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The Orlik–Solomon algebra of a locally geometric poset

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Abstract. Motivated by the intersection data of hyperplane arrangements and their generalizations, we describe a matroidal framework for locally geometric posets. Using this framework, we construct an Orlik–Solomon algebra associated to a locally geometric poset. This construction generalizes that of geometric semilattices, where in the realizable case the Orlik–Solomon algebra arises as the cohomology ring of a hyperplane arrangement complement. In our context, the Orlik–Solomon algebra introduced here is the "combinatorial" contribution to the cohomology of a toric or abelian arrangement complement, which interacts with a "topological" contribution from the cohomology of the ambient space. Using Gröbner basis theory, we establish a no-broken-circuit basis analogous to that of the classical Orlik–Solomon algebra, which provides the tools necessary for a deletion-contraction formula, and deeper understanding of the characteristic polynomial.

Keywords: matroid, geometric lattice, hypersurface arrangement

1 Introduction

The *Orlik–Solomon algebra* is an important object at the intersection of combinatorics, algebra, and topology. By leveraging the combinatorial structure underlying a hyperplane arrangement, an intersection pattern encoded in a *geometric lattice* or *matroid*, Orlik and Solomon [9] observed that the cohomology ring of the complement to a hyperplane arrangement depends only on this combinatorial data. This led them to study the combinatorial and algebraic properties of an analogous algebra for *any* matroid or for any geometric semilattice, regardless of realizability by a hyperplane arrangement.

Our goal is to extend this idea of an Orlik–Solomon algebra from hyperplane arrangements to a broader context of *locally geometric posets* (Definition 1) and *matroid preschemes* (Definition 2), a combinatorial framework that generalizes matroids and encodes the intersection pattern of *abelian arrangements* ([3, Theorem 15.10]). Here, an abelian arrangement is an arrangement of certain subgroups in a connected abelian Lie group,

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including hyperplane arrangements and their toric and elliptic analogues. The cohomology of an abelian arrangement complement is computed using both combinatorial data (from its intersection pattern) and topological data (from the cohomology of the abelian Lie group). See [7, 2] for a description of the cohomology ring in the noncompact cases; the cohomology ring in the compact case is not known but can be computed from a spectral sequence (see eg. [4]). We define the Orlik–Solomon algebra (Definition 6) in a way that captures precisely the combinatorial input (see Remark 1) and makes sense for any locally geometric poset, regardless of realizability by an arrangement.

A key feature of the classical Orlik–Solomon algebra is that its defining ideal admits a combinatorial Gröbner basis [12, Thm. 2.8] corresponding to a well-known vector space basis: the *no-broken-circuit (nbc) basis*. We show (Theorem 2) that the defining ideal of our generalized Orlik–Solomon algebra admits a similarly combinatorial Gröbner basis, corresponding to an analogous nbc basis (Corollary 1). This then provides access to useful tools such as a deletion–contraction formula (Theorem 3) and the characteristic polynomial (Theorem 4).

2 Locally geometric posets

Let $\mathcal{P}$ be a finite partially ordered set, or poset. Given $x \in \mathcal{P}$ we write

$$\mathcal{P}_{\leq x} := \{y \in \mathcal{P} \colon y \leq x\} \quad \text{and} \quad \mathcal{P}_{\geq x} := \{y \in \mathcal{P} \colon y \geq x\}.$$

For a subset $T \subseteq \mathcal{P}$, let the **join**, $\bigvee T$, be the set of minimal upper bounds, and the **meet**, $\bigwedge T$, be the set of maximal lower bounds. That is,

$$\bigvee T := \min\{u \in \mathcal{P} \colon u \geq a,\ \forall a \in T\} \quad \text{and} \quad \bigwedge T := \max\{l \in \mathcal{P} \colon l \leq a,\ \forall a \in T\}.$$

When $T = \{x, y\}$ we write $x \vee y := \bigvee T$ and $x \wedge y := \bigwedge T$. We say that $\mathcal{P}$ is a **(meet-) semilattice** if for all $x, y \in \mathcal{P}$, $|x \wedge y| = 1$. If additionally $|x \vee y| = 1$ for all $x, y \in \mathcal{P}$, we say that $\mathcal{P}$ is a **lattice**.

