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Key Positivity and Temperley-Lieb Immanants

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Abstract. We show that the Temperley-Lieb immanants of flagged Jacobi-Trudi matrices expand positively in the basis of key polynomials, provided that one of the associated partition shapes is strict. This result mirrors the fact that Temperley-Lieb immanants of (unflagged) Jacobi-Trudi matrices are nonnegative linear combinations of Schur polynomials.

As applications, we can generalize two other Schur positivity results to the flagged Schur case. Namely, we give a generalization of the Littlewood-Richardson Rule, showing how a product of flagged skew Schur polynomials expands positively in the basis of key polynomials - given two simple but necessary conditions on the partition shapes. We also deduce a log concavity inequality, similar to a theorem of Lam, Postnikov, and Pylyavskyy.

Our key tools are the recently defined shuffle tableaux of Nguyen and Pylyavskyy. The proof of our main result comes down to showing that flagged shuffle tableaux assemble themselves into Demazure crystals. In order to prove this fact, we develop a new, semi-local set of axioms for proving that a subset of a Type A crystal forms a Demazure crystal.

1 Introduction

Schur polynomials are ubiquitous throughout algebraic combinatorics. Generalizing Schur polynomials are the *flagged Schur polynomials*, denoted $s^b_\lambda(x_1 \dots x_n)$, certain truncations of Schur polynomials which are no longer necessarily symmetric. Flagged Schur polynomials are themselves contained within *key polynomials*. Key polynomials form a basis for $\mathbb{Z}[x_1, x_2, \dots]$, and arise as characters of certain submodules generated by extremal weight spaces under the Borel action of a Lie algebra.

It is therefore interesting to ask which important properties of Schur polynomials are also enjoyed by flagged Schur polynomials, and more generally by key polynomials. For example, the celebrated *Littlewood-Richardson rule* [7] tells us that the product of two Schur polynomials expands as a positive sum of Schur polynomials. This property does not extend to key polynomials (the product of two key polynomials need not be positive

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in the key basis). But, as we will see, this positivity does extend to a large class of skew flagged Schur polynomials.

Schur polynomials can be expressed via *Jacobi-Trudi identities* [3]; Wachs [16] showed that flagged Schur polynomials satisfy similar such identities. Importantly, the *Temperley-Lieb immanants* of generalized Jacobi-Trudi matrices are always Schur positive [13]. This fact has been used to prove several intriguing deep Schur positivity results [5] [11].

Our general goal is to ask when Schur positivity results extend to key positivity, when we replace usual Schur polynomials with flagged Schur polynomials. Our main result is:

Theorem 1.1. Let λ be a *strict* partitions and μ be an arbitrary partition (that is, $\lambda_i > \lambda_{i+1}$ for all $i \leq n$). Then, the Temperley-Lieb immanants of the flagged Jacobi-Trudi matrix $A^b_{\lambda,\mu}$ are key positive.

The condition that λ is strict is actually necessary for Theorem 1.1 to hold, a fact we initially found surprising. It would be interesting to give some kind of representation theoretic interpretation of this necessity (in our proof, our usage of the strictness of λ is combinatorial).

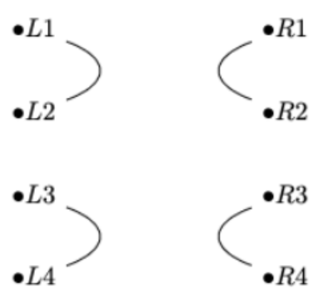
Example 1.2 (Counterexample for nonstrict λ)**.** Let us take $\lambda = (7,7,7,5)$, $\mu = (6,4,3,2)$, and $b = (1,2,2,3)$. Here, λ is not strict, so Theorem 1.1 should not apply. Indeed, we can compute that the Temperley-Lieb immanant corresponding to the matching depicted below is

$$k_{3,6,2} + k_{3,7,1} + k_{3,8} + k_{4,5,2} + k_{4,6,1} + k_{4,7} - k_{5,4,2} - k_{6,3,2} - k_{6,4,1} - k_{7,3,1} - k_{7,4} - k_{8,3}$$

which is far from being key nonnegative. However, when we change λ to $(8,7,6,5)$, the immanant indexed by the same diagram becomes

$$k_{5,4,2} + k_{5,5,1} + k_{5,6} + k_{6,3,2} + k_{6,4,1} + k_{7,3,1} + k_{7,4} + k_{8,3}$$

Magically, this expression is now key positive!



