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The superspace coinvariant ring

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Abstract. Let $\mathbb{F}$ be a field of characteristic 0 and let Ω_n be the ring of regular differential forms on $\mathbb{F}^n$. The natural action of $\mathfrak{S}_n$ on $\mathbb{F}^n$ induces an action on Ω_n. The *superspace coinvariant ring* is the quotient SR_n of Ω_n by the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. We calculate the bigraded character of SR_n, describe a vector space basis of SR_n, and give an 'operator theorem' which characterizes the associated inverse system. Our results prove conjectures of Bergeron, Colmenarejo, Li, Machacek, Reiner, Sagan, Sulzgruber, Swanson, Wallach, and Zabrocki.

Keywords: superspace, coinvariant ring, hyperplane arrangement

1 Introduction

This extended abstract outlines the results in three papers [2, 11, 14] on the superspace coinvariant ring SR_n. We review classical $\mathfrak{S}_n$-coinvariant theory before introducing the object of study.

Let $\mathbb{F}$ be a field of characteristic 0, let $\mathbf{x}_n = (x_1, \ldots, x_n)$ be a list of n variables, and let $\mathbb{F}[\mathbf{x}_n] := \mathbb{F}[x_1, \ldots, x_n]$ be the rank n polynomial ring. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{F}[\mathbf{x}_n]$ by subscript permutation. We write

$$I_n := (\mathbb{F}[\mathbf{x}_n]_+^{\mathfrak{S}_n}) \tag{1.1}$$

for the ideal generated by $\mathfrak{S}_n$-invariant polynomials with vanishing constant term and

$$R_n := \mathbb{F}[\mathbf{x}_n]/I_n \tag{1.2}$$

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for the corresponding quotient ring. The quotient R_n is the *coinvariant ring* associated to the symmetric group; it is a graded $\mathfrak{S}_n$-module.

The ring R_n is among the most important representations in algebraic combinatorics. E. Artin proved [3] that R_n has vector space basis given by the 'substaircase' monomials $\{x_1^{a_1} \cdots x_n^{a_n} \,:\, a_i < i\}$. In particular, the Hilbert series of R_n is given by[1]

$$\mathrm{Hilb}(R_n; q) = [n]!_q. \tag{1.3}$$

Chevalley proved [6] that there is an isomorphism of ungraded $\mathfrak{S}_n$-modules

$$R_n \cong_{\mathfrak{S}_n} \mathbb{F}[\mathfrak{S}_n] \tag{1.4}$$

so that R_n is a graded refinement of the regular representation of $\mathfrak{S}_n$. Lusztig and Stanley [16] gave the following combinatorial formula for the graded character of R_n:

$$\mathrm{grFrob}(R_n; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot s_{\lambda(T)}. \tag{1.5}$$

Here $\mathrm{SYT}(n)$ is the family of standard Young tableaux with n boxes, $\mathrm{maj}(T)$ is the major index of a tableau T, and $\lambda(T) \vdash n$ is the shape of T. Borel proved [5] that R_n presents the cohomology ring $H^*(G/B)$ of the type A_{n-1} complete flag variety.

Now let $\mathbf{x}_n = (x_1, \ldots, x_n), \mathbf{y}_n = (y_1, \ldots, y_n)$ be two lists of n variables. Write $\mathbb{F}[\mathbf{x}_n, \mathbf{y}_n]$ for the rank $2n$ polynomial ring with its 'diagonal' action

$$w \cdot x_i := x_{w(i)}, \quad w \cdot y_i := y_{w(i)} \tag{1.6}$$

of $\mathfrak{S}_n$. Let $DI_n \subseteq \mathbb{F}[\mathbf{x}_n, \mathbf{y}_n]$ be the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. Garsia and Haiman defined [10] the *diagonal coinvariant ring*

$$DR_n := \mathbb{F}[\mathbf{x}_n, \mathbf{y}_n]/DI_n. \tag{1.7}$$

Haiman used the algebraic geometry of Hilbert schemes to prove [9] that the ungraded $\mathfrak{S}_n$-action on DR_n is isomorphic to its permutation action on size n parking functions, up to a sign twist. Haiman also proved [9] that the bigraded Frobenius image of DR_n is given by

$$\mathrm{grFrob}(DR_n; q, t) = \nabla e_n \tag{1.8}$$

where ∇ is the Bergeron-Garsia *nabla operator* on symmetric functions.