A poset $\mathcal{P}$ is **bounded below** if it has a unique minimum element denoted $\hat{0}$. We say that $\mathcal{P}$ is **ranked** if there is an order-preserving map $\rho : \mathcal{P} \to \mathbb{Z}_{\geq 0}$, such that whenever y covers x, $\rho(y) = \rho(x) + 1$. A ranked poset is **pure** if all maximal elements have the same rank. When a ranked poset $\mathcal{P}$ is bounded below, we may assume that $\rho(\hat{0}) = 0$, and let the set of **atoms** of $\mathcal{P}$ be

$$\mathrm{at}(\mathcal{P}) := \{a \in \mathcal{P} : \rho(a) = 1\}.$$

We will abuse this notation and also denote $\mathrm{at}(x) := \mathrm{at}(\mathcal{P}_{\leq x})$ whenever $x \in \mathcal{P}$. When every element of $\mathcal{P}$ is contained in the join of some set of atoms, $\mathcal{P}$ is called **atomic**.

Definition 1. Let $\mathcal{P}$ be a ranked, atomic lattice with rank function ρ. Then we call $\mathcal{P}$ a **geometric lattice** if $\mathcal{P}$ is *semimodular*, i.e. for every $x, y \in \mathcal{P}$,

$$\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y).$$

A ranked, bounded-below poset $\mathcal{P}$ is called **locally geometric** if for all $x \in \mathcal{P}$, $\mathcal{P}_{\leq x}$ is a geometric lattice.

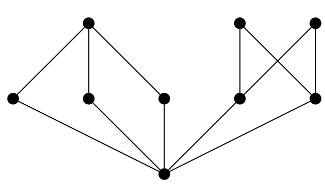
Example 1. For a finite set T, the Boolean lattice $\mathcal{B}(T)$, whose elements are the subsets of T ordered by inclusion, is a geometric lattice. In this case, join corresponds to union; meet corresponds to intersection; and rank corresponds to cardinality.

A **simplicial poset** is a bounded-below poset $\mathcal{P}$ such that for all $x \in \mathcal{P}$, the subposet $\mathcal{P}_{\leq x}$ is isomorphic to the boolean algebra $\mathcal{B}(\mathrm{at}(x))$. As such, a simplicial poset is a locally geometric poset.

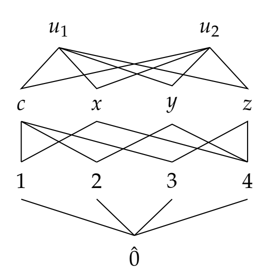
When $\mathcal{P}$ is a simplicial poset, the rank on $\mathcal{P}$ is determined by $|\mathrm{at}(x)|$, thus we will sometimes abbreviate $|x|$ to mean the rank of an element x in a simplicial poset. Furthermore, for $a, x \in \mathcal{P}$, if $a \leq x$ then there is a unique element $x \setminus a \in \mathcal{P}_{\leq x}$ such that $(x \setminus a) \vee a \ni x$ and $(x \setminus a) \wedge a = \hat{0}$. We call this element the complement of a in $\mathcal{P}_{\leq x}$, and it satisfies $|x \setminus a| = |x| - |a|$.

Example 2. A **geometric poset** is a locally geometric poset that satisfies the additional condition: whenever $x \in \mathcal{P}$, $Y \subseteq \mathrm{at}(\mathcal{P})$, $y \in \bigvee Y$, and $\rho(x) < \rho(y) = |Y|$, there exists an $a \in Y$ such that $a \not\leq x$ and $a \vee x \neq \emptyset$.

The posets depicted in Figures 1a and 1b are both locally geometric posets. The second one satisfies the global geometric condition, while the first one does not.



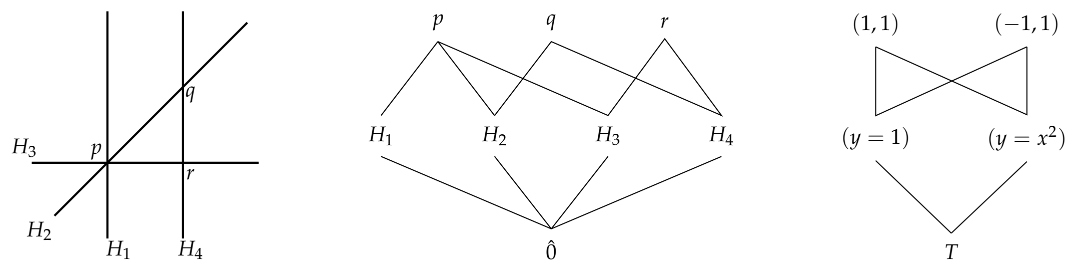
(a) A locally geometric poset



(b) A geometric poset

Figure 1: Examples of locally geometric posets, see Example 2

Example 3. A **geometric semilattice** is a geometric poset that is also a meet-semilattice; this is equivalent to the definition of geometric semilattice from [11]. Given an arrangement $\mathcal{A}$ of affine hyperplanes in a vector space, the collection of nonempty intersections of hyperplanes from $\mathcal{A}$ ordered by reverse inclusion has the structure of a geometric semilattice. An example of an arrangement of lines in the plane is depicted in Figure 2a and its intersection data in Figure 2b.



