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Lattices from Pointed Building Sets: Generalized Ornamentation Lattices

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Abstract. We introduce a novel combinatorial structure called a *pointed building set*, which can be viewed as a family of lattices equipped with compatibility relations. To each pointed building set B, we associate a complete lattice $\mathcal{O}(\mathsf{B})$, called the *ornamentation lattice* of B.

Special cases of this construction have already proven useful in understanding the structure of three families of posets: operahedron lattices, the affine Tamari lattice, and hypergraphic posets of subhypergraphs of the path hypergraph of an increasing tree.

The goal of this extended abstract is to establish the theory of these generalized ornamentations and discuss their application. We examine several natural classes of pointed building sets which recover classical lattices such as the Tamari lattice, the lattice of topologies ordered by coarsening, and the lattice of naturally labeled partial orders. Furthermore, several theoretical directions are explored, including inverse limits and group actions. Notably, this leads to a straightforward construction of inverse limits of Tamari lattices, yielding infinite analogs of the Tamari lattice.

Keywords: lattice, semidistributive, Tamari lattice, inverse limit

1 Introduction

There has been an explosion of study of combinatorial lattice theory in recent decades. In many cases, proving that a particular class of posets form lattices requires a great deal of effort [5, 6, 8]. In this paper we introduce the concept of a *pointed building set*, a simple combinatorial object that is reminiscent of the *building sets* of De Concini and Procesi [7]. Building sets were introduced to understand the "wonderful" compactifications of hyperplane arrangement complements, and have since become a central tool in the study of matroids [2, 11] and generalized permutohedra [14, 15].

Pointed building sets are collections of pointed sets satisfying natural closure properties. To each pointed building set B we associate a poset of functions called *ornamentations*, which will turn out to be a lattice. Recently, special cases of ornamentation lattices have served as a tool to dramatically simplify proofs that several distinct posets

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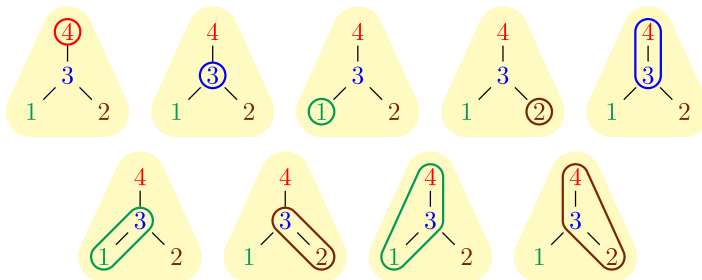

Figure 1: An example of a pointed building set. In this example, all pointed sets are pointed at their minimal element. We color the outline of a set with the same color as its pointed element.

are lattices or otherwise understand their structure [1, 4, 8]. These applications will be discussed in Section 3. The goal of this extended abstract is to provide a common generalization to the constructions in these papers and understand the theory of these generalized ornamentation lattices. We now present the main definitions and theorems.

Definition 1.1. Let $\mathcal{I}$ be an indexing set. A *pointed subset* (S, i) of $\mathcal{I}$ is an ordered pair such that $S \subseteq \mathcal{I}$ and $i \in S$. We say that (S, i) is *pointed* at i. As an abuse of notation, it will generally be useful to treat pointed sets as their underlying set. For example, we will say that $j \in (S, i)$ if $j \in S$ and that $(S, i) \subseteq (T, j)$ if $S \subseteq T$. If $\mathcal{C}$ is a collection of pointed subsets of $\mathcal{I}$, we define $\mathcal{C}|_i := \{(S, j) \in \mathcal{C} \mid j = i\}$.

A *pointed building set* B is a collection of pointed subsets of $\mathcal{I}$ such that the following three axioms hold:

1. B *contains all singletons*, i.e., for all $i \in \mathcal{I}$ we have $(\{i\}, i) \in \mathsf{B}$;
2. B is *transitively closed*, i.e., for all $(S, i), (T, j) \in \mathsf{B}$ if $j \in S$ then $(S \cup T, i) \in \mathsf{B}$;
3. B is *closed under arbitrary pointwise unions*, i.e., for all $i \in \mathcal{I}$, the collection $\mathsf{B}|_i$ is closed under arbitrary unions.

Note that axiom (2) implies axiom (3) if $\mathcal{I}$ is finite. For $i \in \mathcal{I}$, note that axiom (3) guarantees that $\mathsf{B}|_i$ is a complete lattice ordered by inclusion. See Figure 1 for an example of a pointed building set.

