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Descent sets of permutations with only even or only odd cycle lengths

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Abstract. It is known that the number of permutations in the symmetric group $\mathcal{S}_{2n}$ consisting of cycles of odd length is equal to the number of permutations consisting of cycles of even length. We prove a surprising refinement of this equality: the number of permutations in $\mathcal{S}_{2n}$ with a prescribed ascent set J and all cycles of odd length is equal to the number of permutations with descent set J and all cycles of even length. We present two different proofs of this result, as well as a similar statement for $\mathcal{S}_{2n+1}$. The first proof is algebraic, and it involves a new identity on generating functions for higher Lie characters. The second proof is bijective, and it introduces a new bijection between words with distinct Lyndon factors of odd length and words with even Lyndon factors.

Keywords: permutation, descent, cycle, higher Lie character, necklace, Lyndon word

1 Introduction

For a positive integer n, let $\mathcal{S}_n$ denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$. Let $\mathcal{S}_n^o$ be the set of permutations in $\mathcal{S}_n$ all of whose cycles have odd lengths, and let $\mathcal{S}_n^e$ be the set of permutations all of whose cycles have even lengths, except possibly for one fixed point (a cycle of length one). Note that permutations in $\mathcal{S}_n^e$ consist of only cycles of even length if n is even, and they have one fixed point if n is odd. It is known that

$$|\mathcal{S}_n^o| = |\mathcal{S}_n^e| = \begin{cases} (n-1)!!^2 = (n-1)^2(n-3)^2 \cdots 1^2, & \text{if } n \text{ is even;} \\ n!! \cdot (n-2)!! = n(n-2)^2(n-4)^2 \cdots 1^2, & \text{if } n \text{ is odd.} \end{cases}$$

This is proved, for example, in Bóna's book [3], and can also be easily shown using exponential generating functions. See also [9, A001818, A000246] and references therein. For even n, a bijective proof of the equality $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$ appears in [3, Lemma 6.20].

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Our main result is a refinement of the identity $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$. For a permutation $\pi \in \mathcal{S}_n$, denote its descent set by $\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, and its ascent set by $\mathrm{Asc}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$. By definition, $\mathrm{Asc}(\pi) = [n-1] \setminus \mathrm{Des}(\pi)$.

Theorem 1.1. *For any positive integer n and any subset $J \subseteq [n-1]$,*

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) = J\}|.$$

We present two different proofs of this result: one algebraic and one bijective. For the algebraic proof, we show that Theorem 1.1 is equivalent to a new identity on higher Lie characters (Theorem 2.1). For the bijective proof, we combine two known bijections between permutations and multisets of necklaces, and a new bijection for Lyndon factorizations of words.

In Section 2 we provide a summary of the algebraic proof of Theorem 1.1; see [2] for more details of this approach. In Section 3 we present a sketch of the bijective proof; for more details see [5].

2 Algebraic proof

2.1 Outline

For any partition $\lambda \vdash n$, let $\psi^\lambda_{\mathcal{S}_n}$ be the corresponding higher Lie character of $\mathcal{S}_n$; see Subsection 2.2 for a definition. Let $\mathcal{P}_n^o$ be the set of partitions of n with only odd parts, and let $\mathcal{P}_n^e$ be the set of partitions of n with only even parts, except possibly for one part of size one.

We show that Theorem 1.1 is equivalent to the following new identity on higher Lie characters.

Theorem 2.1. *For any positive integer n,*

$$\sum_{\lambda \in \mathcal{P}_n^o} \psi^\lambda_{\mathcal{S}_n} = \mathrm{sign} \otimes \sum_{\lambda \in \mathcal{P}_n^e} \psi^\lambda_{\mathcal{S}_n},$$

where sign *is the sign character of $\mathcal{S}_n$.*

This result follows, in turn, from the two explicit generating functions in Theorem 2.2. For a partition $\nu \vdash n$, let b_j be the number of parts of size j in ν ($\forall j \geq 1$). Let

$$|Z_\nu| = \frac{n!}{\prod_j b_j! j^{b_j}}$$

be the size of the centralizer Z_ν of any element of cycle type ν in $\mathcal{S}_n$. Let $\underline{t} = (t_j)_{j\geq 1}$ be a countable set of indeterminates. Consider the ring $\mathbb{C}[[\underline{t}]]$ of formal power series in these indeterminates, and let $\underline{t}^{c(\nu)} := \prod_j t_j^{b_j}$.