(a) An affine line arrangement (b) A geometric semilattice (c) An intersection poset

Figure 2: Arrangements and their intersection data, see Examples 3 and 4.

Example 4. More generally, any arrangement of subvarieties which *intersect like hyperplanes*, including toric and abelian arrangements as well as blowups of hyperplane arrangements, is encoded in a locally geometric poset ([5, Cor. 4.4.7.], [6, Prop. 2.3.6.]). In this context, the poset contains all connected components of intersections of subvarieties, ordered by reverse inclusion, and when intersections are not connected the poset is not a semilattice. For example, consider the subvarieties of a 2-dimensional torus T cut out by the equations $y = 1$ and $y = x^2$. These two subtori intersect at two points, namely $(1, 1)$ and $(-1, 1)$, and we depict the corresponding intersection poset in Figure 2c.

3 Matroid preschemes

While the definitions in this section and the next are primarily pulled from [3], we redirect emphasis for the sake of this work. In particular, it will be most useful for us to define matroid preschemes in terms of independence, as in [3, Thm. 6.2].

Definition 2. A **matroid prescheme** is a pair $M = (S, \mathcal{I})$, where S is a finite simplicial poset and $\mathcal{I} \subseteq S$ is a subset of **independent** elements satisfying:

(I1) $\mathcal{I}$ is nonempty.

(I2) if $x, y \in S$ with $x \leq y$ and $y \in \mathcal{I}$, then $x \in \mathcal{I}$.

(I3) if $x, y \in \mathcal{I}$, $|x| < |y|$, and $u \in x \vee y$, then there is some atom a such that $a \leq y$, $a \not\leq x$, and $(a \vee x)_{\leq u} \subseteq \mathcal{I}$.

(I4) if $x, y \in S$, $z \in \max(\mathcal{I} \cap S_{\leq x})$, and $z \leq y$, then $x \vee y \neq \emptyset$.

Roughly speaking, conditions (I1) and (I2) say that $\mathcal{I}$ is an *order ideal* of S; condition (I3) guarantees a *local* matroid structure on each Boolean lattice $S_{\leq u}$; and condition (I4) ensures reasonable gluing of these local matroids. There is a stronger (global) version of (I3) given in [3, Thm. 6.2] that one can adopt to define a *matroid scheme*.

Example 5. Whenever the simplicial poset is a lattice (resp. semilattice), a matroid prescheme $(S, \mathcal{I})$ is equivalent to a matroid (resp. semimatroid as in [1]). In any matroid prescheme, the restriction of $\mathcal{I}$ to a Boolean lattice $S_{\leq x}$ defines the independent sets of a matroid on the ground set $\mathsf{at}(x)$.

Example 6. Consider the simplicial poset S whose Hasse diagram is depicted in Figure 3. The pair $(S, \mathcal{I})$, where $\mathcal{I}$ is the collection of elements outlined in rectangles, forms a matroid prescheme.

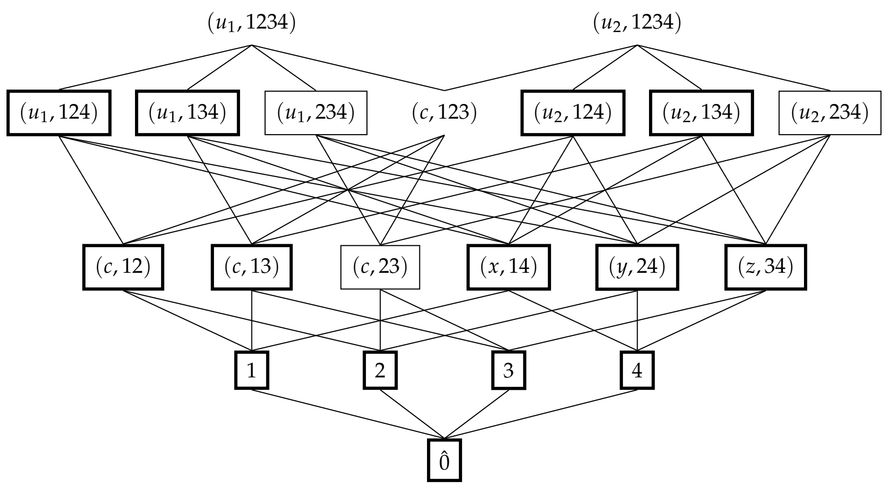


Figure 3: An example of a matroid scheme, with the independent elements in the simplicial poset outlined in rectangles (see Examples 6 to 8). The bolder rectangles mark those independent elements which are nbc (see later Example 12).

