emptyset$. Note that in Equation (1.1) we have $\rho_i - \rho_{i+1} \geq \eta_{i+1} - \rho_{i+1}$ because η/ρ is a horizontal strip.

We now state the first main result of the paper.

Theorem 1.6. *Let Λ be a composition of n and set $\lambda = \mathrm{sort}(\Lambda)$. Let μ be a partition of n. Then $\mathrm{F}_{\mu\Lambda}(q) \neq 0$ if and only if $\lambda \trianglelefteq \mu'$. When $\lambda \trianglelefteq \mu'$, the following hold:*

1. $\mathrm{F}_{\mu\Lambda}(q) = \mathrm{F}_{\mu\lambda}(q) = (q-1)^{n-\ell(\mu)} q^{\binom{n}{2} + \ell(\mu) - \mathsf{n}(\mu) - n}\ b_{\mu'\lambda}(1/q)$.
2. *$\mathrm{F}_{\mu\lambda}(q) \in \mathbb{Z}[q]$, the degree of $\mathrm{F}_{\mu\lambda}(q)$ is $\binom{n}{2} - \mathsf{n}(\mu)$, and its leading coefficient is $\mathrm{K}_{\mu'\lambda}$, the Kostka number of shape μ' and content λ.*
3. *There exists a polynomial $\mathrm{R}_{\mu\lambda}(q) \in \mathbb{Z}[q]$ such that $\mathrm{R}_{\mu\lambda}(0) = 1$, $\mathrm{R}_{\mu\lambda}(1) > 0$, and*

$$\mathrm{F}_{\mu\lambda}(q) = (q-1)^{n-\ell(\mu)} q^{\binom{n}{2} - \binom{\ell(\mu)}{2} - \sum_{j \geq 1} \mu'_j \mu'_{j+1}}\, \mathrm{R}_{\mu\lambda}(q). \tag{1.2}$$

Parts (2) and (3) of Theorem 1.6 extend some of the results of [3] and [9, Theorem 4.1]. The main tool in the proof of Theorem 1.6 is a parabolic variant of Borodin's division algorithm to reduce the counting problem from $\mathfrak{u}_\Lambda(\mathbb{F}_q)$ to a smaller nilradical. The $b_{\mu\lambda}(q)$ are related to the coefficients of Macdonald polynomials up to an explicit normalization factor, as we explain below.

Definition 1.7. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of n. For any partition $\mu \vdash n$, we define

$$a_{\mu\lambda}(q) \overset{\text{def}}{=} \sum_{\mathcal{F}} \prod_{j=1}^{s} \psi_{\mu^j/\mu^{j-1}}(q) \quad \text{with} \quad \psi_{\eta/\rho}(q) \overset{\text{def}}{=} \prod_{i \geq 1} \begin{bmatrix} \eta_i - \eta_{i+1} \\ \eta_i - \rho_i \end{bmatrix}_q, \tag{1.3}$$

where η/ρ is a horizontal strip and $\mathcal{F}$ represents $\mathcal{F}(\mu, \lambda)$ as in Theorem 1.5. By convention, $a_{\mu\lambda}(q) = 0$ when $\mathcal{F} = \emptyset$.

Let $\mathrm{P}_\mu(\mathbf{x}; q, t)$ denote the Macdonald symmetric function in $\mathbf{x} = (x_1, x_2, \ldots)$ indexed by μ. Also, let the $\mathrm{Q}_\mu(\mathbf{x}; q, t)$ denote the duals of the $\mathrm{P}_\mu(\mathbf{x}; q, t)$ with respect to the q,t-scalar product [20, p. 323, Eq. (4.12)]. From [20, p. 346, Eq. (7.13')], one obtains