Theorem 2.2. *For any positive integer n,*

$$\sum_{n\ge 0}\sum_{\lambda\in\mathcal{P}_n^o}\sum_{\nu\vdash n}\psi^{\lambda}_{\mathcal{S}_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\prod_{p\ge 0}\left(\frac{1+t_{2^p}}{1-t_{2^p}}\right)^{1/2^{p+1}} \tag{2.1}$$

$$\sum_{n\ge 0}\sum_{\lambda\in\mathcal{P}_n^e}\sum_{\nu\vdash n}\operatorname{sign}(\nu)\psi^{\lambda}_{\mathcal{S}_n}(\nu)\,\frac{\underline{t}^{c(\nu)}}{|Z_\nu|}=\prod_{p\ge 0}\left(\frac{1+t_{2^p}}{1-t_{2^p}}\right)^{1/2^{p+1}}. \tag{2.2}$$

In particular, setting $t_{2^p}=0$ for all $p\ge 1$, we obtain the following result.

Corollary 2.3. *For any integer $n\ge 2$,*

$$\sum_{\lambda\in\mathcal{P}_n^o}\dim\psi^{\lambda}_{\mathcal{S}_n}=\sum_{\lambda\in\mathcal{P}_n^e}\dim\psi^{\lambda}_{\mathcal{S}_n}=\begin{cases}(n-1)!!^2, & \text{if } n \text{ is even;}\\ n!!\cdot(n-2)!!, & \text{if } n \text{ is odd.}\end{cases}$$

The group of signed permutations B_n can be viewed as the centralizer, in $\mathcal{S}_{2n}$, of a fixed-point-free involution, that is, a permutation of cycle type $(2,\dots,2)$. Noting that its index is $|\mathcal{S}_{2n}|/|B_n|=(2n-1)!!$, Corollary 2.3 (for even n) suggests that the sum of higher Lie characters of $\mathcal{S}_{2n}$ over all partitions in $\mathcal{P}_{2n}^o$ is induced from a character of B_n. We prove that this is indeed the case, with an analogue for $\mathcal{S}_{2n+1}$. Specifically, denote by

$$\eta_{B_n}:=\sum_{\lambda\vdash n}\psi^{(\lambda,\varnothing)}_{B_n},$$

the sum of (type B) higher Lie characters of B_n (see [2, Subsection 7.2] for a definition) corresponding to conjugacy classes with positive cycles only.

Theorem 2.4. *For any integer $n\ge 2$,*

$$\sum_{\lambda\in\mathcal{P}_n^o}\psi^{\lambda}_{\mathcal{S}_n}=\eta_{B_{\lfloor n/2\rfloor}}\uparrow^{\mathcal{S}_n}_{B_{\lfloor n/2\rfloor}}.$$

For applications to root enumeration and connections to perfect matchings see [2].

The equivalence of Theorems 1.1 and 2.1 is explained in Subsection 2.2. In Subsection 2.3 we give some details from the proofs of Theorems 2.2 and 2.4.

2.2 Preliminaries

In this subsection we define the higher Lie characters of $\mathcal{S}_n$, and use basic properties of quasisymmetric functions to explain the equivalence of Theorems 1.1 and 2.1.

We start by defining a higher Lie character for any element (in fact, for any conjugacy class) in the symmetric group $\mathcal{S}_n$.

Let $x \in \mathcal{S}_n$ be an element of cycle type $\lambda \vdash n$ where, for each $i \geq 1$, the partition λ has a_i parts of size i. The centralizer $Z_x = Z_{\mathcal{S}_n}(x)$ satisfies

$$Z_{\mathcal{S}_n}(x) \cong \bigtimes_{i \geq 1} G_i \wr \mathcal{S}_{a_i},$$

where G_i, isomorphic to the cyclic group of order i, is the centralizer in $\mathcal{S}_i$ of a cycle of length i.

Definition 2.5 (Higher Lie characters in $\mathcal{S}_n$)**.** Let $x \in \mathcal{S}_n$ be an element of cycle type $\lambda = (1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$.

(a) For each $i \geq 1$, let ω_i be the linear character on $G_i \wr \mathcal{S}_{a_i}$, which is equal to a primitive irreducible character on the cyclic group G_i, and trivial on the wreathing group $\mathcal{S}_{a_i}$. Let

$$\omega^x := \bigotimes_{i \geq 1} \omega_i,$$

a linear character on Z_x.