We give two quick applications for Theorem 1.1, described more fully in section 4. First, we consider products of skew flagged Schur polynomials. Reiner-Shimozono [12] showed that skew flagged Schur polynomials are key positive, so it is natural to wonder about products of skew flagged Schur polynomials as well.

Theorem 1.3. Let λ/μ and ν/ρ be skew shapes satisfying two properties:

1. For all i, $\lambda_i \neq \nu_i$
2. λ and ν are *interlacing*: that is, $\lambda_i \geq \nu_{i+1}$ and $\nu_i \geq \lambda_{i+1}$.

Then, the product $s^b_{\lambda/\mu}s^b_{\nu/\rho}$ is key positive. Furthermore, there is a relatively simple combinatorial algorithm for its key expansion.

Second, we prove a log concavity inequality for flagged Schur polynomials, extending the log concavity result of Lam, Postnikov and Pylyavskyy [5]. This inequality was the starting point for this project.

Corollary 1.4. Let λ/μ and ν/ρ be skew shapes satisfying the same two properties as in 4.1. Then, the difference

$$s^b_{(\lambda/\mu)\vee(\nu/\rho)}s^b_{(\lambda/\mu)\wedge(\nu/\rho)} - s^b_{\lambda/\mu}s^b_{\nu/\rho}$$

is key positive.

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2 Background

2.1 Flagged Schur polynomials

A *flag* b is a vector $(b_1 \dots b_n)$ of weakly increasing positive integers. We say that a tableau T of shape λ, where λ has n parts, is *flagged by* b if every entry in row i of T is bounded above by b_i.

Definition 2.1. For a flag b, the *flagged Schur polynomial* s^b_λ is given by

$$\sum_{T\in \mathrm{SSYT}(\lambda)\text{ flagged by }b} \mathrm{wt}(T)$$

(The same definitions apply to a skew shape λ/μ rather than a straight shape).

Example 2.2. Let us compute $s^{(1,3)}_{(3,1)}$. There are only two semistandard Young tableaux of shape $(3,1)$ flagged by $(1,3)$. Their weights contribute the monomials $x_1^3x_2$ and $x_1^3x_2$. Therefore,

$$s^{(1,3)}_{(3,1)} = x_1^3x_2 + x_1^3x_3$$

The tableau below is an example of a semistandard Young tableau which is *not* flagged by b, because the 2 in row 1 is greater than b_1:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Every Schur polynomial in n variables is also a flagged Schur polynomial: just let $b = (n, n, n \ldots)$.

2.2 Key Polynomials

Key polynomials are indexed by *weak compositions* $\alpha = (\alpha_1, \alpha_2 \ldots)$. Alternatively, they are indexed by a pair (λ, w), where λ is a (nonstrict) partition and w is a permutation. When w is the longest permutation w_0, then κ_{λ,w_0} is the Schur polynomial s_λ. When w is the identity, $\kappa_{\lambda,w}$ is the single monomial x^λ. In general, key polynomials can be regarded as certain truncations of Schur polynomials.

2.3 Jacobi-Trudi Matrices

A *generalized Jacobi-Trudi matrix* $A_{\lambda,\mu}$ is an $n \times n$ matrix of the form

$$(h_{\lambda_i - \mu_j})^n_{i,j=1}$$

for partitions λ, μ. Here, $h_{\lambda_i-\mu_j}$ is a homogeneous symmetric function.

The Jacobi-Trudi identity expresses skew Schur polynomials as minors of generalized Jacobi-Trudi matrices. Specifically, if $\Delta_{I,J}(M)$ denotes the minor with row set I and column set J, we have

$$s_{\lambda/\mu} = \Delta_{I,J}(A_{\lambda,\mu})$$

Here, the associated row and column sets are given by $I = \{\mu_k + 1 < \mu_{k-1} + 2 < \cdots < \mu_1 + k\}$ and $J = \{\lambda_k + 1 < \lambda_{k-1} + 2 \cdots < \lambda_1 + k\}$.