Example 1.2. Let $D = (V, E)$ be a directed graph. For $v \in V$, a subset $S \subseteq V$ is called v-connected if for all $u \in S$ there exists a directed path in S from v to u. Let

$$\mathsf{B}(D) := \{(S, v) \mid S \text{ is } v\text{-connected in } D\}.$$

We call $\mathsf{B}(D)$ the *digraphical pointed building set* of D. Figure 1 is a digraphical pointed building set when all edges are oriented upwards.

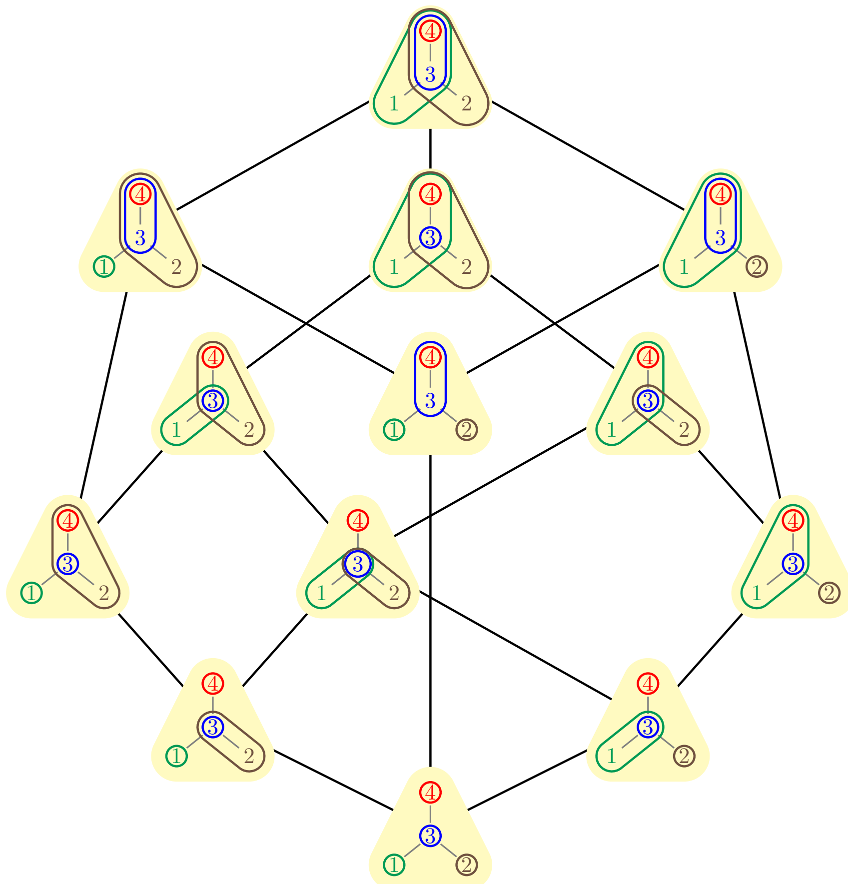

Figure 2: The ornamentation lattice of the pointed building set in Figure 1.

Definition 1.3. Given an indexing set $\mathcal{I}$ and a pointed building set B on $\mathcal{I}$, an *ornamentation* of B is a function $\rho : \mathcal{I} \to \mathsf{B}$ such that

1. $\rho(i)$ *is pointed at* i, i.e., for all $i \in \mathcal{I}$, $\rho(i) \in \mathsf{B}|_i$;
2. ρ is *transitively closed*, i.e., for all $i, j \in \mathcal{I}$ if $j \in \rho(i)$ then $\rho(j) \subseteq \rho(i)$.

Let $\mathcal{O}(\mathsf{B})$ be the set of ornamentations of B. $\mathcal{O}(\mathsf{B})$ has a natural partial order where $\rho_1 \preceq \rho_2$ if for all $i \in \mathcal{I}$, we have $\rho_1(i) \subseteq \rho_2(i)$. The following is our main result and justifies calling $(\mathcal{O}(\mathsf{B}), \preceq)$ the *ornamentation lattice* of B.

Theorem 1.4. *$(\mathcal{O}(\mathsf{B}), \preceq)$ is a complete lattice.*

See Figure 2 for the ornamentation lattice of Figure 1. Note that for $i, j \in \mathcal{I}$ and an ornamentation ρ, it is not necessarily true that $\rho(i)$ and $\rho(j)$ are nested or disjoint.