(b) Define the corresponding *higher Lie character* to be the induced character

$$\psi^x_{\mathcal{S}_n} := \omega^x \uparrow_{Z_x}^{\mathcal{S}_n}.$$

(c) The character $\psi^x_{\mathcal{S}_n}$ depends only on the conjugacy class C (equivalently, the cycle type λ) of x, and can therefore be denoted by $\psi^C_{\mathcal{S}_n}$ or $\psi^\lambda_{\mathcal{S}_n}$.

In the group of signed permutations, B_n, the conjugacy classes are parametrized by bipartitions of n, $\{(\lambda, \mu) \mid \lambda \vdash k;\ \mu \vdash n-k \text{ for some } k\}$. For each class (λ, μ), pick a representative x and an appropriate linear character ω of the centralizer Z_x for which the induced character $\omega \uparrow_{Z_x}^{B_n} = \psi_{B_n}^{(\lambda,\mu)}$ is the higher Lie character.

For each subset $D \subseteq [n-1]$ define the *fundamental quasisymmetric function*

$$\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leq i_2 \leq \ldots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Given any subset $A \subseteq \mathcal{S}_n$, define the quasisymmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} \mathcal{F}_{n,\mathrm{Des}(\pi)}.$$

In their seminal paper [7], Gessel and Reutenauer proved the following.

Theorem 2.6 ([7, Theorem 3.6]). *For every partition $\lambda \vdash n$, let C_λ be the conjugacy class of permutations in $\mathcal{S}_n$ of cycle type λ. Then*

$$\mathcal{Q}(C_\lambda) = \mathrm{ch}\left(\psi^\lambda_{\mathcal{S}_n}\right),$$

where ch *is the Frobenius characteristic map and $\psi^\lambda_{\mathcal{S}_n}$ is the higher Lie character from Definition 2.5.*

The remarks preceding [7, Theorem 4.1] imply the following variant.

Corollary 2.7. *For every partition $\lambda \vdash n$,*

$$\mathrm{ch}\left(\mathrm{sign} \otimes \psi^\lambda_{\mathcal{S}_n}\right) = \sum_{\pi \in C_\lambda} \mathcal{F}_{n,[n-1]\setminus \mathrm{Des}(\pi)}.$$

The linear independence of $(\mathcal{F}_{n,D})_{D\subseteq[n-1]}$ then yields the following.

Corollary 2.8. *Theorem 1.1 and Theorem 2.1 are equivalent.*

2.3 Generating functions for higher Lie character values

In this subsection we give some details from the proofs of Theorems 2.2 and 2.4.

Let λ and ν be two partitions of n. For each integer $i \geq 1$, let a_i be the number of parts of size i in λ. Similarly, for each integer $j \geq 1$, let b_j be the number of parts of size j in ν. Thus

$$\sum_i ia_i = \sum_j jb_j = n.$$

Let $\underline{s} = (s_i)_{i\geq 1}$ and $\underline{t} = (t_j)_{j\geq 1}$ be two countable sets of indeterminates. Consider the ring $\mathbb{C}[[\underline{s}, \underline{t}]]$ of formal power series in these indeterminates, and let $\underline{s}^{c(\lambda)} := \prod_i s_i^{a_i}$ and $\underline{t}^{c(\nu)} := \prod_j t_j^{b_j}$. We prove the following main result.

Theorem 2.9. *For any positive integer n,*

$$\sum_{n\geq 0}\sum_{\lambda\vdash n}\sum_{\nu\vdash n} \psi^\lambda_{\mathcal{S}_n}(\nu)\,\frac{\underline{s}^{c(\lambda)}\underline{t}^{c(\nu)}}{|Z_\nu|} = \exp\left(\sum_i\sum_j\sum_{e|\,\gcd(i,j)} \mu(e)\,\frac{s_i^{j/e}t_j^{i/e}}{ij/e}\right),$$

where $\mu(\cdot)$ is the classical Möbius function.

This is a refined generating function for higher Lie character values. Substituting

$$s_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even} \end{cases}$$

and using known Möbius function identities yields Equation (2.1).

We also prove the following signed analogue of Theorem 2.9.

Theorem 2.10. *For any positive integer n,*

$$\sum_{n\geq 0}\sum_{\lambda\vdash n}\sum_{\nu\vdash n}\operatorname{sign}(\nu)\psi^{\lambda}_{\mathcal{S}_n}(\nu)\,\frac{\underline{s}^{c(\lambda)}\underline{t}^{c(\nu)}}{|Z_\nu|}=\exp\left(\sum_i\sum_j\sum_{e\mid\gcd(i,j)}(-1)^{i(j-1)/e}\mu(e)\,\frac{s_i^{j/e}t_j^{i/e}}{ij/e}\right),$$

where $\mu(\cdot)$ is the classical Möbius function.

