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Downs and Ups in the Weyl Algebra

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Abstract. Motivated by a question and some conjectures of Richard Stanley, we explore the equivalence classes of words in the Weyl algebra, $\mathbf{k}\left\langle D, U \mid DU - UD = 1 \right\rangle$. We show that each class is generated by the swapping of adjacent *balanced subwords*, i.e., those which have the same number of D's as U's, and give several other characterizations. These allow us to deduce a number of enumerative results about the number of such equivalence classes and their sizes. We extend these results to the class of c-Dyck words, where every prefix has at least c times as many U's as D's. We also connect our results to previous work on rook theory.

Keywords: Weyl algebra, words, lattice paths, rook placements, Ferrers boards, Dyck words, monoid kernel, bond percolation, PBW bases, down-up algebra, noncommutative algebra, rings, combinatorics, finite fields

1 Introduction

In 2023 Richard Stanley proposed the following problem (private communication). Let $\mathcal{M}$ denote the monoid freely generated by the two non-commuting variables D and U. Consider the action of $\mathcal{M}$ on the polynomial ring $\mathbb{Q}[x]$ in which D acts as differentiation ($\frac{d}{dx}$) and U acts as multiplication by x. This action is known to satisfy the relation

$$DU - UD = 1, \tag{1.1}$$

which is the defining relation of the *Weyl algebra* (also known as the *Heisenberg–Weyl algebra* [2] due to its obvious quantum-physical significance).

Consider two words in $\mathcal{M}$ to be *Weyl equivalent* (often shortened to *"equivalent"*) if they act equally on $\mathbb{Q}[x]$ (that is, become equal in the Weyl algebra). It can be shown that equivalent words have the same number of U's and the same number of D's. Thus

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we can ask: *How many distinct equivalence classes are there* for words with k many D's and $n-k$ many U's? We call this number $a(n,k)$. For example, among the six words with two D's and two U's, there is one equivalence: $DUUD = UDDU$, so that $a(4,2) = 5$.

We prove explicit formulas for $a(n,k)$ and for $\sum_k a(n,k)$ originally conjectured by Stanley (Section 3). Along the way, we study the equivalence from several directions and give several equivalent descriptions of it. Call a word in $\mathcal{M}$ *balanced* if it has the same number of U's as D's. We show (in Theorem 2.2, another conjecture of Stanley) that two words v and w are equivalent if and only if one can be obtained from another by a series of *balanced commutations*, i.e., by a sequence of swaps of adjacent balanced factors. In the example above, the transposition of DU with UD gives the equivalence. This and several further criteria serve as the linchpin for the enumerative results.

Further connections are contained in the full version of this paper [8], including to

- The *differential posets* [14] defined by Stanley, which leverage (1.1) and some cleverly-defined machinery to obtain results enumerating chains in such posets (equivalent to Young tableaux in Young's Lattice).
- Similar results and questions for the *down-up algebras* of Benkart and Roby [1].
- *Bond percolation on the directed square lattice*, explaining why our numbers $\sum_k a(n,k)$ are closely related to the OEIS sequence [12, A006727] within statistical physics.
- Open questions for Weyl algebras when the field has positive characteristic.

The outline of the paper is as follows. In Section 2 we formally define the monoid $\mathcal{M}$, the Weyl algebra $\mathcal{W}$, and some related objects. Then we state our main result (Theorem 2.2), which gives several different criteria for words in the monoid $\mathcal{M}$ to be equivalent (i.e., to represent the same operator in $\mathcal{W}$). A second result (Theorem 2.3) says that each balanced word u is equivalent to its "reverse toggle-image" (i.e., to the word obtained by reading u right-to-left and replacing each D by U and vice versa).

Enumerative results – including the formulas for $a(n,k)$ and for $\sum_k a(n,k)$ – are then obtained in Section 3. The formula for $\sum_k a(n,k)$ (Corollary 3.5) is surprisingly intricate, despite involving nothing more complicated than Fibonacci numbers. Then we turn our attention to *c-Dyck words*, where every prefix has at least c times as many U's as D's, finding analogous results for this situation, and exploring some interesting special cases. Finally, we give an explicit formula for the *size* of the Weyl-equivalence class of a word w.