The rank n *superspace ring* is the bigraded algebra

$$\Omega_n := \mathbb{F}[x_1, \ldots, x_n] \otimes_{\mathbb{F}} \wedge\{\theta_1, \ldots, \theta_n\} \tag{1.9}$$

[1]We use the q-notation $[n]_q := \frac{1-q^n}{1-q}$, $[n]!_q := [n]_q[n-1]_q \cdots [1]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}$.

of regular differential forms on $\mathbb{F}^n$.[2] The defining action of $\mathfrak{S}_n$ on $\mathbb{F}^n$ induces a diagonal action

$$w \cdot x_i = x_{w(i)}, \quad w \cdot \theta_i = \theta_{w(i)} \tag{1.10}$$

of $\mathfrak{S}_n$ on Ω_n. Our object of study is as follows.

Definition 1. *The* superspace coinvariant ideal *is the two-sided ideal $SI_n \subseteq \Omega_n$ generated by the space $(\Omega_n)_+^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariants with vanishing constant term. The* superspace coinvariant ring *is the quotient*

$$SR_n := \Omega_n / SI_n. \tag{1.11}$$

The quotient SR_n is a bigraded $\mathfrak{S}_n$-module with respect to x-degree and θ-degree. It was introduced by the Fields Institute Combinatorics Group in 2018.[3] Write $\mathcal{OP}_n$ for the family of ordered set partitions of $[n] := \{1, \ldots . n\}$. For example, one has

$$\mathcal{OP}_3 = \left\{ \begin{array}{cccccc} (1 \mid 2 \mid 3), & (2 \mid 1 \mid 3), & (1 \mid 3 \mid 2), & (2 \mid 3 \mid 1), & (3 \mid 1 \mid 2), & (3 \mid 1 \mid 2), \\ (1\,2 \mid 3), & (1\,3 \mid 2), & (2\,3 \mid 1), & (1 \mid 2\,3), & (2 \mid 1\,3), & (3 \mid 1\,2), \\ (1\,2\,3) \end{array} \right\}.$$

The Fields Group predicted that the structure SR_n is governed by the combinatorics of $\mathcal{OP}_n$.

Conjecture 1. (Fields Conjectures [19]) *The superspace coinvariant ring SR_n enjoys the following properties.*

1. *The vector space dimension of SR_n is $\#\mathcal{OP}_n$.*

2. *The bigraded Hilbert series of SR_n is $\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n,k)$ where q tracks x-degree, z tracks θ-degree, and $\mathrm{Stir}_q(n,k)$ is a q-Stirling number.*

3. *As ungraded $\mathfrak{S}_n$-modules, one has $SR_n \cong_{\mathfrak{S}_n} \mathbb{F}[\mathcal{OP}_n] \otimes \mathrm{sign}$ where* sign *is the 1-dimensional sign representation of $\mathfrak{S}_n$.*

4. *The bigraded Frobenius image of SR_n is $\mathrm{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot C_{n,k}(\mathbf{x}; q)$ where $C_{n,k}(\mathbf{x}; q)$ is a symmetric function whose definition is given in Section 5.*

Shortly after Conjecture 1 appeared, other conjectures about SR_n were proposed.

- Sagan and Swanson [15] gave a conjectural monomial basis of SR_n extending Artin's substaircase monomial basis of R_n.

- Swanson and Wallach [18] made an 'operator conjecture' on the inverse system $SI_n^{\perp}$ associated to SR_n.

[2] The 'super' refers to supersymmetry in physics, where the x_i correspond to states of bosons and the θ_i correspond to states of fermions.

[3] N. Bergeron, L. Colmenarejo, S. X. Li, J. Machacek, R. Sulzgruber, and M. Zabrocki

- Reiner [12] made a conjecture on the restriction of SR_n from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ mirroring the usual combinatorial recursion relating ordered set partitions of $[n]$ to ordered set partitions of $[n-1]$.

The inscrutable Gröbner theory of the quotient ring SR_n made these conjectures resistant to the 'straightening' techniques often used to analyze quotient rings in algebraic combinatorics. Finding an elementary proof of any of these conjectures is an open problem. This extended abstract tells the story of how these conjectures were proven.

2 Background

2.1 Symmetric Functions and Representation Theory

We write $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ for the graded ring of symmetric functions in an infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{F}(q, t, z)$. Bases of the degree n component Λ_n are indexed by partitions $\lambda \vdash n$. We will use the elementary basis $\{e_\lambda\}$ and the Schur basis $\{s_\lambda\}$ of Λ_n. Let $\langle -, - \rangle$ be the Hall inner product on Λ with respect to which the Schur basis is orthonormal.