Here, substituting

$$s_i=\begin{cases}1, & \text{if } i \text{ is even;}\\ s_1, & \text{if } i=1;\\ 0, & \text{if } i>1 \text{ is odd}\end{cases}$$

and collecting the coefficients of s_1^0 and s_1^1 on both sides yields Equation (2.2).

The derivation of Theorems 2.9 and 2.10 themselves starts with the following basic formula, for the values of higher Lie characters as induced characters:

$$\psi^x_{\mathcal{S}_n}(g)=\frac{|C_x|}{|C_g|}\sum_{z\in C_g\cap Z_x}\omega^x(z)\qquad(\forall x,g\in\mathcal{S}_n).$$

Here $\psi^x_{\mathcal{S}_n}$, ω^x and Z_x are as in Definition 2.5, and C_x (respectively, C_g) is the conjugacy class of x (respectively, g) in $\mathcal{S}_n$. The proof continues with a detailed analysis of the possible cycle structures of $z\in C_g\cap Z_x$, both as an element of $\mathcal{S}_n$ (conjugate to g) and as an element of Z_x, which is a group isomorphic to a direct product of wreath products. The Möbius function arises as a sum of complex roots of unity, which are the values $\omega^x(z)$ for suitable elements z (in a simplified setting).

Finally, Theorem 2.4 is derived in a similar manner from corresponding results for the Coxeter groups of type B_n, obtained in [1].

3 Bijective proof

In this section we present a bijective proof of Theorem 1.1. We will prove the following.

Theorem 3.1. *For any positive integer n and any subset $S\subseteq[n-1]$, there exists an explicit bijection*

$$f_S:\{\pi\in\mathcal{S}_n^o:\operatorname{Asc}(\pi)\subseteq S\}\to\{\pi\in\mathcal{S}_n^e:\operatorname{Des}(\pi)\subseteq S\}.\tag{3.1}$$

This bijection implies that $|\{\pi\in\mathcal{S}_n^o:\operatorname{Asc}(\pi)\subseteq S\}|=|\{\pi\in\mathcal{S}_n^e:\operatorname{Des}(\pi)\subseteq S\}|$ for any $S\subseteq[n-1]$. Theorem 1.1 follows then from this equality by the principle of inclusion-exclusion.

The proof of Theorem 3.1 is structured as follows. In Subsection 3.1, we describe two bijections between permutations and multisets of necklaces. In Subsection 3.2 we interpret the resulting multisets of necklaces as words whose Lyndon factorization has only even factors on one hand, and words whose Lyndon factorization has only odd and distinct factors on the other hand. In Subsection 3.3 we construct an explicit bijection between these two sets of words.

3.1 From permutations to multisets of necklaces

Let $S = \{s_1, s_2, \ldots, s_{k-1}\} \subseteq [n-1]$, where $s_1 < s_2 < \cdots < s_{k-1}$. Denote its associated composition by $\alpha = \alpha(S) = (s_1, s_2 - s_1, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1})$, and define the monomial $\mathbf{x}^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i}$.

Fix an alphabet $A = \{a_1, a_2, \ldots, a_k\}$, with total order $a_1 < a_2 < \cdots < a_k$. In the examples, we will denote the letters by $\mathtt{a} < \mathtt{b} < \mathtt{c} < \cdots$ instead. Denote by $\mathcal{W} = A^*$ the set of finite words over A, and by $\mathcal{W}_n$ the set of those of length n. Define two words $u, v \in \mathcal{W}$ to be *conjugate* if they are cyclic rotations of each other, that is, there exist words r and s such that $u = rs$ and $v = sr$. A *necklace* is a conjugacy class of words in $\mathcal{W}$. A nonempty word u is *primitive* if it is not the power of another word, i.e., it is not of the form $u = r^j$ for $j \geq 2$. A necklace is *primitive* if it is the conjugacy class of a primitive word.

Let $\mathcal{M}_n$ be the set consisting of all multisets of primitive necklaces of total length n. Given $M \in \mathcal{M}_n$, its *cycle structure* is the partition of n whose parts are the lengths of the necklaces in the multiset, and its *weight* is the monomial $\mathrm{wt}(M) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$, where α_i is the number of times that a_i appears in M.