It is established in the proof of [3, Thm. 11.6] that to any locally geometric poset $\mathcal{P}$, one may construct an associated matroid prescheme $M(\mathcal{P})$ in the following way.

Definition 3. Let $\mathcal{P}$ be a locally geometric poset with rank function ρ. Then the **associated matroid prescheme** $M(\mathcal{P}) = (S, \mathcal{I})$ is obtained as follows. The simplicial poset is given by

$$S = S(\mathcal{P}) = \{(x, T)\colon T \subseteq \mathsf{at}(\mathcal{P}), x \in \vee T\}$$

viewed as a subposet of $\mathcal{P} \times 2^{\mathsf{at}(\mathcal{P})}$, and the independent elements are

$$\mathcal{I} = \mathcal{I}(\mathcal{P}) = \{(x, T) \in S\colon \rho(x) = |T|\}.$$

Example 7. Let $\mathcal{P}$ be the locally geometric poset from Figure 1b. Then the associated matroid prescheme $M(\mathcal{P})$ is precisely the matroid prescheme from Example 6 (see also Figure 3), where we abbreviate the atoms (i, i) by i.

4 Circuits and flats

Next, we extend common matroid notions, including circuits and flats, to our context as needed. This is used to then express an equivalence between (simple) matroid preschemes and locally geometric posets.

Definition 4. In a matroid prescheme $(S, \mathcal{I})$, any minimal element of $S \setminus \mathcal{I}$ is called a **circuit**. An element $u \in S$ such that there exists a unique circuit c with $c \leq u$ is called **unicircular**; we may write c-**unicircular** to distinguish this unique circuit.

Definition 5. In a matroid prescheme $M = (S, \mathcal{I})$, we call an element $f \in S$ a **flat** if, for all circuits c: if there exists an element $x \leq f$ such that c covers x, then $c \leq f$. The **poset of flats** is the subposet of S whose elements are the flats of M.

Example 8. In Figure 3, $(c, 123)$ is the only circuit. As a circuit, $(c, 123)$ is necessarily $(c, 123)$-unicircular. Additionally, $(u_1, 1234)$ and $(u_2, 1234)$ are $(c, 123)$-unicircular.

The flats are given by $\hat{0}$, the atoms 1, 2, 3, and 4, the rank-2 elements $(x, 14)$, $(y, 24)$, and $(z, 34)$, as well as all the three unicircular elements previously mentioned. Notice that the poset of flats of this matroid scheme is isomorphic to the geometric poset from which it arose, given in Figure 1b, via $(x, \mathrm{at}(x)) \mapsto x$.

In general, a locally geometric poset $\mathcal{P}$ is the poset of flats of its associated matroid prescheme $M(\mathcal{P})$. Matroid schemes arising this way are **simple**, that is, every atom is a flat and not a circuit. As such, and stated in the following theorem, we may consider locally geometric posets and simple matroid preschemes to be equivalent.

Theorem 1. *[3, Thm 11.6] A poset is locally geometric if and only if it is isomorphic to the poset of flats of a matroid prescheme. Furthermore, each locally geometric poset is the poset of flats of a unique simple matroid prescheme, up to isomorphism.*

5 The Orlik–Solomon algebra

Let $\mathcal{P}$ be a locally geometric poset, $(S, \mathcal{I})$ its associated matroid prescheme, and E the graded-commutative $\mathbb{Q}$-algebra generated by e_α for $\alpha \in \mathcal{I} \setminus \{\hat{0}\}$, where $\deg(e_\alpha) = |\alpha|$. Fix a linear order on the atoms of $\mathcal{I}$, and write $\mathrm{at}(\alpha)$ in increasing order accordingly. We may then define the function $\mathrm{sgn} : \mathcal{I} \times \mathcal{I} \to \{-1, 1\}$ which takes $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$ to the sign of the permutation needed to sort $(\mathrm{at}(\alpha), \mathrm{at}(\beta))$ into ascending order.