The structure of this extended abstract is as follows. In Section 2, we give numerous examples of both pointed building sets and ornamentation lattices. In Section 3, we discuss three different significant applications of ornamentation lattices in recent papers. In Section 4, we discuss various poset and lattice theoretic properties about ornamentation lattices. In Section 5, we study ornamentations that are invariant under group actions on their base set. This allows us to provide a simple description of the *cyclic Tamari lattice* [4], which answers a question of Barkley and Defant.

In Section 6, we do some light category theory and show that, in some sense, direct limits of pointed building sets induce inverse limits of ornamentation lattices. As a consequence, we obtain simple models for inverse limits of Tamari lattices. For details, theorems, and proofs, we refer the reader to [16].

2 Examples

Example 2.1. Let $G = (V, E)$ be a graph. Let

$$\mathsf{B}(G) = \{(C, i) \subseteq V \mid C \text{ is connected and } i \in C\}.$$

We call $\mathsf{B}(G)$ the *graphical pointed building set* of G.

Definition 2.2. Let $\mathcal{I}$ be a finite set. An (unpointed) *building set* [7, 15] X on $\mathcal{I}$ is a collection of subsets of $\mathcal{I}$ satisfying

1. for all $i \in \mathcal{I}$, $\{i\} \in X$;
2. for all $I, J \in X$, if $I \cap J \neq \emptyset$, then $I \cup J \in X$.

Example 2.3. Let X be an (unpointed) building set on $\mathcal{I}$. Then take

$$\mathsf{B}(X) := \{(S, i) \mid S \in X \text{ and } i \in S\}.$$

When $\mathcal{I}$ is the set of vertices of a finite graph and X is the *graphical* building set of $\mathcal{I}$, i.e., the set of all connected subgraphs, then $\mathsf{B}(X)$ is the graphical pointed building set of $\mathcal{I}$.

Example 2.4. Let X be an (unpointed) building set on $[n] := \{1, \ldots, n\}$. Then take

$$\mathsf{B}(X) := \{(S, i) \mid S \in X \text{ and } i = \min S\}.$$

Example 2.5. For $a, b \in \mathbb{Z}$, let $[a, b] := \{i \in \mathbb{Z} \mid a \leq i \leq b\}$ and $[a, \infty) := \{i \in \mathbb{Z} \mid a \leq i\}$.

1. If $\mathcal{I} = [n]$ and $\mathsf{B} = \{([a, b], a) \mid a, b \in \mathbb{Z} \text{ such that } 1 \leq a \leq b \leq n\}$, then $\mathcal{O}(\mathsf{B})$ is the *Tamari lattice* as proven by Huang and Tamari [12].

2. If $\mathcal{I} = \mathbb{Z}^+$ and $\mathsf{B} = \{([a,b],a) \mid a,b \in \mathbb{Z}^+ \text{ such that } a \leq b\} \cup \{([a,\infty),a) \mid a \in \mathbb{Z}^+\}$, then we call $\mathcal{O}(\mathsf{B})$ the *infinite Tamari lattice* and denote it Tam_∞.

3. If $\mathcal{I} = \mathbb{Z}$ and $\mathsf{B} = \{([a,b],a) \mid a,b \in \mathbb{Z} \text{ such that } a \leq b\} \cup \{([a,\infty),a) \mid a \in \mathbb{Z}\}$, then we call $\mathcal{O}(\mathsf{B})$ the *bi-infinite Tamari lattice* and denote it $\mathrm{Tam}_{\pm\infty}$.

Note that each of these Tamari lattices is a digraphical ornamentation lattice of a (possibly infinite) directed path.

Example 2.6. Let K_n be the complete graph on n vertices and let $\mathsf{B}(K_n)$ be the graphical pointed building set on K_n. Then $\mathcal{O}(\mathsf{B}(K_n))$ is in bijection with transitive digraphs on $[n]$ which is enumerated in [17] by the sequence A000798. To build a transitive digraph from an ornamentation ρ, take the edge set $\{(i,j) \mid i \in [n] \text{ and } j \in \rho(i)\}$. Conversely, given a transitive digraph D, let $\rho_D(i)$ be the set of all $j \in [n]$ that are reachable from i via a directed path (including i itself).

It is known that transitive digraphs are in bijection with topologies [10] as follows. Let D be a transitive directed graph on $[n]$. Then the collection $\{\rho_D(i) \mid i \in [n]\}$ forms a basis for the corresponding topology. This shows that $\mathcal{O}(\mathsf{B}(K_n))$ is isomorphic to the lattice of topologies ordered by coarsening.