We can similarly express flagged Schur polynomials $s^b_{\lambda/\mu}$ as minors of a *flagged* Jacobi-Trudi matrix $A^b_{\lambda,\mu}$. The flagged Jacobi-Trudi matrix $A^b_{\lambda,\mu}$ is obtained by replacing each $h_{\lambda_i-\mu_j}$ in row i with a homogeneous symmetric polynomial *in b_i variables* [16].

2.4 Temperley-Lieb Immanants

As mentioned above, the determinant of any (generalized) Jacobi-Trudi matrix is a skew Schur polynomial, hence Schur positive. Determinants are examples of elements of Lusztig's *dual canonical bases* in type A [8]. Rhoades-Skandera [14] showed that *all* elements of the dual canonical bases evaluate to Schur nonnegative expressions on generalized Jacobi-Trudi matrices. Their proof relies on deep earlier work of Haiman [4].

In general, there is no known combinatorial proof of this fact. But for a subset of dual canonical bases known as *Temperley-Lieb immanants*, introduced by Rhoades-Skandera [13], there is a combinatorial proof by Nguyen-Pylyavskyy [10]. These Temperley-Lieb immanants have been previously used to prove intriguing Schur positivity facts, such as a log concavity inequality of Lam-Postnikov-Pylyavskyy [5].

We do not define Temperley-Lieb immanants here, although we will describe the combinatorial rule given by Nguyen-Pylyavskyy in the next section. We will only recall that an immanant is generally a function I evaluated on matrices of the form

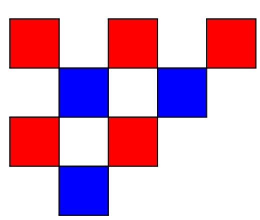
$$I(M) = \sum_{w \in S_n} f(w) \prod_{i=1}^{n} m_{i,w(i)}$$

for some function $f : S_n \to \mathbb{C}$. For the determinant of a matrix, f is simply the sign function. Meanwhile, the permanent of a matrix is given by taking f to be constant at 1. A certain class of functions $f : S_n \to \mathbb{C}$ defined via the *Temperley-Lieb algebra* give us Temperley-Lieb immanants.

2.5 Shuffle Tableaux

Given two skew diagrams λ/μ and ν/ρ, their *shuffle diagram* $(\lambda/\mu) \circledast (\nu/\rho)$ is the diagram we obtain by 'interlacing the rows of the two shapes' (informally. For more details, see [10]).

Example 2.3. Let $\lambda = (3, 2)$ and $\mu = (2, 1)$. The shuffle diagram is



where the red boxes are 'contributed' by λ and the blue boxes are 'contributed' by μ.

Shuffle tableaux were first defined by Nguyen-Pylyavskyy [10] in order to study Temperley-Lieb immanants. A shuffle tableau is just a labeling of the boxes of $(\lambda/\mu) \circledast (\nu/\rho)$ satisfying the usual rules - labels increase weakly along rows and strictly down columns. So, a shuffle tableau simply corresponds to a pair of tableaux, one of shape λ/μ, one of shape ν/ρ.

For Nguyen and Pylyavskyy, the magic of shuffle tableaux was that they can coherently be assigned 'Temperley-Lieb types.' Using this idea, they give a combinatorial rule for Schur expansions of Temperley-Lieb immanants. For us, the magic of shuffle tableaux is also that they 'behave well under flagging.'

Theorem 2.4 (Nguyen-Pylyavskyy)**.** There exist natural raising and lowering operators e_i, f_i such that the set of shuffle tableaux of shape $(\lambda/\mu) \circledast (\nu/\rho)$ assemble into type A crystals. Furthermore, these crystal operators preserve the Temperley-Lieb type of the tableaux.

These crystal operators are similar but not identical to the operators on usual skew tableaux. We direct the reader to [9] for a more complete description of the crystal operators.

2.6 Demazure Crystals

Our key technical result will be that, under the right conditions, shuffle tableaux *flagged by b* sit inside of shuffle tableau crystals as *Demazure subsets*. (Here, a shuffle tableau is flagged by a vector b if, as usual, its ith row contains no entries greater than b_i).

Recall that a *crystal* consists of a set of objects (for example, semistandard tableaux of a given shape λ), a *weight function* on the set of objects outputting weight vectors, (for example, taking the content of a tableau), and certain *raising and lowering operators* $\{e_i\}, \{f_i\}$. The lowering operators f_i should decrease the ith entry of the weight vector by 1, while increasing the $i+1$st entry of the weight vector by 1.