Definition 6. Let $J \subseteq E$ be the ideal generated by the following:

(i) $e_\alpha e_\beta$ when $\alpha \vee \beta \not\subseteq \mathcal{I}$ or $\alpha \wedge \beta \neq \{\hat{0}\}$;

(ii) $e_\alpha e_\beta - \mathrm{sgn}(\alpha, \beta) \sum_{\gamma \in \alpha \vee \beta} e_\gamma$ when $\alpha \vee \beta \subseteq \mathcal{I}$ and $\alpha \wedge \beta = \{\hat{0}\}$; and

(iii) $\sum_{a \in \mathrm{at}(\zeta)} \mathrm{sgn}(a, \omega \setminus a) e_{\omega \setminus a}$ when ω is a ζ-unicircular element.

We call $J = J(\mathcal{P})$ the Orlik–Solomon ideal of $\mathcal{P}$, and $A = A(\mathcal{P}) := E/J$ the Orlik–Solomon algebra of $\mathcal{P}$.

Example 9. In the case that S is a (semi)lattice, this is a different presentation for the Orlik–Solomon algebra of a geometric (semi)lattice or (semi)matroid [9]. Let us compare with an illustrative example, from Example 3 (see also Figure 2b). Since the meet and, when it exists, the join are always unique in a geometric semilattice, we abbreviate $\bigwedge T = \{t\}$ by $\bigwedge T = t$ and $\bigvee T = \{t\}$ by $\bigvee T = t$ in this example.

In our presentation, there are 9 generators and 46 relations. The generators of E correspond to the 9 elements of $\mathcal{I} \setminus \{0\}$: e_1, e_2, e_3, and e_4 in degree 1, and $e_{(p,12)}$, $e_{(p,13)}$, $e_{(p,23)}$, $e_{(q,24)}$, and $e_{(r,34)}$ in degree 2. In the ideal J, there are 30 type (i) generators, including $e_1 e_1$ because $1 \wedge 1 = 1 \neq \hat{0}$; $e_{(p,12)} e_{(q,24)}$ because $(p, 12) \wedge (q, 24) = 2 \neq \hat{0}$; and $e_1 e_{(p,23)}$ because $1 \vee (p, 23) = (p, 123) \notin \mathcal{I}$. There are 15 type (ii) generators, including $e_1 e_4$ because $1 \wedge 4 = \hat{0}$ and $1 \vee 4 = \emptyset \subseteq \mathcal{I}$; and $e_1 e_2 - e_{(p,12)}$ because $1 \wedge 2 = \hat{0}$ and $1 \vee 2 = (p, 12) \in \mathcal{I}$. Finally, there is only one generator of type (iii): $e_{(p,23)} - e_{(p,13)} + e_{(p,12)}$ corresponding to the unicircular element $(p, 123)$.

Alternatively, Orlik and Solomon would use 4 generators, namely a_1, a_2, a_3, and a_4, in degree 1, and 6 relations. Five of the relations arise from sets of atoms with no common upper bound: $a_1 a_2 a_4$, etc. One relation arises from the circuit: $a_2 a_3 - a_1 a_3 + a_1 a_2$.

The isomorphism between the two quotient algebras is given by $a_i \mapsto e_i$. Notice that the type (ii) relations are essential for having a well-defined inverse. For example, the inverse maps both $e_{(p,12)}$ and $e_1 e_2$ to $a_1 a_2$, which is desired because both are indexed by the set of atoms $\{1, 2\}$. In general, when the poset $\mathcal{P}$ is not a semilattice, the fact that joins need not be unique can obstruct the Orlik–Solomon algebra from being generated in degree 1.

Example 10. Now consider the matroid prescheme shown in Figure 3. In this case, E has 16 generators: e_1, e_2, e_3, and e_4 in degree 1; $e_{(c,12)}$, $e_{(c,13)}$, $e_{(c,23)}$, $e_{(x,14)}$, $e_{(y,24)}$, and $e_{(z,34)}$ in degree 2; and $e_{(u_1,124)}$, $e_{(u_1,134)}$, $e_{(u_1,234)}$, $e_{(u_2,124)}$, $e_{(u_2,134)}$, $e_{(u_2,234)}$ in degree 3. The ideal J is generated by over 100 elements; we will point out a few illustrative ones.

For example, the type (ii) generator

$$e_4 e_{(c,23)} - e_{(u_1,234)} - e_{(u_2,234)} \tag{5.1}$$

arises from the non-unique join $4 \vee (c, 23) = \{(u_1, 234), (u_2, 234)\}$, so that in the quotient we have the product $e_4 e_{(c,23)} = e_{(u_1,234)} + e_{(u_2,234)}$ (compare with the last example, where $e_1 e_2 = e_{(p,12)}$ in the quotient).