In [7, Lemma 3.4], Gessel and Reutenauer describe a bijection between words and multisets of necklaces. We will interpret it as a map on permutations; specifically, as a bijection

$$\Phi_S : \{\pi \in \mathcal{S}_n : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}$$

that preserves the cycle structure. Given $\pi \in \mathcal{S}_n$ with $\mathrm{Des}(\pi) \subseteq S$, write π in cycle form, and replace entries $1, \ldots, s_1$ with a_1, entries $s_1 + 1, \ldots, s_2$ with a_2, and so on, finally replacing entries $s_{k-1} + 1, \ldots, n$ with a_k. This operation turns each cycle of π into a necklace. Let $\Phi_S(\pi)$ be the resulting multiset of necklaces, and note that it has weight $\mathbf{x}^{\alpha(S)}$. It is shown in [7] that these necklaces are primitive, and that Φ_S is a bijection.

To describe the inverse map, let $M \in \mathcal{M}_n$ with $\mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}$, and label each element of each necklace by the periodic sequence obtained by reading the necklace starting at that element. Then, order these sequences lexicographically; if some necklaces appear with multiplicity, break the ties in a consistent way by first ordering the repeated necklaces. This order assigns a number from 1 to n to each element of each necklace, which produces the permutation $\Phi_S^{-1}(M)$ in cycle form.

Let $\mathcal{M}_n^e$ be the set of elements in $\mathcal{M}_n$ which consist of necklaces of even length only, except possibly for one necklace of length one. When restricting Φ_S to the right-hand side of Equation (3.1), we obtain the following result.

Proposition 3.2. *The map Φ_S restricts to a bijection*

$$\Phi_S : \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

Example 3.3. Let $S = \{4,7\}$, and let $\pi = 45672381 = (3,6)(2,5)(1,4,7,8) \in \mathcal{S}_8^e$, which has $\mathrm{Des}(\pi) = \{4,7\} \subseteq S$. Then $\Phi_S(\pi) = (\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{a},\mathtt{b},\mathtt{c})$. To recover π from this multiset of necklaces, replace the elements of the necklaces by their periodic labels,

$$(\mathtt{abab}\ldots,\mathtt{baba}\ldots)(\mathtt{abab}\ldots,\mathtt{baba}\ldots)(\mathtt{aabc}\ldots,\mathtt{abca}\ldots,\mathtt{bcaa}\ldots,\mathtt{caab}\ldots)$$

and order these labels lexicographically, consistently breaking the ties.

To deal with the left-hand side of Equation (3.1), we use a different bijection that has appeared in work of Gessel, Restivo and Reutenauer [6], and is also a special case of a bijection due to Steinhardt [10].

In [6, Section 3], the authors describe a bijection Ξ between words of length n and multisets of necklaces of total length n that satisfy two conditions: each necklace is either primitive or of the form (uu) for some primitive word u of odd length, and each necklace of odd length appears with multiplicity at most one. We will interpret Ξ as a map on permutations. When restricting to permutations that consist of only odd cycles, the image are multisets of primitive necklaces of odd length, all of which are distinct. Denote by $\mathcal{M}_n^o$ the set of elements in $\mathcal{M}_n$ which consist of distinct necklaces of odd length. A slight modification of Ξ gives a bijection

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}$$

that preserves the cycle structure.

The description of the map Ξ_S is very similar to that of Φ_S, but the key difference appears when describing their inverses. Define the *alternating lexicographic order* on words, denoted by $<_{\mathrm{alt}}$, by letting $\mathtt{u}_1\mathtt{u}_2\mathtt{u}_3\ldots <_{\mathrm{alt}} \mathtt{v}_1\mathtt{v}_2\mathtt{v}_3\ldots$ if either $\mathtt{u}_1 < \mathtt{v}_1$, or $\mathtt{u}_1 = \mathtt{v}_1$ and $\mathtt{v}_2\mathtt{v}_3\ldots <_{\mathrm{alt}} \mathtt{u}_2\mathtt{u}_3\ldots$. Given $M \in \mathcal{M}_n^o$ with $\mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}$, label each element of each necklace by the periodic sequence obtained by reading the necklace starting at that element. Then, order these sequences according to $<_{\mathrm{alt}}$. There will be no ties, since all the necklaces in M are primitive and distinct. This order assigns a number from 1 to n to each element of each necklace, which produces the permutation $\Xi_S^{-1}(M)$ in cycle form.