$\mathrm{P}_\mu(\mathbf{x};q,0)=\sum_\nu a_{\mu\nu}(q)m_\nu(\mathbf{x})$, where $m_\nu(\mathbf{x})$ is the monomial symmetric polynomial associated to ν. Then we have [22, Theorem 1.6]

$$b_{\mu\lambda}(q)=\left(\prod_{i\geq 1}\frac{[\lambda_i]_q!}{[\mu_i-\mu_{i+1}]_q!}\right)a_{\mu\lambda}(q). \tag{1.4}$$

We denote the coefficient of $\mathbf{x}^\lambda=x_1^{\lambda_1}x_2^{\lambda_2}\cdots$ in $\Phi(\mathbf{x})$ by $[\mathbf{x}^\lambda]\Phi(\mathbf{x})$. Then using Equation (1.4), we can state the formula for $\mathrm{F}_{\mu\lambda}(q)$ in Theorem 1.6 as follows.

Theorem 1.8. *Let μ and $\lambda=(\lambda_1,\dots,\lambda_s)$ be partitions of n and $\lambda\trianglelefteq\mu'$. Then*

$$\begin{aligned}\mathrm{F}_{\mu\lambda}(q)&=\frac{(q-1)^{n-\ell(\mu)}q^{\sum_{i\geq 1}\binom{\mu_i'-\mu_{i+1}'}{2}+\binom{n}{2}+\ell(\mu)-\mathsf{n}(\mu)-\mathsf{n}(\lambda')-n}[\lambda]_q!}{\prod_{i\geq 1}[\mu_i'-\mu_{i+1}']_q!}[\mathbf{x}^\lambda]\mathrm{P}_{\mu'}(\mathbf{x};1/q,0)\\&=(q-1)^n q^{\binom{n}{2}-\mathsf{n}(\mu)-\mathsf{n}(\lambda')-n}[\lambda]_q!\,[\mathbf{x}^\lambda]\mathrm{Q}_{\mu'}(\mathbf{x};1/q,0),\end{aligned}$$

where $[\lambda]_q!\stackrel{\text{def}}{=}[\lambda_1]_q!\dots[\lambda_s]_q!$.

The formula in Theorem 1.8 that relates $\mathrm{F}_{\mu\lambda}(q)$ to $\mathrm{P}_{\mu'}(\mathbf{x};1/q,0)$ has also been proved recently by Karp and Thomas [16]. The *transformed Hall–Littlewood functions* $\mathrm{H}_\mu(\mathbf{x};q)$ are defined by $\mathrm{H}_\mu(\mathbf{x};q)\stackrel{\text{def}}{=}\sum_\eta \mathrm{K}_{\eta\mu}(q)s_\eta(\mathbf{x})$, where the $s_\eta(\mathbf{x})$ are the Schur functions and the $\mathrm{K}_{\eta\mu}(q)$ are the Kostka polynomials [20, Chapter III, Section 6]. The *modified Hall–Littlewood functions* are defined by $\tilde{\mathrm{H}}_\mu(\mathbf{x};q)\stackrel{\text{def}}{=}q^{\mathsf{n}(\mu)}\mathrm{H}_\mu(\mathbf{x};1/q)$.

In order to state the next result, we start by reviewing the definition of the chromatic quasisymmetric functions introduced in an influential paper by Shareshian and Wachs [23]. Let $\mathsf{G}=(V,E)$ be a graph with V a set of positive integers. Given a subset S of $\mathbb{Z}_{\geq 1}$, a proper S-coloring of G is a function $\kappa\colon V\to S$ such that $\kappa(i)\neq\kappa(j)$ whenever $\{i,j\}\in E$. Let $\mathcal{C}(\mathsf{G})$ be the set of proper $\mathbb{Z}_{\geq 1}$-colorings of G. Stanley [24] defined the chromatic symmetric function of G as

$$X_{\mathsf{G}}(\mathbf{x}):=\sum_{\kappa\in\mathcal{C}(\mathsf{G})}\prod_{v\in V}x_{\kappa(v)},$$