The two unicircular elements $(u_1, 1234)$ and $(u_2, 1234)$, which share the circuit $(c, 123)$ and set of atoms $\{1, 2, 3, 4\}$, give rise to the type (iii) generators

$$e_{(u_1,234)} - e_{(u_1,134)} + e_{(u_1,124)} \qquad \text{and} \qquad e_{(u_2,234)} - e_{(u_2,134)} + e_{(u_2,124)}. \tag{5.2}$$

Additionally, the circuit $(c, 123)$ is unicircular itself, and thus gives rise to its own type (iii) generator

$$e_{(c,23)} - e_{(c,13)} + e_{(c,12)}. \tag{5.3}$$

In this example, omitting the unicircular relations of (5.2) would not generate all of J, meaning that it is not enough to generate J using only circuits for type (iii) relations (as is the case when $\mathcal{P}$ is a geometric semilattice). In particular, from a computation we will see in Example 11, the sum of the two expressions in (5.2) would land in the ideal but not each individually.

Remark 1. Consider the case that the geometric poset $\mathcal{P}$ encodes the intersection data of an abelian arrangement in a connected abelian Lie group $X \cong G^n$, given by finitely many kernels of characters in $\mathrm{Hom}(X, G)$. Let B be the quotient of $A(\mathcal{P}) \otimes_{\mathbb{Q}} H^*(X, \mathbb{Q})$ by an ideal generated by $e_{(x,T)} \otimes \chi^*(z)$ where $\chi \in \mathrm{Hom}(X, G)$ is constant on x and $z \in H^{>0}(G)$. We relate the algebra B to the cohomology of the arrangement complement as follows.

When $G = \mathbb{C}^\times$, there is a filtration on the rational cohomology of the toric arrangement complement whose associated graded algebra is isomorphic to B (this reinterprets [10, Thm. 4.6]). When $G \cong G' \times \mathbb{R}^2$ for some even-dimensional abelian Lie group G', we can prove that B is isomorphic to the rational cohomology of the complement (see a presentation for cohomology in [2, Thm. 5.9]). When G is compact, then much less is known about cohomology of an arrangement complement, but at least when $G = S^1 \times S^1$, then equipping B with a differential yields a model for the rational cohomology ring (generalizing [4, Thm. 4.1]).

6 The no-broken-circuit basis

There is a well-known combinatorial basis of the classical Orlik–Solomon algebra, called the nbc basis. Here, we obtain an analogous basis using Gröbner basis theory (see eg. [8, Chapter II.4] for an approach to Gröbner bases in graded-commutative algebras).

To consider Gröbner bases in this context, we must clarify what ordering we will use on our algebra E. Fix a linear order on the atoms of $\mathcal{I}$, and consider a linear extension $\leq_*$ on the poset $\mathcal{I}$ in which $\alpha \leq_* \beta$ if either

1. $|\alpha| < |\beta|$, or
2. $|\alpha| = |\beta|$ and the minimal atom in the symmetric difference $\mathrm{at}(\alpha) \triangle \mathrm{at}(\beta)$ is in $\mathrm{at}(\alpha)$.

Then we may write $\mathcal{I} \setminus \hat{0}$ as a chain:

$$\alpha_1 <_* \cdots <_* \alpha_d,$$

and any monomial in E may be written as

$$e_\alpha^{\overline{m}} := e_{\alpha_1}^{m_1} \cdots e_{\alpha_d}^{m_d}$$

for $\overline{m} := (m_1, \ldots, m_d) \in \mathbb{N}^d$. We will also write $\overline{\alpha} := (|\alpha_1|, \ldots, |\alpha_d|) \in \mathbb{N}^d$, so that $\overline{m} \cdot \overline{\alpha}$ is the degree of the monomial $e_\alpha^{\overline{m}}$.

This induces a graded-lexicographic ordering $\preceq_*$ on E such that for monomials $e_\alpha^{\overline{m}}$ and $e_\alpha^{\overline{n}}$ in E, we have $e_\alpha^{\overline{m}} \prec_* e_\alpha^{\overline{n}}$ if either

1. $\overline{m} \cdot \overline{\alpha} < \overline{n} \cdot \overline{\alpha}$, or
2. $\overline{m} \cdot \overline{\alpha} = \overline{n} \cdot \overline{\alpha}$ and the leftmost nonzero entry of $\overline{n} - \overline{m}$ is positive.