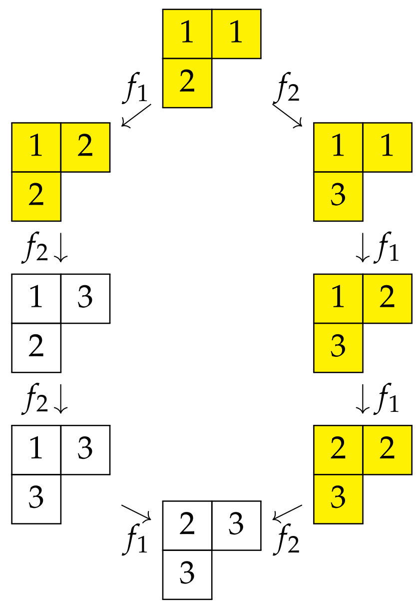
For us, the importance of crystals is that the character of a (connected, type A) crystal is a Schur polynomial.

Stembridge [15] gave a powerful characterization of type A crystals in terms of six local axioms. This characterization gives a useful general strategy for showing that a certain generating function is Schur positive: use Stembridge's criterion to show that the generating objects assemble into crystals.

Nguyen and Pylyavskyy proved that shuffle tableaux satisfy Stembridge's local axioms, and thus form type A crystals. In order to study key positivity, we will be interested in certain truncations of crystals, known as *Demazure crystals* [6]. The set of tableaux of a fixed shape λ satisfying a fixed flag b is a classic example of a Demazure crystal.

While the character of a full, untruncated crystal is a Schur polynomial, the character of a Demazure crystal is a key polynomial.

Example 2.5. Below is the crystal graph on SSYT$(2,1)$ with entries in $\{1,2,3\}$ and edges labeled by lowering operators. The yellow highlighted tableaux give the Demazure subset defined by the flag $(2,3)$. Summing the contents of all the tableaux gives us $s_{(2,1)}$. Summing the contents of the yellow subset gives us $s_{(2,1)}^{(2,3)}$, which is also the key polynomial $\kappa_{(2,1),s_1 s_2}$.



We fix some notation for the future: the operator f_i^* is obtained by composing the lowering operator f_i with itself as many times as possible. For example, in the crystal graph above,

$$f_1\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

but

$$f_1^*\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

The operator f_i^* acts by permuting weight vectors by s_i (in the example above, applying f_1^* transforms the weight vector $(2,0,1)$ to $(0,2,1)$).

Given the background above, our goal becomes to prove the following:

Proposition 2.6. Let $(\lambda/\mu) \circledast (\nu/\rho)$ be a shuffle diagram, where λ and ν are strict partitions. Then, the set of shuffle tableaux of shape $(\lambda/\mu) \circledast (\nu/\rho)$ which are flagged by b, equipped with the usual shuffle tableau crystal operators, decomposes as a disjoint union of Demazure crystals.

3 Proof Ideas: Characterizing Demazure Crystals

Our quest becomes proving Proposition 2.6. A typical strategy in proving that a subset of a crystal is Demazure is to map the crystal isomorphically onto a better understood crystal, like that of straight semistandard Young tableaux. (For example, jeu de taquin can be viewed as a crystal isomorphism from skew tableaux to straight tableaux). However, it proves difficult to find a direct crystal isomorphism from shuffle tableaux to straight semistandard Young tableaux; finding such an isomorphism remains an interesting open problem.

So, we need another approach in order to show 2.6.

The details of this approach are a bit involved, so for brevity, we only give a high-level overview.

A nonempty subset $X \subseteq B(\lambda)$ of a crystal is a Demazure subset if and only if it satisfies a short list of properties. For example, a few necessary conditions are:

1. X must contain the highest weight element u_λ
2. X must be upwards closed (ie if $x \in X$, then $e_i(x) \in X$ for all raising operators e_i)
3. More generally, the notion of 'extremality' previously studied by Assaf and Gonzalez [1] [2]
4. X must contain a unique lowest weight element under the dominance order on weights.

Of the properties above (which form a slightly incomplete list), the most work by far is required to prove item 4. For instance, in example 2.5, the yellow tableau on the lower right is the unique lowest weight tableau).