Irreducible representations of the symmetric group $\mathfrak{S}_n$ are in one-to-one correspondence with partitions of n. Write V^λ for the irreducible $\mathfrak{S}_n$-module corresponding to $\lambda \vdash n$. If V is any finite-dimensional $\mathfrak{S}_n$-module, there are unique multiplicities $c_\lambda \geq 0$ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda V^\lambda$. The *Frobenius image* of V is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda \cdot s_\lambda \tag{2.1}$$

obtained by replacing each irreducible with its corresponding Schur function. More generally, let $V = \bigoplus_{i \geq 0} V_i$ be a graded representation of $\mathfrak{S}_n$ with each V_i finite-dimensional. The *graded Frobenius image* of V is the symmetric function

$$\mathrm{grFrob}(V; q) := \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot q^i. \tag{2.2}$$

Still more generally, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$-module, we write

$$\mathrm{grFrob}(V; q, z) := \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i z^j \tag{2.3}$$

for the *bigraded Frobenius image* of V.

The *Hilbert series* of a graded vector space $V = \bigoplus_{i \geq 0} V_i$ is given by

$$\mathrm{Hilb}(V; q) := \sum_{i \geq 0} \dim V_i \cdot q^i. \tag{2.4}$$

Similarly, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded vector space the *(bigraded) Hilbert series* of V is

$$\mathrm{Hilb}(V; q, z) := \sum_{i,j \geq 0} \dim(V_{i,j}) \cdot q^i z^j. \tag{2.5}$$

2.2 Supercommutative Algebra

For $1 \leq i \leq n$, the partial derivative operator $\partial/\partial x_i$ acts on the first tensor factor of the superspace ring $\Omega_n = \mathbb{F}[x_1, \dots, x_n] \otimes \wedge\{\theta_1, \dots, \theta_n\}$. The *contraction operator* (or *fermionic partial derivative*) $\partial/\partial\theta_i$ acts on the second tensor factor of Ω_n as follows. Consider distinct indices $1 \leq j_1, \dots, j_r \leq n$ so that $\theta_{j_1} \cdots \theta_{j_r}$ is a nonzero exterior monomial. One has

$$\partial/\partial\theta_i : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{s-1}\theta_{j_1} \cdots \theta_{j_{s-1}}\theta_{j_{s+1}} \cdots \theta_{j_r} & \text{if } j_s = i, \\ 0 & \text{if } j_1, \dots, j_r \neq i. \end{cases} \tag{2.6}$$

The operators $\partial/\partial x_i$ and $\partial/\partial\theta_i$ satisfy the same relations as Ω_n, viz.

$$(\partial/\partial x_i)(\partial/\partial x_j) = (\partial/\partial x_j)(\partial/\partial x_i) \quad (\partial/\partial x_i)(\partial/\partial\theta_j) = (\partial/\partial\theta_j)(\partial/\partial x_i)$$

$$(\partial/\partial\theta_i)(\partial/\partial\theta_j) = -(\partial/\partial\theta_j)(\partial/\partial\theta_i)$$

for all $1 \leq i, j \leq n$. For any superspace element $f = f(x_1, \dots, x_n, \theta_1, \dots, \theta_n)$, we thus have a well-defined operator $\partial f : \Omega_n \to \Omega_n$ given by

$$\partial f := f(\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial\theta_1, \dots, \partial/\partial\theta_n).$$

Let $I \subseteq \Omega_n$ be a bihomogeneous ideal. The *inverse system* (or *harmonic space*) is the subspace $I^\perp \subseteq \Omega_n$ given by

$$I^\perp := \{g \in \Omega_n \,:\, (\partial f)(g) = 0 \text{ for all } f \in I\}. \tag{2.7}$$

One can show that $I^\perp$ is a bigraded subspace of Ω_n with the same Hilbert series as Ω_n/I. If $G \subseteq GL_n(\mathbb{F})$ is a finite matrix group, then G acts on Ω_n by linear substitutions. If $I \subseteq \Omega_n$ is stable under the action of G, one can show that $I^\perp$ is also stable under the action of G and that one has an isomorphism $\Omega_n/I \cong_G I^\perp$ of bigraded G-modules.