In Section 4, we extend Theorem 2.2 by three further equivalent statements, which come from the part of combinatorics known as rook theory. Namely, the equivalence of two words u and v reveals itself as a stronger version of the *rook equivalence* of two Ferrers boards B_u and B_v induced by these words (see Theorem 4.2); it furthermore is equivalent to the equality of certain matrix counting functions over finite fields (Theorem 4.3).

2 Definitions and main structural results

2.1 The monoid $\mathcal{M}$, the Weyl algebra $\mathcal{W}$, words, and paths

Let $\mathcal{M}$ be the free (noncommutative) monoid generated by two symbols D and U. Its elements are the words with letters D and U, such as $DUUDDUDD$.

Let $\mathbf{k}$ be a field of characteristic 0, and let $\mathcal{W}$ be the Weyl algebra over $\mathbf{k}$ with generators D and U and relation $DU - UD = 1$. This algebra $\mathcal{W}$ acts on the univariate polynomial ring $\mathbf{k}[x]$ in a standard way: D acts as the derivative operator $\dfrac{\partial}{\partial x}$, whereas U acts as multiplication by x. It is known that this action is faithful, and $\mathcal{W}$ has a basis $\left(D^i U^j\right)_{i,j\in\mathbb{N}}$ as well as a basis $\left(U^j D^i\right)_{i,j\in\mathbb{N}}$ [10, 13] (where $\mathbb{N} := \{0, 1, 2, \ldots\}$).

Let $\phi : \mathcal{M} \to \mathcal{W}$ be the canonical monoid (homo)morphism from $\mathcal{M}$ to the monoid $(\mathcal{W}, \cdot, 1)$ that sends D to D and U to U. This morphism ϕ is **not** injective, since (for example) $DUUD = UDDU$ in $\mathcal{W}$ (but not in $\mathcal{M}$). Thus, one may naturally wonder what pairs of words $u, v \in \mathcal{M}$ have equal images under ϕ. In the parlance of monoid theory, this is asking about the *kernel* of the monoid morphism ϕ – that is, the equivalence relation "$\phi(u) = \phi(v)$" on $\mathcal{M}$ – or about the fibers of φ.

Definition 2.1. A word $v \in \mathcal{M}$ is said to be a *factor* of a word $w \in \mathcal{M}$ if there exist words $u, u' \in \mathcal{M}$ (possibly empty) such that $w = uvu'$. When u is empty, v is a *prefix*; and when u' is empty, v is a *suffix* of w.

For example, the word DUD is a prefix of $DUDUDD$ and is a suffix of $UDDUD$. Furthermore, the word DUD is a factor of $UDUDDD$, but neither a prefix nor a suffix. The word DUD is not a factor of $DUUD$ (since a factor must appear contiguously).

We represent words by *diagonal paths* (paths with diagonal steps, either NE ($\nearrow$) or SE ($\searrow$)), whose vertices lie in $\mathbb{Z}^2$ in the obvious way. (See [8, Sec. 2] for details.) Steps are identified with their starting points. We write $\mathbf{p} = (p_0, p_1, \ldots, p_k)$, where each $p_i \in \mathbb{Z}^2$, and each step is either NE or SE. The *height* of $(i, j) \in \mathbb{Z}^2$ is just $\operatorname{ht}(i, j) = j$. The *initial height* of $\mathbf{p}$ is $\operatorname{ht}(p_0)$, and the *final height* of $\mathbf{p}$ is $\operatorname{ht}(p_k)$.

For example, if $\mathbf{p} = (p_0, p_1, p_2, p_3)$ is a diagonal path with $p_0 \nearrow p_1 \searrow p_2 \searrow p_3$, then its vertices are p_0, p_1, p_2, p_3; its only NE-step is p_0; and its SE-steps are p_1 and p_2.