Example 2.7. Direct the edges of K_n from i to j for $i < j$ and let $\mathsf{B}(K_n)$ be the digraphical pointed building set of K_n. Then $\mathcal{O}(\mathsf{B}(K_n))$ is in bijection with the set of naturally labeled posets which is enumerated in [17] by the sequence A006455. To obtain a naturally labeled poset from an ornamentation $\rho \in \mathcal{O}(\mathsf{B}(K_n))$, take relations $i \leq j$ for all $j \in \rho(i)$. This recovers the lattice of natural partial orders of Avann [3].

Example 2.8. Let X be a set and let $Y \subseteq 2^X$ be any collection of subsets of X that is closed under unions. We take $\mathcal{I} = X \sqcup \{\hat{0}\}$ where $\sqcup$ denotes disjoint union. Define

$$\mathsf{B} = \{(\{i\},i) \mid i \in \mathcal{I}\} \cup \{(Z \sqcup \{\hat{0}\},\hat{0}) \mid Z \in Y\}.$$

It is known that every lattice $L = (X, \preceq)$ is isomorphic to a union-closed family of sets ordered under inclusion. In particular, one may consider the union-closed family $Y = \{S_x \mid x \in X\}$ where $S_x = \{y \in L \mid x \not\preceq y\}$. Then $\mathcal{O}(\mathsf{B})$ is isomorphic to L.

As all lattices are ornamentation lattices, one can view the construction of ornamentation lattices as a method for combining the different lattices $\mathsf{B}|_i$ into a new lattice. However, in practice, natural classes of pointed building sets have practical applications.

3 Applications

3.1 Operahedron lattices

Let T be a rooted plane tree with the root drawn at the bottom. In [13], Laplante-Anfossi defined the operahedron poset as follows. A *nest* of T is a set of vertices that induces a connected subgraph of T. Two sets are said to be *nested* if one is contained in the other. A *maximal nesting* of T is a collection $\mathcal{N}$ of tubes of T such that

- $\mathsf{V} \in \mathcal{N}$;
- each nest in $\mathcal{N}$ has cardinality at least 2;
- any two tubes in $\mathcal{N}$ are either nested or disjoint;
- $|\mathcal{N}| = |\mathsf{T}| - 1$.

Let $\mathrm{MN}(\mathsf{T})$ be the set of maximal nestings of T. Laplante-Anfossi defined a natural partial order $(\mathrm{MN}(\mathsf{T}), \preceq)$ and conjectured that it was a lattice. For $\mathcal{N}, \mathcal{N}' \in \mathrm{MN}(\mathsf{T})$, we say that $\mathcal{N} \prec \mathcal{N}'$ if one can form $\mathcal{N}'$ from $\mathcal{N}$ by moving a single nest upwards (called an associahedron move) or rightwards (called a permutohedron move). See Figure 3 for an example of these cover relations. For more details of this construction, see [8, 13].