Our main novel technique is a 'semi-local' approach to showing that a subset X contains a unique lowest weight element. This technique is inspired by the following simple observation:

Remark 3.1. Suppose we already knew X was a Demazure crystal, with heighest weight element u_λ. Then, the unique lowest weight element of X can be constructed via a recursive algorithm as follows. Find the smallest integer a_1 such that $b_2 = f_n^* f_{n-1}^* \cdots f_{a_1}^*(b_1)$ is still in X. Next, find the smallest a_2 such that $f_{n-1}^* f_{n-2}^* \cdots f_{a_2}^*(b_2)$ is still in X. Continue like this for each i.

For brevity, we will not further explain our rather technical characterization of Demazure crystals here, but it essentially re-expresses the idea of the 'greedy algorithm' for maximizing weights above. Most of the work behind our proof of Theorem 1.1 comes down to showing that flagged shuffle tableaux satisfy our characterization. This is also the step that breaks down when our partition shapes are not strict.

With these black boxes, we can now finish the proof of Theorem 1.1, by concluding that flagged shuffle tableaux assemble into Demazure subsets of (unflagged) shuffle tableaux.

4 Applications

We believe that shuffle tableaux will prove a particularly useful model for studying flagged Schur polynomials in the future. Here is a short philosophical argument as to why.

In order to take the product of s_λ and s_μ (for straight shapes λ and μ), we can put the two shapes together into a larger skew shape by just placing λ northeast of μ. This observation allows us to easily relate skew Schur polynomials to products of Schur polynomials.

But if we are interested in *flagged* semistandard Young tableaux, this simple operation no longer makes sense, since putting together λ and μ in this way does not preserve the non-decreasing-ness of the flag.

However: if we instead *shuffle the two diagrams together* (assuming the two diagrams are assigned the same nondecreasing flag b), so that their rows interlace with each other, the resulting flag is still increasing; it becomes $(b_1, b_1, b_2, b_2 \dots)$. The philosophy is therefore that, to study flagged tableaux, one should shuffle.

We solidify these ideas by phrasing a generalized Littlewood-Richardson Rule. (In the statements, we will assume the flags on all flagged Schur polynomials are identical, but this constraint can be loosened quite a bit).

4.1 A Generalized Flagged Littlewood-Richardson Rule

Reiner-Shimozono [12] showed that *flagged skew Schur polynomials are key-positive*, giving a generalization of the Littlewood-Richardson rule. But for Schur polynomials, more can be said: products $s_{\lambda/\mu} s_{\nu/\rho}$ are also Schur positive, since products of straight Schur polynomials are Schur positive.

It is thus natural to try to generalize Reiner-Shimozono's theorem to products of skew flagged Schur polynomials

$$s^b_{\lambda/\mu} s^b_{\nu/\rho}$$

Theorem 4.1. Let λ/μ and ν/ρ be skew shapes satisfying the same two properties:

1. For all i, $\lambda_i \neq \nu_i$
2. λ and ν are *interlacing*: that is, $\lambda_i \geq \nu_{i+1}$ and $\nu_i \geq \lambda_{i+1}$.

Then, $s^b_{\lambda/\mu} s^b_{\nu/\rho}$ is key positive. Furthermore, there is a relatively simple combinatorial algorithm for its key expansion.

Perhaps surprisingly, if the hypotheses on the shapes $\lambda/\mu, \nu/\rho$ are not true, the product $s^b_{\lambda/\mu}s^b_{\nu/\rho}$ is *not* key positive in general. In this way, products of skew flagged Schur polynomials are more subtle than products of skew (unflagged) Schur polynomials.

We take a moment to sketch our combinatorial algorithm for the expansion, with an example. To express $s^b_{\lambda/\mu}s^b_{\nu/\rho}$ as a sum of key polynomials, perform the following steps:

1. Make the shuffle diagram $(\lambda/\mu) \circledast (\nu/\rho)$.

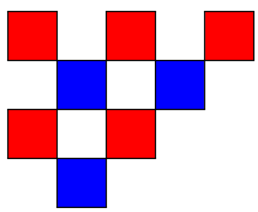
2. Find all highest-weight shuffle tableaux T of this shape. Each one will contribute a key polynomial to the sum $k_{\lambda,w}$, where λ is given by the content of T.