The *reading word* $\mathrm{w}(\mathbf{p})$ of a diagonal path $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is defined to be the word $w_0 w_1 \cdots w_{k-1} \in \mathcal{M}$, where $w_i = U$ when $p_i \nearrow p_{i+1}$ and D when $p_i \searrow p_{i+1}$. For instance, if $\mathbf{p} = (p_0, p_1, p_2, p_3, p_4)$ with $p_0 \nearrow p_1 \searrow p_2 \searrow p_3 \searrow p_4$, then $\mathrm{w}(\mathbf{p}) = UDDD$.

For any word $w \in \mathcal{M}$, there is a unique diagonal path $\mathbf{p}$ that starts at $(0, 0)$ and has reading word $\mathrm{w}(\mathbf{p}) = w$. We will call this path $\mathbf{p}$ the *standard path* of w. For example, the path shown in Figure 1 is the standard path of $UDUUDDD$. The final height of $\mathbf{p}$ will be called the final height of w.

If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is any diagonal path, and $w = \mathrm{w}(\mathbf{p})$ its reading word, then we associate three Laurent polynomials (in z) to $\mathbf{p}$ (and to w when $p_0 = (0, 0)$):

- the *height polynomial* $H(\mathbf{p}, z) = H(w, z) = \sum_{i=0}^{k} z^{\mathrm{ht}(p_i)}$;
- the *NE-height polynomial* $H_{\mathrm{NE}}(\mathbf{p}, z) = H_{\mathrm{NE}}(w, z) = \sum_{p_i \text{ is an NE-step of } \mathbf{p}} z^{\mathrm{ht}(p_i)}$;
- and the *SE-height polynomial* $H_{\mathrm{SE}}(\mathbf{p}, z) = H_{\mathrm{SE}}(w, z) = \sum_{p_i \text{ is an SE-step of } \mathbf{p}} z^{\mathrm{ht}(p_i)}$.

For the diagonal path $\mathbf{p}$ shown in Figure 1 its initial height is 0, its final height is -1, its height polynomial is $H(\mathbf{p}, z) = z^{-1} + 3z^0 + 3z^1 + z^2$, its NE-height polynomial is $H_{\mathrm{NE}}(\mathbf{p}, z) = 2z^0 + z^1$, its SE-height polynomial is $H_{\mathrm{SE}}(\mathbf{p}, z) = z^0 + 2z^1 + z^2$, and its reading word is $\mathrm{w}(\mathbf{p}) = UDUUDDD$.

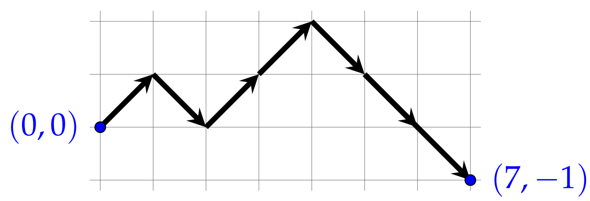

Figure 1: A diagonal path.

We furthermore define two useful maps, both of which we call ω:

- Let $\omega : \mathcal{M} \to \mathcal{M}$ be the monoid anti-morphism that sends U and D to D and U. Thus, acting on a word $w \in \mathcal{M}$, it reverses the word and toggles every letter.

  For example, $\omega(UDD) = UUD$.
- Let $\omega : \mathcal{W} \to \mathcal{W}$ be the $\mathbf{k}$-algebra anti-morphism that sends U and D to D and U (easily seen to be well-defined). For example, $\omega(UDD) = UUD$ (but now in $\mathcal{W}$).

Both maps ω are involutions in the sense that $\omega \circ \omega = \mathrm{id}$. Such anti-morphisms that are their own inverses are also called *anti-involutions*. Moreover, the two ω's commute with ϕ, justifying our using the same letter for both: $\omega \circ \phi = \phi \circ \omega$.

2.2 Balanced words, commutations and flips

Finally, we define the concept of a *balanced word* and two equivalence relations on words:

- A word $w \in \mathcal{M}$ is said to be *balanced* if it has the same number of D's and U's. For example, $DUUDDU$ is balanced, whereas $DUUUD$ is not.