The *chromatic quasisymmetric function* of $\mathsf{G}=(V,E)$ is a q-deformation of Stanley's chromatic symmetric function defined by:

$$X_{\mathsf{G}}(\mathbf{x};q):=\sum_{\kappa\in\mathcal{C}(\mathsf{G})}q^{\mathrm{asc}(\kappa)}\prod_{v\in V}x_{\kappa(v)},$$

where $\mathrm{asc}(\kappa):=|\{\{i,j\}\in E(\mathsf{G}):i<j\text{ and }\kappa(i)<\kappa(j)\}|$, is the number of ascents of the coloring κ. The chromatic quasisymmetric function is not symmetric in general. However, for the graphs $\mathsf{G}(\mathsf{h})$, where $\mathsf{h}\in\mathcal{H}_n$, Shareshian and Wachs' theorem [23, Theorem

4.5] shows that $X_{\mathsf{G}(\mathsf{h})}(\mathbf{x}, q)$ is a symmetric function with coefficients in $\mathbb{Z}[q]$. Therefore, $X_{\mathsf{G}(\mathsf{h})}(\mathbf{x}; q)$ can be expressed in terms of the elementary symmetric functions:

$$X_{\mathsf{G}(\mathsf{h})}(\mathbf{x}; q) = \sum_{\lambda \vdash n} c_{\lambda \mathsf{h}}(q) e_\lambda(\mathbf{x}).$$

Let $\langle \cdot, \cdot \rangle$ denote the Hall scalar product [20, p. 63, Eq. (4.5)].

Theorem 1.9. *Let $\mathsf{h} \in \mathcal{H}_n$ be a Hessenberg function and let μ be a partition of n. Then $\mathrm{F}_{\mu \mathsf{h}}(q) \in \mathbb{Z}[q]$ is a polynomial in q and*

$$\mathrm{F}_{\mu \mathsf{h}}(q) = |\mathfrak{u}_\mathsf{h}| \sum_{\lambda \vdash n} \frac{c_{\lambda \mathsf{h}}(q)}{[\lambda]_q!} \frac{\mathrm{F}_{\mu\lambda}(q)}{|\mathfrak{u}_\lambda|} = \frac{q^{n^2 - |\mathsf{h}|}(q-1)^n}{|\mathrm{Z}_\mathrm{G}(I + J_\mu)|} \langle \tilde{\mathrm{H}}_\mu(\mathbf{x}; q), X_{\mathsf{G}(\mathsf{h})}(\mathbf{x}; q) \rangle, \tag{1.5}$$

where $\mathrm{Z}_G(I + J_\mu)$ is the centralizer of $I + J_\mu$ in $\mathrm{GL}_n(\mathbb{F}_q)$.

The strategy of the proof of Theorem 1.9 is a reduction process from $\mathrm{F}_{\mu \mathsf{h}}(q)$ to the $\mathrm{F}_{\mu\lambda}(q)$ using the *modular law* of Abreu and Nigro [1].

As in Theorem 1.6, it is interesting to determine for which μ and h we have $\mathrm{F}_{\mu \mathsf{h}}(q) \neq 0$. In Theorem 1.10, we address this problem and indeed prove a stronger result over an arbitrary field. Let $\mathcal{P}$ be any poset of cardinality n and let $c_k(\mathcal{P})$ denote the maximum possible cardinality of a union $\bigcup_{i=1}^k C_i$ where C_i $(1 \leq i \leq k)$ is a chain in $\mathcal{P}$. For each $i \geq 1$, define integers λ_i by $c_k(\mathcal{P}) = \lambda_1 + \cdots + \lambda_k$ for $k \geq 1$. Then it can be shown that the sequence λ_i $(i \geq 1)$ is weakly decreasing and a partition of n (see [4, Theorem 2.1]). This partition is called the Greene-Kleitman shape of $\mathcal{P}$, denoted λ_P. The Greene-Kleitman shape of $\mathcal{P}_\mathsf{h}$ is denoted λ_h for simplicity. A simple algorithm for determining λ_h essentially goes back to an early influential paper of Gerstenhaber [11, p. 535] (also see [8, Proposition 6.8]).