3. For each highest weight T, find the corresponding w following the algorithm outlined in 3.1.

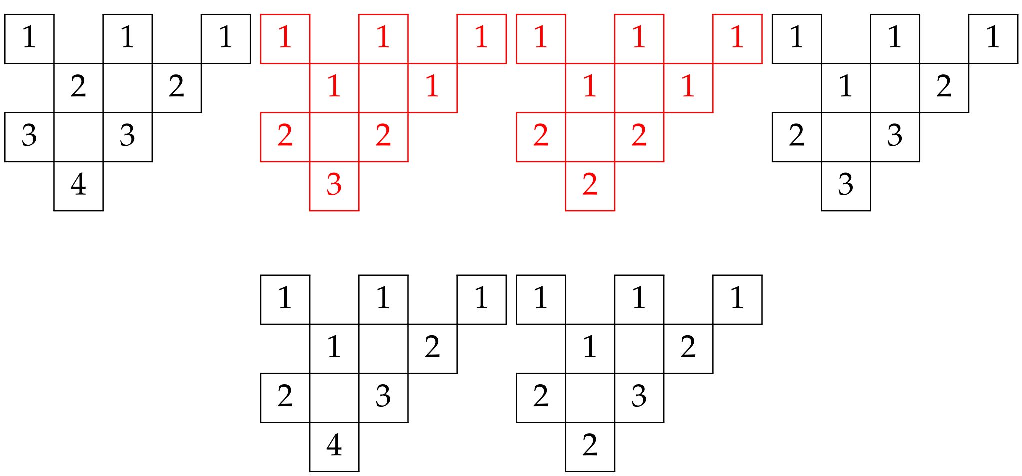
The most computationally inefficient step of this algorithm is step 2. In principle, one would have to run through all shuffle tableaux T, and for each one, check whether every e_i gives us $e_i(T) = 0$. But, thanks to the bijection described in [9], this step is actually much easier than that (we must only find all 'D-peelable (straight) tableaux.')

Example 4.2. Let us expand the product $s^b_\lambda s^b_\nu$, with $\lambda = (3,2)$, $\nu = (2,1)$, and flag $b = (1,3)$, according to this rule.

The relevant shuffle diagram is:



The flagged shuffle tableaux we seek have entries at most 1 in the first two rows and at most 3 in the last two rows. Using [9], we can quickly compute the highest weight elements supported on this diagram:



Only the red highlighted shuffle tableaux are flagged by $(1,1,3,3)$, so there will be two key polynomials in the expansion. Furthermore, using remark 3.1, the permutation s_2 is associated to both tableaux. Therefore, we have

$$s^{(1,3)}_{(3,2)}s^{(1,3)}_{(2,1)} = k_{(5,2,1),s_2} + k_{(5,3),s_2} = k_{(5,1,2)} + k_{(5,0,3)}$$

4.2 A Log Concavity Inequality

We recall our log concavity inequality for flagged Schur polynomials, which was the original motivating question for this project:

Corollary 4.3. Let λ/μ and ν/ρ be skew shapes satisfying the same two properties as in Theorem 4.1

Then, the difference

$$s^b_{\lambda/\mu\vee\nu/\rho}s^b_{\lambda/\mu\wedge\nu/\rho} - s^b_{\lambda/\mu}s^b_{\nu/\rho}$$

is key-positive.

Here, the operations $\vee$ and $\wedge$ are defined on skew diagrams as follows. For partitions λ, μ, we let $\lambda \vee \mu$ be the partition $(\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2)\dots)$. We define $\lambda \wedge \mu$ similarly, replacing maxima with minima. Then, for skew shapes, we write

$$(\lambda/\mu) \wedge (\nu/\rho) = \lambda \wedge \nu/\mu \wedge \rho$$

Example 4.4. Let $\lambda = (3,1)$, $\nu = (2,2)$, and let μ, ρ be empty. Let the flag b be $(2,4)$. Then, $\lambda \vee \mu = (3,2)$, $\lambda \wedge \mu = (2,1)$. We can compute that

$$s^{(2,4)}_{(3,2)}s^{(2,4)}_{(2,1)} - s^{(2,4)}_{(3,1)}s^{(2,4)}_{(2,2)} = k_{3,3,1,1} + k_{3,4,0,1} + k_{4,4}$$

which is indeed key positive.

The proof is almost immediate from Theorem 1.1, following a similar strategy to the proof for the unflagged case of Lam-Postnikov-Pylyavskyy [5].

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