3 Operator Theorem and Basis Transfer

There is a beautiful description of the inverse system $I_n^\perp$ attached to the classical coinvariant ideal $I_n \subseteq \mathbb{F}[\mathbf{x}_n]$. Write $\delta_n \in \mathbb{F}[\mathbf{x}_n]$ for the Vandermonde determinant

$$\delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{3.1}$$

Steinberg proved [17] that $I_n^\perp$ is the smallest linear subspace of $\mathbb{F}[\mathbf{x}_n]$ containing δ_n which is closed under the partial derivative operators $\partial/\partial x_i$ for $i = 1, \dots, n$. Our first result is a similar characterization of the inverse system $SI_n^\perp \subseteq \Omega_n$ of the superspace coinvariant ideal. For $j \geq 1$, let $d_j : \Omega_n \to \Omega_n$ be the *higher Euler operator*

$$d_j(f) := \sum_{i=1}^{n} \frac{\partial^j f}{\partial x_i^j} \cdot \theta_i. \tag{3.2}$$

When $j = 1$, the operator $d_1 = d$ is the usual boundary operator in de Rham cohomology. The following result was conjectured by Swanson and Wallach [18].

Theorem 1. (Operator Theorem [14, R.–W.]) *The inverse system $SI_n^\perp \subseteq \Omega_n$ is the smallest linear subspace of Ω_n which ...*

- *contains the Vandermonde determinant δ_n,*
- *is closed under the partial derivative operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$, and*
- *is closed under the higher Euler operator d_j for all $j \geq 1$.*

Theorem 1 is similar in form to a result of Haiman's characterizing the inverse system $DI_n^\perp \subseteq \mathbb{F}[\mathbf{x}_n, \mathbf{y}_n]$ in the case of two commuting sets of variables. Haiman proved [9] that $DI_n^\perp$ is the smallest linear subspace of $\mathbb{F}[\mathbf{x}_n, \mathbf{y}_n]$ containing δ_n (in the x-variables) which is closed under the derivatives $\partial/\partial x_i, \partial/\partial y_i$ together with the 'higher polarization operators'

$$\rho_j(f) := \sum_{i=1}^{n} \frac{\partial^j f}{\partial x_i^j} \cdot y_i \qquad (j \geq 1). \tag{3.3}$$

The proof of Theorem 1 in [14] is deeply intertwined with the proof of the following result. For a subset $J \subseteq [n]$, define a polynomial $f_J \in \mathbb{F}[\mathbf{x}_n]$ by

$$f_J := \prod_{j \in J} \left(x_j \times \prod_{j < k \leq n} (x_j - x_k) \right). \tag{3.4}$$

We have the colon ideal $(I_n : f_J) := \{g \in \mathbb{F}[\mathbf{x}_n] : g \cdot f_J \in I_n\} \subseteq \mathbb{F}[\mathbf{x}_n]$. The next result transfers vector space bases from the purely commutative quotients $\mathbb{F}[\mathbf{x}_n]/(I_n : f_J)$ to the superspace coinvariant ring SR_n. This trades a single problem in supercommutative algebra for 2^n problems in commutative algebra (where one has more tools).

Theorem 2. (Basis Transfer [14, R.–W.]) *For each $J \subseteq [n]$ let $\mathcal{B}_J \subseteq \mathbb{F}[\mathbf{x}_n]$ be a set of homogeneous polynomials. Suppose that $\mathcal{B}_J$ descends to a vector space basis of $\mathbb{F}[\mathbf{x}_n]/(I_n : f_J)$ for all $J \subseteq [n]$. Then the set*

$$\mathcal{B} := \bigsqcup_{J \subseteq [n]} \mathcal{B}_J \cdot \theta_J$$

descends to a vector space basis of SR_n, where the product $\theta_J := \prod_{j \in J} \theta_j$ is taken in increasing order.

Theorem 2 leads to an expression for the bigraded Hilbert series of SR_n. The *q-Stirling number* is defined by the recursion

$$\mathrm{Stir}_q(n,k) := \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k) \tag{3.5}$$

together with the initial condition $\mathrm{Stir}_q(0,k) := \delta_{k,0}$ (Kronecker delta). This is a q-analogue of the usual Stirling number $\mathrm{Stir}(n,k)$ counting k-block set partitions of $[n]$. The following result proves Parts 1 and 2 of the Fields Conjecture 1.

Theorem 3. ([14, R.-W.]) *The bigraded Hilbert series of SR_n is given by*

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n,k).$$

The proofs of Theorems 1, 2 and 3 make use of so-called *J-staircases*. Given a subset $J \subseteq [n]$, the J-staircase $\mathrm{st}(J) := (\mathrm{st}(J)_1, \ldots, \mathrm{st}(J)_n)$ is defined recursively by

$$\mathrm{st}(J)_1 := \begin{cases} 0 & 1 \in J, \\ 1 & 1 \notin J, \end{cases} \qquad \mathrm{st}(J)_{i+1} := \begin{cases} \mathrm{st}(J)_i & i+1 \in J, \\ \mathrm{st}(J)_i + 1 & i+1 \notin J. \end{cases} \tag{3.6}$$

When $n = 7$ and $J = \{3,5,6\}$ then $\mathrm{st}(J) = (1,2,\underline{2},3,\underline{3},\underline{3},4)$ where underlines indicate elements of J.