Proposition 3.4. *The map Ξ_S is a bijection*

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

Example 3.5. Let $S = \{4,7\}$, and let $\pi = 86325417 = (5)(3)(2,6,4)(1,8,7) \in \mathcal{S}_8^o$, which has $\mathrm{Asc}(\pi) = \{4,7\} \subseteq S$. Then $\Xi_S(\pi) = (\mathtt{b})(\mathtt{a})(\mathtt{a},\mathtt{b},\mathtt{a})(\mathtt{a},\mathtt{c},\mathtt{b})$. To recover π from this multiset of necklaces, replace the elements by their periodic labels,

$$(\mathtt{bbb}\ldots)(\mathtt{aaa}\ldots)(\mathtt{aba}\ldots,\mathtt{baa}\ldots,\mathtt{aab}\ldots)(\mathtt{acb}\ldots,\mathtt{cba}\ldots,\mathtt{bac}\ldots)$$

and order these labels according to $<_{\mathrm{alt}}$.

Propositions 3.2 and 3.4 reduce the proof of Theorem 3.1 to finding a bijection

$$\{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\} \to \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathbf{x}^{\alpha(S)}\}. \quad (3.2)$$

3.2 Lyndon factorizations

Next we interpret the sets in Equation (3.2) in terms of Lyndon factorizations of words. We use $<$ to denote the lexicographic order on $\mathcal{W}$. A primitive word in $\mathcal{W}$ is called a *Lyndon word* if it is lexicographically smaller than all the other words in its conjugacy class. Lyndon words are in one-to-one correspondence with primitive necklaces, since each conjugacy class of primitive words has a unique lexicographically smallest element. Denote by $\mathcal{L}$ the set of Lyndon words in $\mathcal{W}$.

Theorem 3.6 ([8, Theorem 5.1.5])**.** *Every $w \in \mathcal{W}$ has a unique* Lyndon factorization*, that is, an expression $w = \ell_1\ell_2 \dots \ell_m$ where $\ell_i \in \mathcal{L}$ for all i, and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m$.*

We use vertical bars to indicate that $w = \ell_1|\ell_2|\dots|\ell_m$ is the Lyndon factorization of w, and call the words ℓ_i the *Lyndon factors* of w. By identifying primitive necklaces with Lyndon words, Theorem 3.6 gives a straightforward bijection between $\mathcal{M}_n$ and $\mathcal{W}_n$, where each necklace in $M \in \mathcal{M}_n$ becomes a Lyndon factor of the associated word $w \in \mathcal{W}_n$. For instance, the multiset $(\mathtt{b})(\mathtt{a})(\mathtt{a},\mathtt{b},\mathtt{a})(\mathtt{a},\mathtt{c},\mathtt{b})$ in Example 3.5 corresponds to the word $\mathtt{b}|\mathtt{acb}|\mathtt{aab}|\mathtt{a}$.

Define the *weight* of a word $w \in \mathcal{W}$ to be the monomial $\mathrm{wt}(w) = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_k^{\alpha_k}$, where α_i is the number of times that a_i appears in w. The *length* of w is $|w| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Note that if $w = \ell_1\ell_2\dots\ell_m$, then $\mathrm{wt}(w) = \mathrm{wt}(\ell_1)\,\mathrm{wt}(\ell_2)\cdots\mathrm{wt}(\ell_m)$.

Let $\mathcal{W}_n^o$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have odd length and are distinct. Let $\mathcal{W}_n^e$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have even length, except possibly for one factor which has length one.

The above correspondence between multisets of necklaces and words gives straightforward bijections between $\mathcal{M}_n^o$ and $\mathcal{W}_n^o$, and between $\mathcal{M}_n^e$ and $\mathcal{W}_n^e$. With this identification, finding a bijection for Equation (3.2) is equivalent to finding a bijection

$$\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = \mathbf{x}^{\alpha}\} \to \{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = \mathbf{x}^{\alpha}\}. \quad (3.3)$$

3.3 A bijection from odd and distinct to even Lyndon factorizations

Next we describe a weight-preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$. The definition of Ψ relies on the notion of standard factorization of a Lyndon word.

Lemma 3.7 ([8, Prop. 5.1.3])**.** *A word $w \in \mathcal{W}$ is a Lyndon word if and only if $|w| = 1$ or $w = rs$ with $r, s \in \mathcal{L}$ and $r < s$. Additionally, if $w \in \mathcal{L}$ with $|w| \geq 2$, and s is the longest proper suffix of w that belongs to $\mathcal{L}$, then $r \in \mathcal{L}$ and $r < rs < s$.*

When s is the longest proper suffix of w that belongs to $\mathcal{L}$, the expression $w = rs$ is called the *standard factorization* of w. We will denote this by $w = r \dot{|} s$.