- Given two words $v, w \in \mathcal{M}$, we say that v is obtained from w by a *balanced commutation* if and only if we can write v and w as $v = pxyq$ and $w = pyxq$, where $p, q \in \mathcal{M}$ are two words and where $x, y \in \mathcal{M}$ are two balanced words. This means that v can be obtained from w by swapping two balanced factors that abut each other in w. We define an equivalence relation $\overset{\text{bal}}{\sim}$ on $\mathcal{M}$ as the transitive closure of this relation.

  For instance, the following three words are all equivalent (over/under-lining and colors indicate which balanced subwords are being swapped): $\underline{DU}\,\overline{DDUU}DUUD \overset{\text{bal}}{\sim} \overline{DDUU}\,\underline{DU}DUUD = DDUUDU\underline{DU}\,\overline{UD} \overset{\text{bal}}{\sim} DDUUDU\overline{UD}\,\underline{DU}$.

- Given two words $v, w \in \mathcal{M}$, we say that v is obtained from w by a *balanced flip* if and only if we can write v and w as $v = pxq$ and $w = p\omega(x)q$, where $p, q, x \in \mathcal{M}$ are words such that x is balanced. This means that v can be obtained from w by picking a balanced factor and applying the involution ω to it. We define an equivalence relation $\overset{\text{flip}}{\sim}$ on $\mathcal{M}$ as the transitive closure of this relation. For instance, $D\underline{UDDU}U \overset{\text{flip}}{\sim} D\underline{DUUD}U$.

2.3 Main result: Equivalent descriptions of Weyl equivalence

Everything is now in place to state our main result, which gives several (necessary and sufficient) criteria for when two words $u, v \in \mathcal{M}$ have the same image under ϕ.

Theorem 2.2. Let u and v be two words in $\mathcal{M}$. Then, the following seven statements are equivalent:

- $\mathcal{S}_1$: We have $\phi(u) = \phi(v)$.
- $\mathcal{S}_2$: The elements $\phi(u)$ and $\phi(v)$ act equally on the polynomial ring $\mathbf{k}[x]$. (That is, we have $(\phi(u))(p) = (\phi(v))(p)$ for each polynomial $p \in \mathbf{k}[x]$.)
- $\mathcal{S}_3$: The words u and v have the same final height and satisfy $H_{\text{NE}}(u, z) = H_{\text{NE}}(v, z)$.
- $\mathcal{S}_3'$: The words u and v have the same final height and satisfy $H_{\text{SE}}(u, z) = H_{\text{SE}}(v, z)$.
- $\mathcal{S}_4$: The words u and v have the same final height and satisfy $H(u, z) = H(v, z)$.
- $\mathcal{S}_5$: We have $u \overset{\text{bal}}{\sim} v$.
- $\mathcal{S}_6$: We have $u \overset{\text{flip}}{\sim} v$.

As a consequence of Theorem 2.2, we obtain:

Theorem 2.3. Let $u \in \mathcal{M}$ be a balanced word. Then, $\phi(u) = \phi(\omega(u))$ and $u \overset{\text{bal}}{\sim} \omega(u)$.

The proof of our main result requires significant build-up and preparation, and we only give the briefest sketch of it here. See [8, Sec. 3–5] for all the details. One key is the following definition and proposition, giving a *normal form* for elements of $\mathcal{M}$.

A word $w \in \mathcal{M}$ is said to be *rising* if it has at least as many U's as it has D's (i.e., its final height is ≥ 0), and *falling* if it has at least as many D's as it has U's (i.e., its final height is ≤ 0). Thus, each word $w \in \mathcal{M}$ is rising or falling or both. Moreover, the balanced words $w \in \mathcal{M}$ are exactly the words $w \in \mathcal{M}$ that are both rising and falling.

A *down-zig* means a word of the form UD^kU for some $k \geq 2$. A rising word $w \in \mathcal{M}$ is said to be *up-normal* if it contains no down-zig as a factor. For example, the rising word $UUDUDUDD$ is up-normal, whereas the rising word $UUDDUU$ is not.

Proposition 2.4. ([8, Prop. 4.4]) Let $w \in \mathcal{M}$ be a rising word. Then there exists a unique up-normal word $t \in \mathcal{M}$ such that $t \overset{\text{bal}}{\sim} w$.