In order to prove the above theorems, one finds [14] explicit homogeneous polynomials $p_{J,1}, \ldots, p_{J,n} \in \mathbb{F}[\mathbf{x}_n]$ with $\deg(p_{J,i}) = \mathrm{st}(J)_i$ so that $(I_n : f_J) = (p_{J,1}, \ldots, p_{J,n})$. The $p_{J,i}$ appearing in [14] have a complicated definition involving partial derivatives of complete homogeneous symmetric polynomials in partial variable sets. The ideal equality $(I_n : f_J) = (p_{J,1}, \ldots, p_{J,n})$ implies that colon ideal $(I_n : f_J)$ is the unit ideal when $1 \in J$ and $p_{J,1}, \ldots, p_{J,n}$ is a regular sequence when $1 \notin J$. In particular, one has the Hilbert series

$$\mathrm{Hilb}(\mathbb{F}[\mathbf{x}_n]/(I_n : f_J); q) = \prod_{i=1}^{n} [\mathrm{st}(J)_i]_q. \tag{3.7}$$

Theorem 3 follows from Equation (3.7) and Theorem 2.

4 Sagan–Swanson basis

Theorem 2 and Equation (3.7) suggest a monomial basis of SR_n. For $J \subseteq [n]$, let

$$\mathcal{A}_n(J) := \{x_1^{a_1} \cdots x_n^{a_n} \,:\, a_i < \mathrm{st}(J)_i\} \tag{4.1}$$

be the set of monomials in the x-variables which fit under the J-staircase. These reduce to the usual Artin monomials $\{x_1^{a_1} \cdots x_n^{a_n} \,:\, a_i < i\}$ when $J = \varnothing$. The following result was conjectured by Sagan and Swanson [15].

Theorem 4. ([2, A.–C.–K.–M.–R.]) *For all $J \subseteq [n]$, the set $\mathcal{A}_n(J)$ descends to a vector space basis of $\mathbb{F}[\mathbf{x}_n]/(I_n : f_J)$. Furthermore, the set*

$$\mathcal{A}_n := \bigsqcup_{J \subseteq [n]} \mathcal{A}_n(J) \cdot \theta_J.$$

of superspace monomials descends to a vector space basis of the superspace coinvariant ring SR_n.

The second part of Theorem 4 follows from the first part and Theorem 2. For example, if $n = 3$ we have the following vector space basis of SR_3.

$J \subseteq [3]$	$\mathrm{st}(J)$	monomials in $\mathcal{A}_3$
$\varnothing$	$(1,2,3)$	$\{x_2x_3^2,\ x_2x_3,\ x_3^2,\ x_2,\ x_3,\ 1\}$
$\{2\}$	$(1,1,2)$	$\{x_3\theta_2,\ \theta_2\}$
$\{3\}$	$(1,2,2)$	$\{x_2x_3\theta_3,\ x_2\theta_3,\ x_3\theta_3,\ \theta_3\}$
$\{2,3\}$	$(1,1,1)$	$\{\theta_2\theta_3\}$

The subsets $J \subseteq [3]$ containing 1 have $\mathrm{st}(J)_1 = 0$ and so do not contribute any monomials to $\mathcal{A}_3$.

The proof of Theorem 4 in [2] is indirect. Indeed, the authors do not know an elementary proof that $\mathcal{A}_n$ spans SR_n using the relations of SI_n. Rather, the proof of Theorem 4 uses *Solomon–Terao (ST) algebras* associated to hyperplane arrangements as defined by Abe, Maeno, Murai, and Numata [1]. It is shown that the $\mathbb{F}[\mathbf{x}_n]/(I_n : f_J)$ are instances of these ST algebras for 'southwest arrangements'. The southwest arrangements group into free triples leading to short exact sequences among their ST algebras which inductively give the desired bases $\mathcal{A}_n(J)$ of the $\mathbb{F}[\mathbf{x}_n]/(I_n : f_J)$.