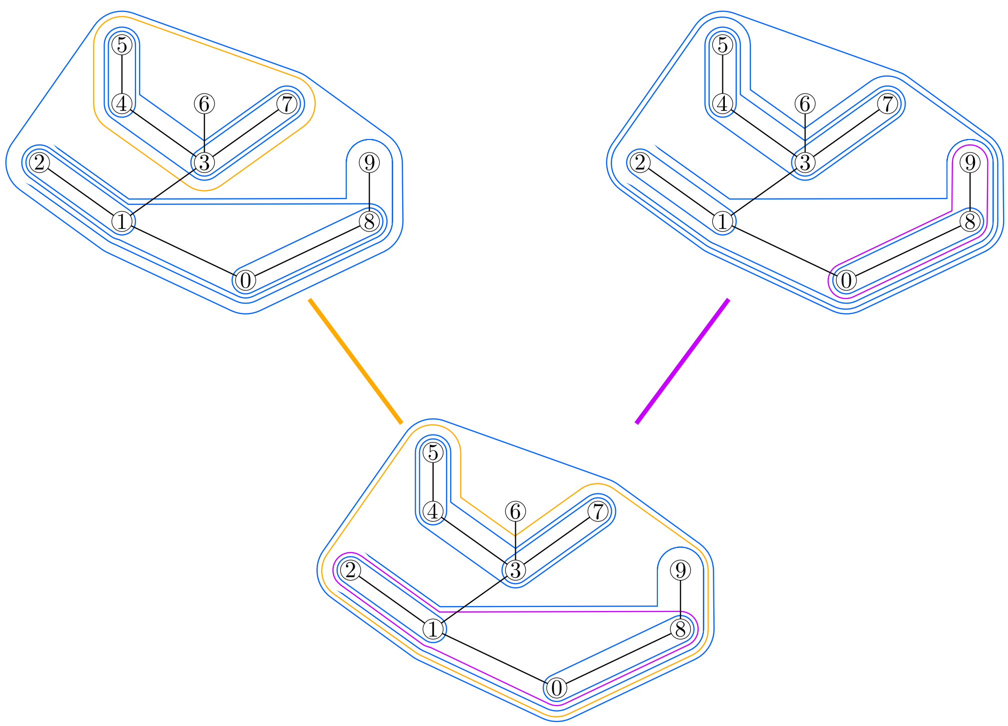

Figure 3: Three maximal nestings of a tree. The cover relation on the left corresponds to an associahedron move, while the cover relation on the right corresponds to a permutohedron move.

$(\mathrm{MN}(\mathsf{T}), \preceq)$ specializes to the Tamari lattice when T is a chain and the weak Bruhat order on S_n when T is a claw.

In [8, Theorem 1.1], ornamentations were used to prove Laplante-Anfossi's conjecture that $(\mathrm{MN}(\mathsf{T}), \preceq)$ is a lattice. In particular, let r be the root of T and orient the edges in the rooted forest $\mathsf{F} := \mathsf{T} - \{r\}$ upwards. A (poset but not lattice) embedding

$$\Psi : \mathrm{MN}(\mathsf{T}) \hookrightarrow \mathcal{L}(\mathsf{F}) \times \mathcal{O}(\mathsf{B}(\mathsf{F}))$$

provided a critical structural lemma that enabled the proof, where $\mathcal{L}(\mathsf{F})$ is the lattice of linear extensions of F and $\mathsf{B}(\mathsf{F})$ is the digraphical pointed building set of F. The ornamentation factor $\mathcal{O}(\mathsf{B}(\mathsf{F}))$ is described as follows: for each vertex $v \in \mathsf{F}$, let $\rho_{\mathcal{N}}(v)$ be the largest tube in $\mathcal{N}$ whose root is v; if no such tube exists, let $\rho_{\mathcal{N}}(v) = \{v\}$. This defines an ornamentation $\rho_{\mathcal{N}} \in \mathcal{O}(\mathsf{F})$.