The main idea of the proof is that every word that is not up-normal can be transformed to a lexicographically smaller word by a balanced commutation. The lexicographically smallest representative is the up-normal word t in the proposition.

We show [8, Prop. 4.2] that each up-normal word has the form $D^a (UD)^{r_1} U (UD)^{r_2} U \cdots (UD)^{r_h} U D^b$ for unique nonnegative integers $h, a, b, r_1, r_2, \ldots, r_h$. Thus, such a word is uniquely determined by its final height and NE-height polynomial.

3 Enumeration

Two words $u, v \in \mathcal{M}$ will be called *Weyl-equivalent* if $\phi(u) = \phi(v)$. Obviously, Weyl equivalence is an equivalence relation. Theorem 2.2 (and, later, Theorem 4.2) provides some necessary and sufficient criteria for Weyl equivalence. In particular, the $\mathcal{S}_1 \iff \mathcal{S}_5$ part of Theorem 2.2 shows that Weyl equivalence is precisely the relation $\overset{\text{bal}}{\sim}$.

In this section, we prove several enumerative results regarding the equivalence classes of Weyl equivalence (henceforth just called "equivalence classes").

3.1 Counting equivalence classes by numbers of D's and U's

First, we consider equivalence classes of words with a given number of D's and U's. For $0 \leq k \leq n$, let $a(n,k)$ be the number of equivalence classes of words with k many D's and $n-k$ many U's. The ω involution creates the following symmetry:

Proposition 3.1. We have $a(n,k) = a(n, n-k)$ for any integers $0 \leq k \leq n$.

In particular, $a(n,0) = a(n,n) = 1$. Our first real result about the $a(n,k)$ is the following recursion.

Lemma 3.2. ([8, Lemma 6.2]) For $n > 2k \geq 0$, we have $a(n,k) = a(n-1,k) + a(n-2,k-1)$. Here, $a(n-2,-1)$ is interpreted as 0 when $k = 0$.

The proof uses the unique up-normal representative from Proposition 2.4. If the associated standard path has only one vertex of maximum height, we remove the last U to obtain a bijection with classes counted by $a(n-1,k)$. Otherwise, we remove the last U and the D right before it to obtain a bijection with classes counted by $a(n-2,k-1)$.

Second, we prove explicit formulas in two special cases, which together with the previous lemma characterize the numbers $a(n,k)$ for all n and k.

Lemma 3.3. ([8, Lemma 6.3]) For $k > 0$, we have

$$a(2k,k) = (k+3)2^{k-2} \quad \text{and} \quad a(2k+1,k) = (k+2)2^{k-1}.$$

The combination of these two lemmas can be used to yield explicit formulas for the bivariate generating function of $a(n,k)$, given in the full version of the paper.

Corollary 3.4. ([8, Cor. 6.5]) For all n and k with $0 \leq k \leq n/2$, we have

$$a(n,k) = \sum_{j=0}^{k} (k-j+1) \binom{n-k-1}{j}.$$

$n \backslash k$	0	1	2	3	4	5	6	7	8	$\sum_k a(n,k)$
0	1									1
1	1	1								2
2	1	2	1							4
3	1	3	3	1						8
4	1	4	5	4	1					15
5	1	5	8	8	5	1				28
6	1	6	12	12	12	6	1			50
7	1	7	17	20	20	17	7	1		90
8	1	8	23	32	28	32	23	8	1	156

Table 1: Table of the values of $a(n,k)$ and total number of equivalence classes.

Corollary 3.5. The total number of equivalence classes of words of length $n > 0$ is

$$\sum_{k=0}^{n} a(n,k) = 2F_{n+4} - \begin{cases} (3n+42)2^{n/2-3}, & \text{if } n \text{ is even,} \\ (n+15)2^{(n-3)/2}, & \text{if } n \text{ is odd,} \end{cases}$$

where F_n is the n-th Fibonacci number. See Table 1.