Example 11. The largest monomial, or **leading term**, in (5.1) is $e_4 e_{(c,23)}$. In (5.2), the leading terms are $e_{(u_1,234)}$ and $e_{(u_2,234)}$, and in (5.3), the leading term is $e_{(c,23)}$.

In this case, multiplying (5.3) by e_4 and subtracting from (5.1) will cancel leading terms, so that the S-polynomial of (5.1) and (5.3) is

$$-e_{(u_1,234)} - e_{(u_2,234)} + e_4 e_{(c,13)} - e_4 e_{(c,12)}.$$

As the latter two terms are the leading terms of their own respective type (ii) relations, one can determine that this S-polynomial reduces to the sum of the two equations in (5.2), hence reduces to 0 in J.

More generally, we prove that any pair of generators of J as defined in Definition 6 are reducible by utilizing the combinatorics of the underlying matroid prescheme. This yields the following theorem.

Theorem 2. *Let $\mathcal{P}$ be a locally geometric poset. The type (i), (ii), and (iii) generators of the Orlik–Solomon ideal $J = J(\mathcal{P})$ as defined in Definition 6 form a Gröbner basis for J.*

Definition 7. Let $M = (S, \mathcal{I})$ be a matroid prescheme with a linear ordering on the atoms of S. Let ζ be a circuit in S with $a = \min(\mathrm{at}(\zeta))$. Then $\zeta \setminus a$ is called a **broken circuit**. An element $\alpha \in S$ is called a **no-broken-circuit element**, or *nbc*-element, if there exists no broken circuit in $S_{\leq \alpha}$. Let $nbc(M) \subseteq S$ be the collection of nbc elements.

Example 12. In Figure 3, the only broken circuit is $(c, 23) = (c, 123) \setminus 1$. As such, the *nbc* elements are exactly the elements not above $(c, 23)$, which are all boxed in bold.

Corollary 1. *Let $\mathcal{P}$ be a locally geometric poset with associated matroid prescheme $M(\mathcal{P}) = (S, \mathcal{I})$. Then the Orlik–Solomon algebra $A(\mathcal{P})$ has a vector space basis given by*

$$\{e_\alpha \colon \alpha \in nbc(M)\}.$$

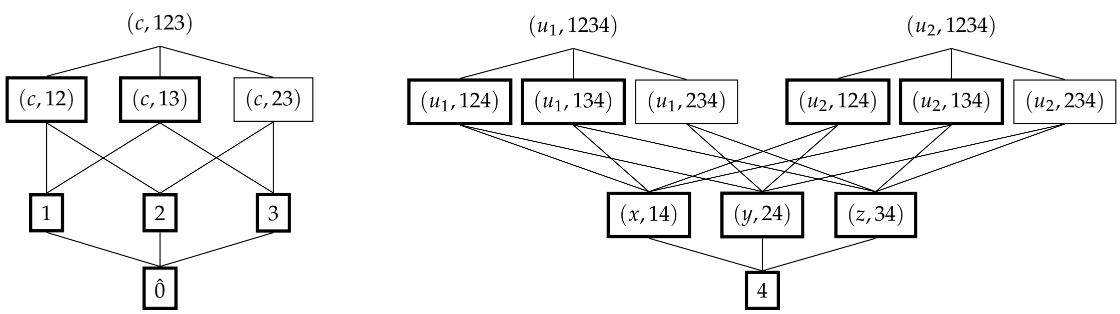
Proof sketch. The quotient algebra E/J has a vector space basis given by the monomials of E not divisible by leading terms of the Gröbner basis. Observe that the leading term of type (i) and (ii) generators imply that no monomials in the vector space basis can be a product of two or more generators of E. The key step is then to show that any $\alpha \in \mathcal{I}$ is *nbc* if and only if there exists no ζ-unicircular element ω such that $\alpha = \omega \setminus a$ where $a = \min_\prec(\mathrm{at}(\zeta))$, which index the leading terms of type (iii) generators. $\square$

7 Deletion and contraction

In the classical case, one leverages the relationship of the deletion and contraction in a matroid to establish properties of the Orlik–Solomon algebra inductively. We adapt these notions of deletion and contraction to matroid preschemes to do the same. As we are primarily concerned with matroid preschemes that arise from locally geometric posets, we assume throughout that our matroid preschemes are simple.