Theorem 1.10. *Let $\mathbb{F}$ be an arbitrary field, and let $\mathsf{h} \in \mathcal{H}_n$ be a Hessenberg function with Greene–Kleitman shape λ_h. Let $\mathcal{C}_\mu = \mathcal{C}_\mu(\mathbb{F})$ denote the $\mathrm{GL}_n(\mathbb{F})$-conjugacy class of J_μ. Then, for any partition μ of n, we have $\mathcal{C}_\mu \cap \mathfrak{u}_\mathsf{h}(\mathbb{F}) \neq \emptyset$ if and only if $\mu \trianglelefteq \lambda_\mathsf{h}$.*

Let $\mathsf{U}_\mathsf{h} = \mathsf{U}_\mathsf{h}(\mathbb{F}_q) \stackrel{\text{def}}{=} \{I + A : A \in \mathfrak{u}_\mathsf{h}(\mathbb{F}_q)\}$ denote the unipotent subgroup of $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$ associated with h. Understanding $\mathrm{F}_{\mu \mathsf{h}}(q)$ reduces to analyzing the induced representation of the trivial representation from $\mathsf{U}_\mathsf{h}(q)$ to $\mathrm{GL}_n(\mathbb{F}_q)$. This approach was explored by Gagnon, among other things, in his recent and inspiring paper [10]. To describe the connection to Gagnon's work, we need to recall Zelevinsky's construction [27] which introduces a family of Hopf algebras, known as PSH-algebras, that unify the representation theories of S_n and $\mathrm{GL}_n(\mathbb{F}_q)$. In Gagnon's notation, Zelevinsky's construction produces a graded Hopf algebra isomorphism

$$\mathbf{p}_{\{1\}} \colon \mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) \to \mathrm{Sym}, \quad \delta_\lambda \mapsto \tilde{\mathrm{P}}_\lambda(\mathbf{x}; q) := q^{-\mathsf{n}(\lambda)} \mathrm{P}_\lambda(\mathbf{x}; q^{-1}),$$

where Sym is the ring of symmetric functions with coefficients in $\mathbb{Q}(q)$ and $\mathrm{P}_\lambda(\mathbf{x};t) \stackrel{\text{def}}{=} \mathrm{P}_\lambda(\mathbf{x};0,t)$ denotes a Hall–Littlewood function. Here $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ is the PSH-algebra of class functions on the family $(\mathrm{GL}_n(\mathbb{F}_q))_{n\geq 1}$. Using this, Gagnon proves [10, Theorem 3.1] that if χ_{h} denotes the character of the induced representation $\mathrm{Ind}_{\mathrm{U}_{\mathsf{h}}}^{G}\mathbf{1}$, then

$$\chi_{\mathsf{h}} = (q-1)^n\,\mathbf{p}_{\{1\}}^{-1}(X_{\mathsf{G}(\mathsf{h})}(\mathbf{x};q)).$$

Gagnon then applies this identity to derive a formula for χ_{h} which can be reproved using Theorem 1.9. We remark that $\tilde{\mathrm{P}}_\lambda(\mathbf{x};q)$ is dual to the modified Hall–Littlewood basis $\tilde{\mathrm{H}}_\lambda(\mathbf{x};q)$ with respect to the Hall scalar product.

In the rest of this extended abstract, we explain the applications of our results on $\mathrm{F}_{\mu\lambda}(q)$ and $\mathrm{F}_{\mu\mathsf{h}}(q)$. Our first application is in counting the number of points on nilpotent Hessenberg varieties. Varieties of this kind have been studied in [26, 13, 21]. We recall the definition of a nilpotent Hessenberg variety from [21]. In what follows, the conjugate of a Hessenberg function $\mathsf{h} \in \mathcal{H}_n$ is defined by

$$\mathsf{h}' : [n] \to [n], \quad i \mapsto \#\{j \in [n] : \mathsf{h}(j) \geq n+1-i\}.$$