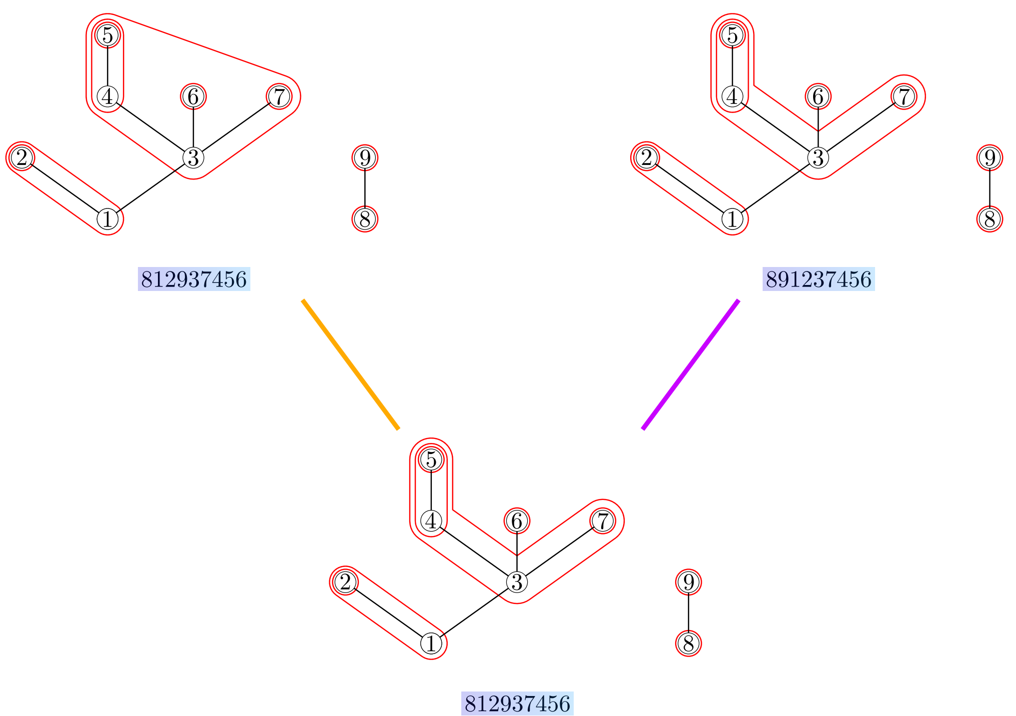

Figure 4: Applying Ψ to the maximal nestings in Figure 3 yields these three pairs, each of which consists of an ornamentation and a linear extension.

3.2 The affine Tamari lattice

An early draft of this paper was communicated to the authors of [4] who used our generalized ornamentations to understand the structure of the *affine Tamari lattice* ATam_n. ATam_n turns out to be isomorphic to $\mathcal{O}(\mathsf{B}(C_n))$, the digraphical ornamentation lattice of an oriented n-cycle. Barkley and Defant also show that the Hasse diagram of $\mathcal{O}(\mathsf{B}(C_n))$ is an orientation of the 1-skeleton of the type-D associahedron.

The isomorphism of $\mathcal{O}(\mathsf{B}(C_n))$ and ATam_n was used to calculate that the maximal length of a chain in ATam_n is $\binom{n+1}{2}$. This result was then used to construct a classification of the possible lengths of maximal green sequences for the completed path algebra of the oriented n-cycle.

3.3 Intreeval hypergraphic lattices

A further application of generalized ornamentations was found in [1]. In particular, let D be a directed (but not necessarily rooted) tree with vertex labels increasing along paths, let $\mathbb{H}(D)$ be the hypergraph consisting of all directed paths of D, and let $\mathbb{H}'$ be a subhypergraph of $\mathbb{H}(D)$. For a set of vertices $X \subseteq D$, define the simplex

$$\Delta_X := \operatorname{conv}\{e_i \mid i \in X\}.$$

The hypergraphic polytope $\Delta_{\mathbb{H}'}$ is the Minkowski sum $\Delta_{\mathbb{H}'} := \sum_{E \in H'} \Delta_E$.

The Hasse diagram of the *hypergraphic poset* $P_{\mathbb{H}'}$ is a particular orientation of the 1-skeleton of $\Delta_{\mathbb{H}'}$. In [1, Proposition 6.27], a surjective map $\mathcal{O}(\mathsf{B}(D)) \to P_{\mathbb{H}'}$ is used to completely classify when the hypergraphic poset of $\mathbb{H}'$ is a lattice.

4 Lattice properties

In this section, we discuss certain lattice properties of interest.

Definition 4.1. A pointed building set B is *acyclic* if whenever $(S, i), (T, j) \in \mathsf{B}$ such that $i \in T$ and $j \in S$ then $i = j$.

Remark 4.2. The digraphical pointed building set of an acyclic directed graph is acyclic.

Lemma 4.3. *Suppose* B *is finite and acyclic, and let* $\rho, \sigma \in \mathcal{O}(\mathsf{B})$ *such that* $\rho \lessdot \sigma$. *Then there exists a unique* $i \in \mathcal{I}$ *such that* $\rho(i) \subsetneq \sigma(i)$ *and for all other* $j \in \mathcal{I}, \rho(j) = \sigma(j)$. *Furthermore, at least one of the following two holds:*

- *there exists* $(S, i) \in \mathsf{B}|_i$ *such that* $\sigma(i) = \left(\bigcup_{j \in S} \rho(j),\ i\right)$
- $\rho(i) \lessdot \sigma(i)$ *in* $\mathsf{B}|_i$.

Definition 4.4. Let P be a poset with unique minimal element $\hat{0}$. An element $x \in P$ is called an *atom* if it covers $\hat{0}$. A complete lattice is called *atomic* (or sometimes *atomistic*) if every element is a join of a (possibly infinite) set of atoms.

Proposition 4.5. *$\mathcal{O}(\mathsf{B})$ is atomic if and only if $\mathsf{B}|_i$ is atomic for each $i \in \mathcal{I}$.*

Proposition 4.6. *$\rho \in \mathcal{O}(\mathsf{B})$ is join-irreducible if and only if*

- *$\rho = \rho_{(S,i)}$ for some $(S,i) \in \mathsf{B}$ and*
- *(S,i) is join-irreducible in $\mathsf{B}|_i$.*

Definition 4.7. A lattice L is *semidistributive* if for all $x, y, z \in L$, whenever $x \vee y = x \vee z$ we have $x \vee (y \wedge z) = x \vee y$ and whenever $x \wedge y = x \wedge z$ we have $x \wedge (y \vee z) = x \wedge y$.