Given $w \in \mathcal{W}_n^o$, we will build $\Psi(w)$ by repeatedly updating a pair of words (O, E). Initially, $(O, E) = (w, -)$, where $-$ denotes the empty word. Each step moves some subword from O to the beginning of E. At any time, all the Lyndon factors of O are odd and distinct, and all the Lyndon factors of E are even. At the end of the algorithm, $(O, E) = (-, \Psi(w))$. Define a word, denoted by ∞, which satisfies $w < \infty$ for any $w \in \mathcal{W}$.

Definition 3.8 (The map Ψ)**.** On input $w \in \mathcal{W}_n^o$, initially set $(O, E) = (w, -)$, and iterate the following step as long as $|O| \geq 2$:

- Let $O = o_1|o_2|\ldots|o_m$ be the Lyndon factorization of O. Say that o_m is *splittable* if $|o_m| \geq 2$ and its standard factorization $o_m = r \dot{|} s$ satisfies $s < o_{m-1}$ (with the convention $o_{m-1} = \infty$ if $m = 1$). Update (O, E) to

$$(O', E') = \begin{cases} (o_1 o_2 \ldots o_{m-1} r,\ sE) & \text{if } o_m \text{ is splittable and } r \text{ is odd,} \quad \texttt{(S)} \\ (o_1 o_2 \ldots o_{m-1} s,\ rE) & \text{if } o_m \text{ is splittable and } r \text{ is even,} \quad \texttt{(P)} \\ (o_1 o_2 \ldots o_{m-2},\ o_m o_{m-1} E) & \text{if } o_m \text{ is not splittable.} \quad \texttt{(F)} \end{cases}$$

If we reach $|O| = 1$ (this case only occurs when n is odd), move this letter to the Lyndon factorization of E by inserting it as a new factor, in the unique location that keeps the factors weakly decreasing from left to right.

Once O is empty, let $\Psi(w) = E$.

The steps of type `(S)`, `(P)` and `(F)` are named after *suffix*, *prefix* and *flip*, respectively. Note that, since $o_m = rs$ has odd length, r is odd if and only if s is even. The proof of the following theorem is omitted from this extended abstract, but can be found in [5].

Theorem 3.9. *The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.*

Let us give a few examples of the map Ψ. An implementation in SageMath of this bijection, as well as of its inverse, is available in [4].

Example 3.10. Let $w =$ `dadccdbccc` $\in \mathcal{W}_{10}^o$, which has Lyndon factorization $o_1|o_2 =$ `d`$|$`adccdbccc`. The rightmost factor has standard factorization $o_2 = r \dot{|} s =$ `adccd`$\dot{|}$`bccc`. Since $s < o_1$, the word o_2 is splittable, and since s is even, we apply `(S)` and move it to E.

Now $O = o_1|o_2 =$ `d`$|$`adccd`, and $o_2 = r \dot{|} s =$ `ad`$\dot{|}$`ccd`. Again $s < o_1$, so o_2 is splittable, and since r is even, we apply `(P)` and move it to E.

In the third iteration, $O = o_1|o_2 =$ `d`$|$`ccd`, and $o_2 = r \dot{|} s =$ `c`$\dot{|}$`cd`. Again $s < o_1$, so o_2 is splittable, and since s is even, we apply `(S)` and move it to E.

Now $O = \mathtt{d}|\mathtt{c}$, and $o_2 = \mathtt{c}$ is not splittable because it has length one. Applying (F), we move $o_2 o_1 = \mathtt{cd}$ to E, and we obtain $\Psi(w) = \mathtt{cdcdadbccc}$. We can summarize these steps in a table as follows.

O		E
d∣adccd¦bccc		—
d∣ad¦ccd	(S)	bccc
d∣c¦cd	(P)	adbccc
d∣c	(S)	cdadbccc
—	(F)	cdcdadbccc

The Lyndon factorization $\Psi(w) = \mathtt{cd}|\mathtt{cd}|\mathtt{adbccc}$ in the above example illustrates that the subwords that are moved to E at each step do not necessarily become its Lyndon factors. Next is an example for odd n.

Example 3.11. For $w = \mathtt{ddecedbdbdccdabda} \in \mathcal{W}^o_{17}$, the following table summarizes the computation of $\Psi(w) = \mathtt{dedccedcdbdbdaabd}$.