Definition 1.11 (Nilpotent Hessenberg variety)**.** Let $\mathsf{h} \in \mathcal{H}_n$ be a Hessenberg function, and let X be a nilpotent $n \times n$ matrix with entries in $\mathbb{F}_q$. Let $\mathscr{F}_n$ denote the set of complete flags in $\mathbb{F}_q^n$. The nilpotent Hessenberg variety associated to h and X is defined by

$$\mathcal{H}\mathrm{ess}_{\mathrm{nil}}(\mathsf{h}, X) \stackrel{\text{def}}{=} \left\{ V_\bullet = (V_i)_{1\leq i\leq n} \in \mathscr{F}_n : XV_i \subseteq V_{\mathsf{e}(i)} \right\},$$

where $\mathsf{e}(i) := n - \mathsf{h}'(n+1-i)$.

Example 1.12. For the Hessenberg function $\mathsf{h} = (2,3,5,5,5)$ (see Example 1.2), we have $\mathsf{h}' = (3,3,4,5,5)$ and $\mathsf{e} = (0,0,1,2,2)$.

Theorem 1.13. *Let $\mathsf{h} \in \mathcal{H}_n$ be a Hessenberg function and let X be a nilpotent $n \times n$ matrix with entries in $\mathbb{F}_q$, with Jordan form J_μ. If E_{h} denotes the number of edges of the graph $\mathsf{G}(\mathsf{h})$, then*

$$\#\mathcal{H}\mathrm{ess}_{\mathrm{nil}}(\mathsf{h}, X) = \frac{|\mathrm{Z}_{\mathrm{G}}(I+J_\mu)|}{(q-1)^n q^{\binom{n}{2}}}\mathrm{F}_{\mu\mathsf{h}}(q) = q^{-E_{\mathsf{h}}}\langle \tilde{\mathrm{H}}_\mu(\mathbf{x};q), X_{\mathsf{G}(\mathsf{h})}(\mathbf{x};q)\rangle.$$

Our second application concerns another elegant idea introduced by Kirillov. Define $a_{nk} \stackrel{\text{def}}{=} \#\left\{X \in \mathfrak{u}_n(\mathbb{F}_q) \,:\, X^2 = 0 \text{ and } \mathrm{rank}(X) = k\right\}$. In [17], Kirillov discovered the mysterious relation

$$\sum_{k\geq 0} a_{nk}\, q^{k(k-n)}\, H_{n-2k}(w \mid 1/q) = (2w)^n, \tag{1.6}$$

where the $H_{n-2k}(w \mid 1/q)$ are the continuous q-Hermite polynomials as known in the Askey–Wilson scheme. The proof of Equation (1.6) in [17] is based on an algebraic structure which Kirillov calls "fusion rules" on the set of coadjoint orbits of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathfrak{u}_n(\mathbb{F}_q)$. In [18], Kirillov writes "This algebra is in fact one of the realizations of the Hall algebra, but we shall not discuss this phenomenon here." In Theorem 1.14 below, we obtain a generalization of Kirillov's mysterious Equation (1.6), and show that it follows easily from the Cauchy identity for Macdonald polynomials combined with Theorem 1.8. Recall that the continuous q-Hermite polynomials $H_n(w\,|\,q)$ are defined recursively by $H_{n+1}(w\,|\,q) = 2wH_n(w\,|\,q) + (q^n-1)H_{n-1}(w\,|\,q)$ for $n \geq 0$, with the initial conditions $H_0(w\,|\,q) := 1$ and $H_n(w\,|\,q) := 0$ when $n \leq -1$. One can also show [12, Chapter 13] that

$$H_n(\cos(\theta)\,|\,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}. \tag{1.7}$$