Theorem 4.8. *Suppose that $\mathcal{I}$ is finite and for all $i \in \mathcal{I}$, $B|_i$ is a chain. Then $\mathcal{O}(\mathsf{B})$ is semidistributive.*

Corollary 4.9. *The affine Tamari lattice is semidistributive.*

Corollary 4.10. *Let T be a finite rooted tree with edges directed towards the root and let $\mathsf{B}(\mathsf{T})$ be the digraphical pointed building set of T. Then $\mathcal{O}(\mathsf{B}(\mathsf{T}))$ is semidistributive.*

We note that Corollary 4.9 was separately proven in [4]. See Figure 1 and Figure 2 for an example of Corollary 4.10.

5 Ornamentation lattices under group actions

Let G be a group acting on $\mathcal{I}$.

Definition 5.1. Let B be a pointed building set. For $g \in G$ and $(S,i) \in \mathsf{B}$, define

$$g \cdot (S,i) := (g \cdot S,\ g \cdot i)$$

where $g \cdot S = \{g \cdot j \mid j \in S\}$. We say that B is *G-invariant* if for all $g \in G$ and $(S,i) \in \mathsf{B}$ we have $g \cdot (S,i) \in \mathsf{B}$. Similarly, an ornamentation $\rho \in \mathcal{O}(\mathsf{B})$ is called *G-invariant* if for all $i \in \mathcal{I}$ and $g \in G$ we have $\rho(g \cdot i) = g \cdot \rho(i)$. We denote the set of G-invariant ornamentations of B by $\mathcal{O}_G(\mathsf{B})$.

Proposition 5.2. *If B is G-invariant, then $(\mathcal{O}_G(\mathsf{B}), \preceq)$ is a sublattice of $(\mathcal{O}(\mathsf{B}), \preceq)$.*

Example 5.3. Let $n \in \mathbb{Z}^+$ and let $G = \mathbb{Z}$ act on $\mathcal{I} = \mathbb{Z}$ by translation by n, i.e., $a \cdot k = an + k$. Then for B from the bi-infinite Tamari lattice (see Example 2.5), we have $\mathcal{O}_G(\mathsf{B}) \simeq \mathsf{ATam}_n$.

In [4], Barkley and Defant further define the *cyclic Tamari lattice* CTam_n and conjecture that it can be described in terms of centrally symmetric type-D_{2n} triangulations. The following, combined with their description of ATam_n in terms of type-D_n triangulations, confirms their conjecture.

Theorem 5.4. *Let C_{2n} be an oriented cycle on $2n$ vertices. Let B_n be the digraphical pointed building set of C_{2n} and let $G = \mathbb{Z}/2\mathbb{Z}$ act by rotation. Then $\mathsf{CTam} \simeq \mathcal{O}_G(\mathsf{B}_n)$.*

6 Direct limits of pointed building sets induce inverse limits of ornamentation lattices

Let $\mathcal{A}$ be a directed set, let $\{(\mathcal{I}_\alpha, \mathsf{B}_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of pairs such that

- for all $\alpha \in \mathcal{A}$, $\mathcal{I}_\alpha$ is an indexing set and B_α is a pointed building set over $\mathcal{I}_\alpha$;
- for all $\alpha \leq \beta$ in $\mathcal{A}$, we have $\mathcal{I}_\alpha \subseteq \mathcal{I}_\beta$ and $\mathsf{B}_\alpha \subseteq \mathsf{B}_\beta$;
- for all $\alpha \in \mathcal{A}$, B_α is *locally finite*, i.e., for all $i \in \mathcal{I}_\alpha$, all sets in $\mathsf{B}|_i$ are finite.

Definition 6.1. We define $\varinjlim \mathcal{I}_\alpha := \bigcup_{\alpha \in \mathcal{A}} \mathcal{I}_\alpha$. For a collection of pointed subsets $\mathcal{C}$ of $\mathcal{I}$, define the *pointwise union closure* to be

$$\mathrm{PUC}(\mathcal{C}) := \left\{ \left(\bigcup_{(S,i) \in \mathsf{A}} S,\ i \right) \mid i \in \mathcal{I} \text{ and } \mathsf{A} \subseteq \mathcal{C}|_i \right\}.$$

By an abuse of notation, we define $\varinjlim \mathsf{B}_\alpha := \mathrm{PUC}\left(\bigcup_{\alpha \in \mathcal{A}} \mathsf{B}_\alpha \right)$.