Definition 8. Let $M = (S, \mathcal{I})$ be a simple matroid prescheme, and let a be a distinguished atom of S. Then the **deletion** of S by a is the matroid prescheme $M' := (S_{\not\geq a}, \mathcal{I}_{\not\geq a})$. Furthermore, the **contraction** of S by a is the matroid prescheme $M'' := (S_{\geq a}, \mathcal{I}_{\geq a})$. We will call (M, M', M'') a **triple of matroid preschemes**.

Example 13. Using the matroid prescheme M from Figure 3 and distinguished atom 4, the matroid preschemes M' and M'' are shown in Figures 4a and 4b, respectively. In each, the independent elements are boxed, and the *nbc* elements are boxed in bold.



(a) M' (b) M''

Figure 4: The deletion and contraction of M from Figure 3 with respect to atom 4.

A key observation here is that the disjoint union of $\mathcal{I}_{\not\geq a}$ and $\mathcal{I}_{\geq a}$ is exactly $\mathcal{I}$. Similarly, as we made an appropriate choice of distinguished atom, the disjoint union of $nbc(M')$ and $nbc(M'')$ is exactly $nbc(M)$. In fact, this holds true in general, as is stated in the following lemma.

Lemma 1. *Let (M, M', M'') be a triple of matroid preschemes, where the distinguished atom is the largest in the chosen linear order on atoms. Then the set of nbc-elements of M is the disjoint union of nbc-elements of the deletion M' and nbc-elements of the contraction M'':*

$$nbc(M) = nbc(M') \sqcup nbc(M'')$$

Now given a locally geometric poset $\mathcal{P}$, we can consider deletions and contractions in its associated matroid prescheme $M = M(\mathcal{P})$. Given the triple of matroid preschemes, (M, M', M''), we denote the poset of flats of M' as $\mathcal{P}'$ and of M'' as $\mathcal{P}''$. We can then compare the corresponding Orlik–Solomon algebras. We will denote these as $A' := E(\mathcal{P}')/J(\mathcal{P}')$ and $A'' := E(\mathcal{P}'')/J(\mathcal{P}'')$. The following theorem uses our understanding of the Orlik–Solomon algebra as it relates to the *nbc* elements, and the correspondence between deletion-contraction and *nbc* elements.

Theorem 3. *Let $\mathcal{P}$ be a locally geometric poset with associated matroid prescheme $M = M(\mathcal{P})$ and triple (M, M', M''). Then A' is a subalgebra of A, and there exists a short exact sequence of vector spaces*

$$0 \to A' \to A \to A'' \to 0.$$

Proof sketch. Define $B := \operatorname{span}\{e_\alpha \colon \alpha \in nbc(M)\}$ as the $\mathbb{Q}$-vector space spanned by the *nbc* elements of M, and similarly B' and B'' for M' and M''. Then Lemma 1 gives us a short exact sequence of vector spaces

$$0 \to B' \to B \to B'' \to 0.$$

As A is isomorphic to B as a vector space by Corollary 1, we obtain the short exact sequence desired. Furthermore, the map $A' \to A$ is the map on quotients induced by the inclusion $E' \to E$, hence is an injective algebra homomorphism. □

8 The characteristic polynomial

Recall that for a bounded-below poset $\mathcal{P}$, we may recursively define a Möbius function $\mu : \mathcal{P} \to \mathbb{Z}$ such that $\mu(\hat{0}) = 1$ and for any $x \in \mathcal{P}_{>\hat{0}}$, $\sum\limits_{y \leq x} \mu(y) = 0$. When $\mathcal{P}$ is additionally a pure, ranked poset, then define the **characteristic polynomial** of $\mathcal{P}$ by

$$\chi_{\mathcal{P}}(t) = \sum_{x \in \mathcal{P}} \mu(x) t^{\rho(\mathcal{P}) - \rho(x)}.$$

Since the defining relations in Definition 6 are homogeneous, the Orlik–Solomon algebra is a graded algebra $A(\mathcal{P}) = \oplus_i A^i$. We thus can define its *Hilbert series* as

$$H_{\mathcal{P}}(t) = \sum_{i \geq 0} \dim(A^i) t^i.$$

Theorem 4. *For any pure locally geometric poset $\mathcal{P}$,*

$$H_{\mathcal{P}}(t) = (-t)^{\rho(P)} \chi_{\mathcal{P}}\left(-\frac{1}{t}\right).$$

Proof sketch. By Theorem 3, we have that $H_{\mathcal{P}}(t) = H_{\mathcal{P}'}(t) + tH_{\mathcal{P}''}(t)$. The claim then follows from induction on the rank of $\mathcal{P}$. □

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