It is a classical result that the $H_n(w\,|\,1/q)$ form a sequence of orthogonal polynomials:

$$\frac{1}{2\pi}\int_0^{\pi} H_n(\cos(\theta)\,|\,1/q)H_m(\cos(\theta)\,|\,1/q)(e^{2i\theta},e^{-2i\theta};1/q)_\infty = \frac{(1/q;1/q)_n}{(1/q;1/q)_\infty}\delta_{nm}, \tag{1.8}$$

where δ_{nm} is the Kronecker delta and

$$(e^{2i\theta},e^{-2i\theta};1/q)_\infty \overset{\text{def}}{=} \left(e^{2i\theta};1/q\right)_\infty \left(e^{-2i\theta};1/q\right)_\infty = \prod_{j\geq 0}\left(1-q^{-j}e^{2i\theta}\right)\prod_{j\geq 0}\left(1-q^{-j}e^{-2i\theta}\right).$$

From this, one can conclude the following remarkable formula (due to Kirillov)

$$a_{nk} = M(n,k,q)\frac{1}{2\pi}\int_0^{\pi}(2\cos(\theta))^n H_{n-2k}(\cos(\theta)\,|\,1/q)(e^{2i\theta},e^{-2i\theta};1/q)_\infty\,d\theta, \tag{1.9}$$

where $M(n,k,q) \overset{\text{def}}{=} q^{k(n-k)}\prod_{j=1}^{n-2k}\left(1-q^{-j}\right)^{-1}$. Below we discuss a generalization of Equation (1.6). To this end, first we introduce some notation. Recall that the Chebyshev polynomials of the first kind $T_n(w)$ are defined by $T_n(\cos(\theta)) = \cos(n\theta)$, which are also derived from the recurrence relation $T_{n+1}(w) = 2wT_n(w) - T_{n-1}(w)$ for $n \geq 2$ with $T_0(w) = 1$ and $T_1(w) = w$. For a partition $\rho = (\rho_1,\rho_2,\dots)$, define $T_\rho(w) \overset{\text{def}}{=} T_{\rho_1}(w)T_{\rho_2}(w)\cdots$. For any partition $\rho \vdash n$, we set $z_\rho \overset{\text{def}}{=} \prod_{i\geq 1} i^{m_i}\cdot m_i!$, where $m_i = m_i(\rho)$ is the number of parts of ρ equal to i. Note that z_ρ is the size of the centralizer of a permutation of type ρ in the symmetric group S_n.

For a composition Λ of n, define $a_{nk}(\Lambda) \overset{\text{def}}{=} \#\left\{X \in \mathfrak{u}_\Lambda(\mathbb{F}_q) : X^2 = 0, \mathrm{rank}(X) = k\right\}$. From Theorem 1.6 it follows that $a_{nk}(\Lambda) = a_{nk}(\lambda)$ for $\lambda = \mathrm{sort}(\Lambda)$. Our analysis of $a_{nk}(\lambda)$ starts with the observation that a matrix $X \in \mathfrak{u}_\lambda(\mathbb{F}_q)$, with rank k and $X^2 = 0$, must have a Jordan form of type $\mu = (2^k, 1^{n-2k})$ and so $\mu' = (n-k,k)$, where $0 \leq k \leq n/2$. Therefore, thanks to Theorem 1.8, we have

$$a_{nk}(\lambda) = (q-1)^n q^{nk-k^2-\mathsf{n}(\lambda')-n}\left(\prod_{1\leq i\leq s}[\lambda_i]_q!\right)[\mathbf{x}^\lambda]\mathrm{Q}_{\mu'}(\mathbf{x};1/q,0). \tag{1.10}$$

Evidently we have $a_{nk}(\lambda) = 0$ when $k > n/2$. We prove the following assertion.

Theorem 1.14. *Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of n. Then*

$$\sum_{k=0}^{n} a_{nk}(\lambda) q^{k(k-n)} H_{n-2k}(w \,|\, 1/q) = [\lambda]_q!(q-1)^n q^{-\mathsf{n}(\lambda')} \prod_{i=1}^{s} \sum_{\rho \vdash \lambda_i} \frac{2^{\ell(\rho)} T_\rho(w)}{z_\rho \prod_{i=1}^{\ell(\rho)} (q^{\rho_i} - 1)}.$$

We can use orthogonality to represent $a_{nk}(\lambda)$ as an integral, similar to Equation (1.8). This integral can be evaluated using Jacobi's triple product, but we do not discuss this calculation.