5 Module structure

We turn to the bigraded $\mathfrak{S}_n$-isomorphism type of SR_n. For $k \leq n$, let $C_{n,k}(\mathbf{x};q) \in \Lambda_n$ be the following symmetric function:

$$C_{n,k}(\mathbf{x};q) := \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T) + \binom{n-k}{2} - (n-k)\cdot \mathrm{des}(T)} \begin{bmatrix} \mathrm{des}(T) \\ n-k \end{bmatrix}_q \cdot s_{\lambda(T)}. \tag{5.1}$$

Here $\mathrm{des}(T)$ is the number of descents in a standard Young tableau T. In terms of Macdonald eigenoperators, one has $C_{n,k}(\mathbf{x};q) = \Delta'_{e_{k-1}} e_n \mid_{t \to 0}$ where $\Delta'_{e_{k-1}} : \Lambda \to \Lambda$ is a *delta operator*. There are combinatorial expressions for $C_{n,k}(\mathbf{x};q)$ involving statistics on ordered set partitions; we refer the reader to [8, 13] for the symmetric function avatars of $C_{n,k}(\mathbf{x};q)$.

The bigraded character $\mathrm{grFrob}(SR_n;q,z)$ is built out of the $C_{n,k}(\mathbf{x};q)$. The following result proves Parts 3 and 4 of the Fields Conjecture 1.

Theorem 5. ([11, M.–R.–W.]) *The bigraded Frobenius image of SR_n is given by*

$$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot C_{n,k}(\mathbf{x}; q).$$

As an example, when $n = 3$, Theorem 5 gives the formula

$$\mathrm{grFrob}(SR_3; q, z) = z^0 \cdot (s_3 + q \cdot s_{21} + q^2 \cdot s_{21} + q^3 \cdot s_{111}) \\ + z^1 \cdot (s_{21} + q \cdot (s_{21} + s_{111}) + q^2 \cdot s_{111}) + z^2 \cdot s_{111}.$$

Proof. (Sketch) For $\mu = (\mu_1, \ldots, \mu_r) \vdash n$, let $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_r}$ be the corresponding parabolic subgroup of $\mathfrak{S}_n$. Let $\varepsilon_\mu := \sum_{w \in \mathfrak{S}_\mu} \mathrm{sign}(w) \cdot w \in \mathbb{F}[\mathfrak{S}_n]$ be the group algebra element which antisymmetrizes over $\mathfrak{S}_\mu$. If V is an $\mathfrak{S}_n$-module, one has

$$\langle \mathrm{Frob}(V), e_\mu \rangle = \dim(\varepsilon_\mu \cdot V). \tag{5.2}$$

Since $\{e_\mu : \mu \vdash n\}$ is a basis of Λ_n, the symmetric function $\mathrm{Frob}(V) \in \Lambda_n$ is determined by the numbers $\dim(\varepsilon_\mu \cdot V)$ for $\mu \vdash n$. Equivalently, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ then $\mathrm{grFrob}(V; q, z)$ is determined by the polynomials

$$\langle \mathrm{grFrob}(V; q, z), e_\mu \rangle = \mathrm{Hilb}(\varepsilon_\mu \cdot V; q, z), \quad \mu \vdash n. \tag{5.3}$$

Taking $V = SR_n$ to be the superspace coinvariant ring, we prove that Equation (5.3) coincides with $\langle \sum_{k=1}^n z^{n-k} \cdot C_{n,k}(\mathbf{x}; q), e_\mu \rangle$ for all $\mu \vdash n$.

If f, g are polynomials in q, z, write $f \leq g$ if $g - f$ has nonnegative coefficients. The inequality

$$\mathrm{Hilb}(\varepsilon_\mu \cdot SR_n; q, z) \leq \left\langle \sum_{k=1}^{n} z^{n-k} \cdot C_{n,k}(\mathbf{x}; q), e_\mu \right\rangle \tag{5.4}$$

follows from the basis result of Theorem 4; applying ε_μ to the monomials of $\mathcal{A}_n$ and removing 'obvious' linear dependencies gives a spanning set of $\varepsilon_\mu \cdot SR_n$ which implies (5.4). For the other inequality

$$\mathrm{Hilb}(\varepsilon_\mu \cdot SR_n; q, z) \geq \left\langle \sum_{k=1}^{n} z^{n-k} \cdot C_{n,k}(\mathbf{x}; q), e_\mu \right\rangle \tag{5.5}$$

one switches to the inverse system $SI_n^\perp$ and proves that there are sufficiently many linearly independent elements of $\varepsilon_\mu \cdot SI_n^\perp$ in the appropriate bidegrees. □

Theorem 5 gives the following combinatorial interpretation of the ungraded $\mathfrak{S}_n$-structure of SR_n. Write $\mathcal{OP}_{n,k}$ for the family of ordered set partitions of $[n]$ with k blocks, equipped with its permutation action of $\mathfrak{S}_n$.