Besides counting equivalences clases, one can also ask how large the equivalence classes are. The following theorem ([8, Thm. 6.14]) answers this question for ordinary words, generalizing [7, Theorem 6] (for rook theory) or [5, Proposition 3A and 3B] (for continued fractions and lattice paths).

Theorem 3.6. Let $w \in \mathcal{M}$ be a word with NE-height polynomial $H_{\mathrm{NE}}(w,z) = \sum_{i\in\mathbb{Z}} a_i z^i$ and SE-height polynomial $H_{\mathrm{SE}}(w,z) = \sum_{i\in\mathbb{Z}} b_i z^i$. Then, the size of the equivalence class containing w (that is, the number of words $u \in \mathcal{M}$ satisfying $u \overset{\mathrm{bal}}{\sim} w$) is

$$\prod_{i\geq 0} \binom{a_i + b_{i+2} - 1}{b_{i+2}} \binom{b_{-i} + a_{-i-2} - 1}{a_{-i-2}} \times \begin{cases} \binom{a_0+b_0}{a_0}, & \text{if } w \text{ is balanced;} \\ \binom{a_0+b_0-1}{b_0}, & \text{if } w \text{ is rising and non-balanced;} \\ \binom{a_0+b_0-1}{a_0}, & \text{if } w \text{ is falling and non-balanced.} \end{cases}$$

3.2 c-Dyck words

Let us now turn our attention to restricted words ([8, §6.3]). Fix a constant $c > 0$, and consider words with the property that every prefix has at least c times as many U's as D's. These words form a submonoid $\mathcal{M}_c$ of $\mathcal{M}$. Again, we will be interested in the number of equivalence classes of words of length n in $\mathcal{M}_c$. Note here that not all words equivalent to a word in $\mathcal{M}_c$ are necessarily also in $\mathcal{M}_c$. To give a simple example, the word $UUDUUD$ is in $\mathcal{M}_2$ while the equivalent word $UUUDDU$ is not.

Let $a_c(n,k)$ be the number of equivalence classes of words in the submonoid $\mathcal{M}_c$ that consist of k D's and $n-k$ U's. Note that we must have $n-k \geq ck$, or equivalently $n \geq (c+1)k$. We first show that we can again focus on up-normal words.

Lemma 3.7. Let $c \geq 1$ be a real constant. An equivalence class of words in $\mathcal{M}$ contains words in $\mathcal{M}_c$ if and only if it consists of rising words and its unique up-normal representative is in $\mathcal{M}_c$.

In analogy to Lemma 3.2, the following lemma holds.

Lemma 3.8. For every real constant $c \geq 1$ and every pair (n,k) of positive integers with $n-1 \geq (c+1)k$, we have

$$a_c(n,k) = a_c(n-1,k) + a_c(n-2,k-1). \tag{3.1}$$

The recursion (3.1) is in general false for $c < 1$. For positive integer values of c, there is a fairly simple explicit formula for $a_c(n,k)$ ([8, Thm. 6.11]):

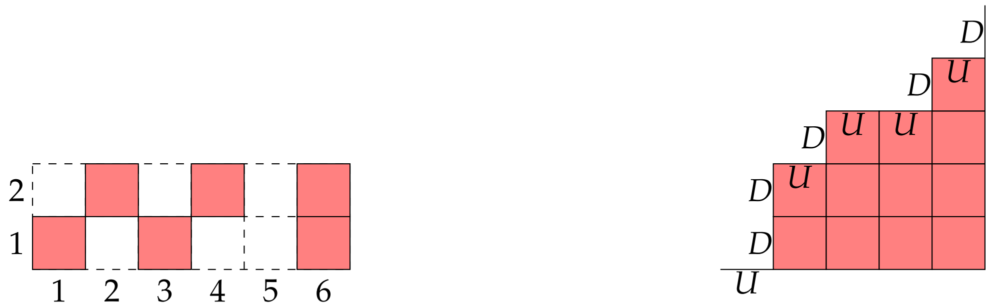

Figure 2: Left: An arbitrary board. Right: The board B_w for $w = UDDUDUUDUD$.