Next, let $f^{\lambda}_{\nu\rho}(t)$ be the Hall-Littlewood structural constants defined in [20, Sec. III.3, p. 215] satisfying $\mathrm{P}_\nu(\mathbf{x}; t)\mathrm{P}_\rho(\mathbf{x}; t) = \sum_\lambda f^{\lambda}_{\nu\rho}(t)\mathrm{P}_\lambda(\mathbf{x}; t)$.

Theorem 1.15. *Let λ be a partition of n. Then for $0 \le k \le n/2$ we have*

$$a_{nk}(\lambda) = q^{nk-k^2-\mathsf{n}(\lambda')} \bar{a}_{nk}(\lambda),$$

where $\bar{a}_{nk}(\lambda)$ is equal to

$$\sum_{j=0}^{k} (-1)^j q^{-\binom{j+1}{2}-(n-2k)j} \begin{bmatrix} n-2k+j \\ j \end{bmatrix}_q \frac{q^{n-2k+2j}-1}{q^{n-2k+j}-1} \sum_{\substack{\nu \vdash n-k+j \\ \rho \vdash k-j}} \frac{q^{\mathsf{n}(\nu')+\mathsf{n}(\rho')} f^{\lambda}_{\nu\rho}(1)}{[\nu]_q! \, [\rho]_q!},$$

with the convention that $(q^0 - 1)/(q^0 - 1) = 1$.

Now define $\mathrm{C}_n = \mathrm{C}_n(q) \stackrel{\text{def}}{=} \# \left\{ X \in \mathfrak{u}_n(\mathbb{F}_q) \,:\, X^2 = 0 \right\}.$

Corollary 1.16. *For $n \ge 1$ we have*

$$\mathrm{C}_n = \sum_{0 \le k_1 + k_2 \le \lfloor \frac{n}{2} \rfloor} q^{nk_1 - k_1^2} \binom{n}{k_1} (-1)^{k_2} q^{\binom{k_2}{2}} \begin{bmatrix} n-2k_1-k_2 \\ k_2 \end{bmatrix}_q \frac{[n-2k_1]_q}{[n-2k_1-k_2]_q}.$$

The problem of computing C_n was considered by Kirillov and Melnikov in [19], where they conjectured a different formula for C_n that was later proved by Ekhad and Zeilberger [7]. The Ekhad–Zeilberger proof of the Kirillov–Melnikov formula is very short, but R. Stanley notes in [25, p. 194] that "no conceptual reason is known for such a simple formula". We give a short proof that Theorem 1.16 implies the Kirillov–Melnikov formula, using a formula for Macdonald polynomials $\mathrm{Q}_{(\mu_1,\mu_2)}(\mathbf{x}\,; q, t)$ due to Jing and Józefiak [14, Eq. (1)].

As a final application of our results, we have the following theorem.

Theorem 1.17. *Let h_1 and h_2 be two Hessenberg functions on $[n]$, and let $\mathsf{U}_{\mathsf{h}_1}$ and $\mathsf{U}_{\mathsf{h}_2}$ be the unipotent subgroups of $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$ associated with h_1 and h_2. Then*

$$\left| \mathsf{U}_{\mathsf{h}_1} \backslash \mathrm{G} / \mathsf{U}_{\mathsf{h}_2} \right| = q^{|\mathsf{h}_1| + |\mathsf{h}_2| - 2n^2} \sum_{\mu \vdash n} |\mathrm{Z}_{\mathrm{G}}(I + J_\mu)| \, \mathrm{F}_{\mu\mathsf{h}_1}(q) \, \mathrm{F}_{\mu\mathsf{h}_2}(q). \tag{1.11}$$

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