Corollary 1. (M.–R.–W. [11]) *We have an isomorphism of ungraded $\mathfrak{S}_n$-modules*

$$SR_n \cong_{\mathfrak{S}_n} \mathbb{F}[\mathcal{OP}_n] \otimes \text{sign}.$$

In fact, for any $k \geq 1$, we have an isomorphism of ungraded $\mathfrak{S}_n$-modules

$$\theta\text{-degree } n-k \text{ part of } SR_n \cong_{\mathfrak{S}_n} \mathbb{F}[\mathcal{OP}_{n,k}] \otimes \text{sign}.$$

The number $\#\mathcal{OP}_{n,k}$ of k-block ordered set partitions of $[n]$ satisfies the recursion

$$\#\mathcal{OP}_{n,k} = k \cdot (\#\mathcal{OP}_{n-1,k-1} + \#\mathcal{OP}_{n-1,k})$$

coming from erasing the largest element n (together with its block, if $\{n\}$ is a singleton). Reiner conjectured [12] a refinement of this recursion in SR_n. In the following result, we write $(SR_n)_d$ for the θ-degree d part of SR_n. Observe that $(SR_n)_d$ is a singly-graded $\mathfrak{S}_n$-module under x-degree.

Corollary 2. (Reiner's Conjecture [11, M.–R.–W.]) *The restriction* $\text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(SR_n)_{n-k}$ *satisfies the following recursion:*

$$\text{grFrob}(\text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(SR_n)_{n-k};q) = \\ [k]_q \cdot \left(\text{grFrob}((SR_{n-1})_{n-k};q) + \text{grFrob}((SR_{n-1})_{n-k-1};q)\right).$$

6 Future Directions

A natural way to extend the above results on SR_n is to consider a larger family of reflection groups. For clarity, we focus on the case where the ground field $\mathbb{F} = \mathbb{R}$ is the real numbers.

Let W act by reflections on $V = \mathbb{R}^n$. Let Ω be the algebra of regular differential forms on V, bigraded by polynomial and exterior degree. The action of W on V induces a bigraded action of W on Ω. Let $SI_W \subseteq \Omega$ be the ideal generated by W-invariant differential forms with vanishing constant term. The *W-superspace coinvariant ring* is the quotient

$$SR_W := \Omega / SI_W. \tag{6.1}$$

Then SR_W is a bigraded W-module with respect to polynomial and exterior degree.

We aim to give a combinatorial model for SR_W; here is a conjecture in this direction. Recall that the *Coxeter complex* of W is the subdivision of $V = \mathbb{R}^n$ formed by the reflecting hyperplanes of W. We write $\mathcal{OP}_W$ for the Coxeter complex and $\mathcal{OP}_{W,k} \subseteq \mathcal{OP}_W$ for its *codimension* k faces. The set $\mathcal{OP}_{W,k}$ carries a natural permutation action of W. We conjecture that SR_W is 'bounded from below' by $\mathcal{OP}_W$ as follows.

Conjecture 2. *Let W be a real reflection group. For any $k \geq 0$, there exists a W-equivariant surjection*

$$\varphi : (SR_W)_k \twoheadrightarrow \mathbb{R}[\mathcal{OP}_{W,k}] \otimes \det$$

where $(SR_W)_k$ is the exterior degree k part of SR_W and det *is the linear determinant character.*

Corollary 1 implies that Conjecture 2 is true, and that φ is an isomorphism, in type A. Bhattacharya [4] calculated the bigraded Hilbert series of SR_W in types BC and, in particular, proved that $\dim(SR_W)_k = \#\mathcal{OP}_{W,k}$ in this case. In type F_4, the characters of $(SR_W)_k$ and $\mathbb{R}[\mathcal{OP}_{W,k}] \otimes \det$ disagree only when $k = 2, 3$. The group F_4 has 25 irreducible characters; we order them $\chi_1, \ldots, \chi_{25}$ as in sage. If $\chi = a_1\chi_1 + \cdots + a_{25}\chi_{25}$ is any character, we have a sequence $(a_1, \ldots, a_{25})$ of multiplicities. The following table displays these multiplicities for $(SR_W)_k$ and $\mathbb{R}[\mathcal{OP}_{W,k}] \otimes \det$.