Theorem 3.9. If c, n, k are positive integers with $n \geq (c+1)k$, then we have

$$a_c(n,k) = \binom{n-k-1}{k} - (c-2)\sum_{j=0}^{k-1}\binom{n-k-1}{j}. \tag{3.2}$$

In the special case $c = 1$, we obtain $a_1(n,k) = \sum_{j=0}^{k}\binom{n-k-1}{j}$, for which we indicate a combinatorial proof in the full version of the paper ([8, Rmk. 6.12]).

In the special case $c = 2$, the sum disappears from (3.2), and we obtain the remarkably simple formula $a_2(n,k) = \binom{n-k-1}{k}$, which also has a combinatorial proof ([8, Rmk. 6.13]).

4 The rook theory connection

Theorem 2.2 classifies equalities between products of D's and U's in the Weyl algebra $\mathcal{W}$. Another approach to this classification problem is to expand any such product in one of the bases $\left(D^i U^j\right)_{i,j\in\mathbb{N}}$ and $\left(U^j D^i\right)_{i,j\in\mathbb{N}}$ of $\mathcal{W}$; the uniqueness of this expansion then allows us to compare two such products by comparing their respective coefficients.

It turns out that this expansion can be done in explicit combinatorial terms using *rook theory*. We do not give a detailed introduction to this subject (see [3] and [10, §2.4.4] for that), but quickly recall the basics we need.

A *cell* means a pair (i,j) of positive integers. Each cell (i,j) will be drawn as a 1×1-square, situated in the Cartesian plane with center at the point (i,j), with its sides parallel to the axes. A *board* means a finite set of cells. For instance, the board $\{(1,1), (2,2), (3,1), (4,2), (6,1), (6,2)\}$ is shown on Figure 2 left.

A *rook placement* of a board B is a subset S of B such that no two cells in S lie in the same row or column (that is, if we interpret B as a chessboard and rooks are placed on the cells in S, they do not attack each other). If B is a board, and if $k \in \mathbb{N}$, then the *rook number* $r_k(B)$ is the number of k-element rook placements of B. For instance, if B is the board on Figure 2 left, then its rook numbers are $r_0(B) = 1$ and $r_1(B) = |B| = 6$ and $r_2(B) = 8$ and $r_k(B) = 0$ for all $k > 2$.

Two boards B and C are said to be *rook-equivalent* if they share the same rook numbers (i.e., if $r_k(B) = r_k(C)$ for all $k \in \mathbb{N}$).

If w is a word in $\mathcal{M}$, then the *Ferrers board* B_w is a special board defined as follows: It is a contiguous set of cells, whose bottom and right boundaries are straight lines, whereas the rest of its boundary is a jagged path that (when walked from southwest to northeast) takes a north-step for each D in w and an east-step for each U in w (reading the word w from left to right). For instance, if $w = UDDUDUUDUD$, then B_w is shown on Figure 2 right (where the D and U labels are signaling the correspondence between the letters of w and the steps of the jagged boundary).[1]

Now a classical result of Navon (originally [11, §2], but see [3, Theorem 20] or [10, Theorem 6.11 for $h = 1$] or [15, Theorem 3.2] for a modern treatment) says the following:

Theorem 4.1. Let $w \in \mathcal{M}$ be any word that contains n many D's and m many U's. Then,

$$\phi(w) = \sum_{k=0}^{\min\{m,n\}} r_k(B_w)\, U^{m-k} D^{n-k} \qquad \text{in } \mathcal{W}.$$

As a consequence, we obtain the following:

Theorem 4.2. Let u and v be two words in $\mathcal{M}$ that have the same number of D's and the same number of U's. Then, the following are equivalent: the statements $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_3'$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ from Theorem 2.2, and the additional statement

- $\mathcal{R}_1$: The boards B_u and B_v are rook-equivalent.

The implication $\mathcal{R}_1 \Longrightarrow \mathcal{S}_1$ in Theorem 4.2 has been implicitly observed in [3, bottom of p. 40]. Note that this implication really requires the assumption about equal numbers of D's and of U's in Theorem 4.2, since (e.g.) the Ferrers boards B_{DUUDU} and B_{DDUU} are rook-equivalent without $\phi(DUUDU)$ equalling $\phi(DDUU)$. For an even starker example, if $w \in \mathcal{M}$ is any word, then the Ferrers boards B_w and $B_{\omega(w)}$ are rook-equivalent (being each other's reflection across a diagonal), but $\phi(w)$ is usually not $\phi(\omega(w))$. We note that this reasoning leads to a new proof of Theorem 2.3.