O		E
dde∣ced∣bdbdccd∣abd∣a		—
dde∣ced∣bd¦bdccd	(F)	aabd
dde∣ced∣bd¦ccd	(P)	bdaabd
dde∣ced∣c¦cd	(P)	bdbdaabd
dde∣ced∣c	(S)	cdbdbdaabd
d¦de	(F)	ccedcdbdbdaabd
d	(S)	deccedcdbdbdaabd
—		de∣d∣ccedcd∣bd∣bd∣aabd

Proof of Theorem 3.1. Combining Theorem 3.9 with Propositions 3.2 and 3.4, and identifying $\mathcal{M}^o_n$ with $\mathcal{W}^o_n$ and $\mathcal{M}^e_n$ with $\mathcal{W}^e_n$ as described in Section 3.2, it follows that the map $f_S = \Phi_S^{-1} \circ \Psi \circ \Xi_S$ is a bijection from $\{\pi \in \mathcal{S}^o_n : \operatorname{Asc}(\pi) \subseteq S\}$ to $\{\pi \in \mathcal{S}^e_n : \operatorname{Des}(\pi) \subseteq S\}$. □

We end with an example of the computation of f_S as a composition of these three bijections.

Example 3.12. Let $n = 17$ and $S = \{2, 5, 8, 15\}$, and let

$$\begin{aligned}\pi &= 3\ 2\ 15\ 13\ 11\ 16\ 14\ 7\ 17\ 9\ 8\ 6\ 5\ 4\ 1\ 12\ 10\\ &= (9,17,10)(6,16,12)(4,13,5,11,8,7,14)(2)(1,3,15) \in \mathcal{S}^o_{17},\end{aligned}$$

which has $\operatorname{Asc}(\pi) = S$, so it belongs to the left-hand side of Equation (3.1). Making the replacements $\{1,2\} \mapsto \mathtt{a}$, $\{3,4,5\} \mapsto \mathtt{b}$, $\{6,7,8\} \mapsto \mathtt{c}$, $\{9,10,11,12,13,14,15\} \mapsto \mathtt{d}$ and $\{16,17\} \mapsto \mathtt{e}$ in the cycle form of π, we get

$$\Xi_S(\pi) = (\mathtt{d},\mathtt{e},\mathtt{d})(\mathtt{c},\mathtt{e},\mathtt{d})(\mathtt{b},\mathtt{d},\mathtt{b},\mathtt{d},\mathtt{c},\mathtt{c},\mathtt{d})(\mathtt{a})(\mathtt{a},\mathtt{b},\mathtt{d}),$$

which can be identified with the word $w = \mathtt{dde}|\mathtt{ced}|\mathtt{bdbdccd}|\mathtt{abd}|\mathtt{a}$. As in Example 3.11, we have

$$\Psi(w) = \mathtt{de}|\mathtt{d}|\mathtt{ccedcd}|\mathtt{bd}|\mathtt{bd}|\mathtt{aabd},$$

which corresponds to the multiset of necklaces

$$(\mathtt{d},\mathtt{e})(\mathtt{d})(\mathtt{c},\mathtt{c},\mathtt{e},\mathtt{d},\mathtt{c},\mathtt{d})(\mathtt{b},\mathtt{d})(\mathtt{b},\mathtt{d})(\mathtt{a},\mathtt{a},\mathtt{b},\mathtt{d}).$$

To apply Φ_S^{-1} to this multiset of necklaces, we replace the elements by their periodic labels

$$(\mathtt{ded}\ldots,\mathtt{ede}\ldots)(\mathtt{ddd}\ldots)(\mathtt{cce}\ldots,\mathtt{ced}\ldots,\mathtt{edc}\ldots,\mathtt{dcd}\ldots,\mathtt{cdc}\ldots,\mathtt{dcc}\ldots)$$
$$(\mathtt{bdb}\ldots,\mathtt{dbd}\ldots)(\mathtt{bdb}\ldots,\mathtt{dbd}\ldots)(\mathtt{aab}\ldots,\mathtt{abd}\ldots,\mathtt{bda}\ldots,\mathtt{daa}\ldots).$$

and order them lexicographically, breaking ties consistently, to get the permutation

$$\begin{aligned} f_S(\pi) = \Phi_S^{-1}(\Psi(\Xi_S(\pi))) &= (15,17)(14)(6,8,16,13,7,12)(5,11)(4,10)(1,2,3,9) \\ &= 2\ 3\ 9\ 10\ 11\ 8\ 12\ 16\ 1\ 4\ 5\ 6\ 7\ 14\ 17\ 1\ 15 \in \mathcal{S}_{17}^{e}, \end{aligned}$$

which has $\mathrm{Des}(f_S(\pi)) = \{5,8,15\} \subseteq S$.

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