k	characters of $(SR_{W(F_4)})_k$ (top) and $\mathbb{R}[\mathcal{OP}_{W(F_4),k}] \otimes \det$ (bottom)
2	$(0,6,1,1,6,0,6,0,0,5,5,12,\mathbf{5},6,8,4,16,4,16,21,10,10,3,14,18)$ $(0,6,1,1,6,0,6,0,0,5,5,12,\mathbf{4},6,8,4,16,4,16,21,10,10,3,14,18)$
3	$(0,4,0,0,2,0,2,0,0,1,1,\ 4,\mathbf{1},0,2,0,4,0,4,6,1,1,0,2,2)$ $(0,4,0,0,2,0,2,0,0,1,1,\ 4,\mathbf{0},0,2,0,4,0,4,6,1,1,0,2,2)$

The eagle-eyed reader will notice a single discrepancy in both cases: the algebraic character $(SR_W)_k$ contains one more copy of the 'unlucky' degree 4 character χ_{13}. This gives some evidence that, although the map φ in Conjecture 2 cannot always be an isomorphism, there is an uncanny resemblance between its domain and target.

In the context of two commuting sets of variables, Conjecture 2 is reminiscent of a famous result of Gordon [7]. Let W be a Weyl group with root lattice Q and Coxeter number h. Then W acts on the 'finite torus' $Q/(h+1)Q$ as well as the coordinate ring ring $\mathbb{R}[V \oplus V^*]$ of $V \oplus V^*$. The *W-diagonal coinvariant ring* is $DR_W := \mathbb{R}[V \oplus V^*]/DI_W$ where DI_W is generated by W-invariants with vanishing constant term. Using Cherednik algebras, Gordon proved [7] that a W-epimorphism $DR_W \twoheadrightarrow \mathbb{R}[Q/(h+1)Q] \otimes \det$ exists. It would be interesting to see if Gordon's ideas could shed light on Conjecture 2.

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References

[1] T. Abe, T. Maeno, S. Murai, and Y. Numata. “Solomon-Terao algebra of hyperplane arrangements”. *J. Math. Soc. Japan* **71**.4 (2019), pp. 1027–1047. DOI.

[2] R. Angarone, P. Commins, T. Karn, S. Murai, and B. Rhoades. “Superspace coinvariants and hyperplane arrangements”. *Adv. Math.* **467** (2025), Paper No. 110185, 36. DOI.

[3] E. Artin. “Galois Theory”. 1944.

[4] S. Bhattacharya. “The superspace coinvariant ring of type B”. 2025. arXiv:2505.24122.

[5] A. Borel. “Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts”. *Ann. of Math. (2)* **57** (1953), pp. 115–207. DOI.

[6] C. Chevalley. “Invariants of finite groups generated by reflections”. *Amer. J. Math.* **77** (1955), pp. 778–782. DOI.

[7] I. Gordon. “On the quotient ring by diagonal invariants”. *Invent. Math.* **153**.3 (2003), pp. 503–518. DOI.

[8] J. Haglund, J. B. Remmel, and A. T. Wilson. “The delta conjecture”. *Trans. Amer. Math. Soc.* **370**.6 (2018), pp. 4029–4057. DOI.

[9] M. Haiman. “Vanishing theorems and character formulas for the Hilbert scheme of points in the plane”. *Invent. Math.* **149**.2 (2002), pp. 371–407. DOI.

[10] M. D. Haiman. “Conjectures on the quotient ring by diagonal invariants”. *J. Algebraic Combin.* **3**.1 (1994), pp. 17–76. DOI.

[11] S. Murai, B. Rhoades, and A. Wilson. “A proof of the Fields Conjectures”. 2025. arXiv: 2505.24027.

[12] V. Reiner. Personal Communication, 2022.

[13] B. Rhoades. “Ordered set partition statistics and the delta conjecture”. *J. Combin. Theory Ser. A* **154** (2018), pp. 172–217. DOI.

[14] B. Rhoades and A. T. Wilson. “The Hilbert series of the superspace coinvariant ring”. Vol. 12. 2024, Paper No. e16, 35. DOI.

[15] B. E. Sagan and J. P. Swanson. “q-Stirling numbers in type B”. *European J. Combin.* **118** (2024), Paper No. 103899, 35. DOI.

[16] R. P. Stanley. “Invariants of finite groups and their applications to combinatorics”. *Bull. Amer. Math. Soc. (N.S.)* **1**.3 (1979), pp. 475–511. DOI.

[17] R. Steinberg. “Invariants of finite reflection groups”. *Canadian J. Math.* **12** (1960), pp. 616–618. DOI.

[18] J. P. Swanson and N. R. Wallach. “Harmonic differential forms for pseudo-reflection groups II. Bi-degree bounds”. *Comb. Theory* **3**.3 (2023), Paper No. 17, 43. DOI.

[19] M. Zabrocki. “A module for the Delta conjecture”. 2019. arXiv:1902.08966.