Theorem 4.2 connects our study of the kernel of ϕ to a classical question, namely: When are two Ferrers boards rook-equivalent? A classical result of Foata and Schützenberger ([6, Theorem 6] or [3, Theorem 7]) shows that each Ferrers board is rook-equivalent to a unique "increasing Ferrers board". These "increasing Ferrers boards" are somewhat similar to our up-normal words, but not quite in bijection, since (as we said) the rook equivalence of B_u and B_v implies $\phi(u) = \phi(v)$ only when we know that u and v have the same number of U's and the same number of D's.

[1] Note that the U at the beginning of w, and the D at the end, do not affect the Ferrers board.

Interestingly, Foata and Schützenberger have their own kind of moves that they use to normalize a Ferrers board modulo rook equivalence: the "(k, k')-transforms" (see [6, Definition 8 bis on page 9]). These appear to be close relatives of our balanced flips.

A recent preprint by Cotardo, Gruica and Ravagnani [4] proves another set of equivalent criteria for the rook-equivalence of two Ferrers boards [4, Corollary 3.2]. It lists six equivalent conditions, one of which (condition 6) is rook-equivalence, whereas another (condition 1) is equivalent to our statement $\mathcal{S}_3$ (albeit this equivalence takes some work to prove) provided that u and v have the same number of U's and the same number of D's, which one may assume (cf. Remark 4.4). The other four conditions are not found in our lists so far. In the following, we state two of these four conditions (4 and 5), as they are rather surprising and reveal an unexpected connection to the theory of finite fields. (Arguably, at least one of them has been foreseen, to some extent, in Haglund's [9].)

First, we introduce the necessary notations. For any finite field F, any nonnegative integers n and k, and any board $B \subseteq \{1, 2, \ldots, n\}^2$, we define $P_k(B/F)$ to be the number of $n \times n$-matrices $A \in F^{n \times n}$ of rank k such that all entries of A in cells outside of B are zero. (This is independent of n.) Now, we claim the following:

Theorem 4.3. Let u and v be two words in $\mathcal{M}$ that have the same number of D's and the same number of U's. Then, the following are equivalent: the statements $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}'_3$, $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ from Theorem 2.2, the statement $\mathcal{R}_1$ from Theorem 4.2, and the following two additional statements:

- $\mathcal{R}_2$: For any finite field F and any $k \in \mathbb{N}$, we have $P_k(B_u/F) = P_k(B_v/F)$.
- $\mathcal{R}_3$: For any finite field F, we have $P_1(B_u/F) = P_1(B_v/F)$.

Remark 4.4. Any word $w \in \mathcal{M}$ satisfies $B_w = B_{Uw} = B_{wD}$. That is, the Ferrers board B_w of a word $w \in \mathcal{M}$ does not change if we insert a U at the beginning of w or a D at the end of w. Thus, we can use Theorem 4.3 to tell whether two Ferrers boards are rook-equivalent; we just need to rewrite them as B_u and B_v, where u and v are two words with the same number of U's and the same number of D's. (This can be achieved by inserting some number of U's at the start of either word and some number of D's at the end of either word.)

5 Some extensions

The question of "when are products of generators equal" can be posed for any algebra given by generators and relations. Having answered it for the Weyl algebra $\mathcal{W}$ over a characteristic-0 field, we consider it natural to extend our analysis to other settings.

Some modifications allow us to extend our criteria to all multivariate Weyl algebras $\mathcal{W}_n$ (see [8, §10.1]) as well as to several of the so-called *down-up algebras* (see [8, §10.3]).

Much more difficult is the case of $\mathcal{W}$ in characteristic $p > 0$, as it creates additional equalities with their own, so far unformalized, combinatorics. This case is still open and appears quite worthy of further study.

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