Remark 6.2. It would be interesting to make this notation rigorous by defining a category of pointed building sets.

Lemma 6.3. $\varinjlim \mathsf{B}_\alpha$ *is a pointed-building set over* $\varinjlim \mathcal{I}_\alpha$.

Definition 6.4. For $\alpha \leq \beta$ in $\mathcal{A}$, define $\Psi_{\beta\alpha} : \mathcal{O}(\mathsf{B}_\beta) \to \mathcal{O}(\mathsf{B}_\alpha)$ by

$$\Psi_{\beta\alpha}(\rho)(i) = (\max\{S \in \mathsf{B}_\alpha|_i \mid S \subseteq \rho(i)\},\ i).$$

For $\alpha \in \mathcal{A}$, we also define projection maps $\pi_\alpha : \mathcal{O}(\varinjlim \mathsf{B}_\beta) \to \mathcal{O}(\mathsf{B}_\alpha)$ by

$$\pi_\alpha(\rho)(i) = (\max\{S \in \mathsf{B}_\alpha|_i \mid S \subseteq \rho(i)\},\ i).$$

We now arrive at the main theorem of this section.

Theorem 6.5. *In the category of posets,* $\varprojlim \mathcal{O}(\mathsf{B}_\alpha) \simeq \mathcal{O}\left(\varinjlim \mathsf{B}_\alpha\right)$.

Corollary 6.6. *Let* $[n]$ *have its usual total order. For* $m \leq n$*, define* $\pi_{nm} : [n] \to [m]$ *by*

$$\pi_{nm}(i) = \begin{cases} i & i \leq m \\ m & i > m. \end{cases}$$

Then $\varprojlim [n] \simeq \omega + 1$*, the positive integers with an upper bound adjoined.*

Corollary 6.7. *Let* Bool_n *be the n-th Boolean lattice, i.e., the subsets of* $[n]$ *ordered by inclusion. For* $m \leq n$*, define* $\pi_{nm} : \mathrm{Bool}_n \to \mathrm{Bool}_m$ *by*

$$\pi_{nm}(S) = S \cap [m].$$

Then $\varprojlim \mathrm{Bool}_n \simeq \mathrm{Bool}_{\mathbb{Z}^+}$ *the set of subsets of* $\mathbb{Z}^+$ *ordered by inclusion.*

Corollary 6.8. *The infinite Tamari lattice* $\mathrm{Tam}_\infty \simeq \varprojlim \mathrm{Tam}_n$*.*

Furthermore, let

$$\mathsf{B}'_n := \begin{cases} \mathsf{B}(\{-k, \ldots, k\}_\leq) & n = 2k+1 \\ \mathsf{B}(\{-k+1, \ldots, k\}_\leq) & n = 2k \end{cases}$$

Observe that $\mathcal{O}(\mathsf{B}'_n) \simeq \mathrm{Tam}_n$. Then under this (different) inverse limit of Tamari lattices we obtain the following corollary.

Corollary 6.9. *The bi-infinite Tamari lattice* $\mathrm{Tam}_{\pm\infty} \simeq \varprojlim \mathcal{O}(\mathsf{B}'_n)$*.*

Remark 6.10. In [9], a direct limit of Tamari lattices is considered, but this fails to be a complete lattice. The author could not find elsewhere in the literature where an inverse limit of Tamari lattices is considered.

7 Questions

Question 7.1. One can view a pointed building set as a collection of lattices with compatibility conditions induced by their embeddings into the Boolean lattice. Can one develop the theory of ornamentation lattices without relying on this embedding? Can lattices other than the Boolean lattice be used?

Remark 7.2. One can remove the axiom that pointed building sets contain all singletons (axiom (1) of Definition 1.1) and the axiom that ornamentations are pointed at their images (axiom (1) of Definition 1.3) and still obtain join-semilattices. It may be interesting to study the combinatorics under these relaxed axioms. It may also be interesting to study a version where sets can have multiple distinguished elements.

Remark 7.3. Building sets have been highly influential in matroid theory. It would be interesting to study if pointed building sets have any application to matroids and in particular to oriented matroids.

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