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Graded Ehrhart Theory of Unimodular Zonotopes

Colin Crowley*[1] and Ethan Partida†[2]

[1]*Department of Mathematics, University of Oregon*
[2]*Department of Mathematics, Brown University*

Abstract. Graded Ehrhart theory is a new q-analogue of Ehrhart theory based on the orbit harmonics method. We study the graded Ehrhart theory of unimodular zonotopes from a matroid-theoretic perspective. Generalizing a result of Stanley (1991), we prove that the graded lattice point count of a unimodular zonotope is a q-evaluation of its Tutte polynomial. We conclude that the graded Ehrhart series of a unimodular zonotope is rational and obeys graded Ehrhart–Macdonald reciprocity. In an algebraic direction, we prove that the harmonic algebra of a unimodular zonotope is a coordinate ring of its associated arrangement Schubert variety. Using the geometry of arrangement Schubert varieties, we prove that the harmonic algebra of a unimodular zonotope is finitely generated and Cohen–Macaulay. We also give an explicit presentation of the harmonic algebra of a unimodular zonotope in terms of generators and relations. Our work answers, in the special case of unimodular zonotopes, two conjectures of Reiner and Rhoades (2024).

Keywords: Ehrhart Theory, Unimodular Zonotopes, Arrangement Schubert Varieties, Zonotopal Algebras

1 Introduction

In this extended abstract, we study the combinatorial, algebraic and geometric aspects of the graded Ehrhart theory of unimodular zonotopes. Graded Ehrhart theory is a new q-analogue of Ehrhart theory introduced by Reiner and Rhoades [16]. For every lattice polytope P and positive integer m, there is a canonical pair of polynomials $i_P(m;q)$ and $\widetilde{i_P}(m;q)$ in $\mathbb{Z}[q]$ that have non-negative coefficients summing to the number of lattice points and interior lattice points in mP, respectively. These polynomials are constructed via the *orbit harmonics* method. The orbit harmonics method is a combinatorial degeneration that turns a finite set of points $\mathcal{Z} \subseteq k^n$ into a graded ring $\mathrm{Orb}(\mathcal{Z})$ whose total dimension is equal to $|\mathcal{Z}|$. When the finite set of points $\mathcal{Z}$ is combinatorially interesting, the dimensions of the graded pieces of $\mathrm{Orb}(\mathcal{Z})$ often have an interesting combinatorial interpretation. For example, if $\mathcal{Z}$ is the set of vertices of the permutahedron,

*crowley@uoregon.edu. Colin Crowley was partially by NSF grant DMS-2039316 and the Simons foundation.

†ethan_partida@brown.edu. Ethan Partida was partially supported by NSF grant DMS-2053288, a U.S. Department of Education GAANN award, and the Simons foundation SFI-MPS-SDF-00015018.

then the ring $\mathrm{Orb}(\mathcal{Z})$ is the coinvariant ring of the symmetric group. The polynomials $i_P(m;q)$ and $\widetilde{i_P}(m;q)$ record the dimensions of the graded pieces of $\mathrm{Orb}(mP \cap \mathbb{Z}^d)$ and $\mathrm{Orb}(\mathrm{interior}(mP) \cap \mathbb{Z}^d)$, respectively.

A compelling feature of Ehrhart theory and, indeed, our present work, is its broad connections to combinatorics, commutative algebra and algebraic geometry. Classical Ehrhart theory is closely related to the enumerative combinatorics of integer-valued polynomials, the commutative algebra of semigroup rings, and the algebraic geometry of toric varieties. In our study of the graded Ehrhart theory of unimodular zonotopes, we draw from, and prove new facts about: the enumerative combinatorics of quantum integer-valued polynomials [10], the commutative algebra of zonotopal and harmonic algebras [11, 16], and the algebraic geometry of arrangement Schubert varieties [1, 14].

The classical Ehrhart theory of unimodular zonotopes is studied in the works of Stanley, Backman–Baker–Yuen, and Beck–Jochemko–McCullough [17, 3, 4]. They show that, among many other things, the Ehrhart theory of a unimodular zontope Z is essentially dictated by its associated matroid M. Our results show that M also dictates the graded Ehrhart theory of Z.

We now summarize our results. As our work draws on many different mathematical areas, we have separated our results into two categories: enumerative and algebraic. We delay our background until Section 3 in order to highlight our results with a large worked example in Section 2. We expand upon the enumerative and algebraic aspects of our work in Sections 4 and 5, respectively. We close by defining zonotopal algebras and using them to prove Theorems 1 and 3 in Section 6.

1.1 Enumerative Results

Our main enumerative theorem expresses the graded lattice point counts $i_Z(m;q)$ and $\widetilde{i_Z}(m;q)$ of a unimodular zonotope as q-evaluations of the Tutte polynomial. This theorem, upon setting $q=1$, specializes to a result of Stanley [17] that expresses the ordinary lattice point counts $i_Z(m;1)$ and $\widetilde{i_Z}(m;1)$ as an evaluation of the Tutte polynomial.

Theorem 1. *Let $m \geq 1$ be an integer, $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope and M be the matroid of A. We have that,*

$$i_Z(m;q) = q^{(n-d)m}[m]_q^d T_M\Big(\frac{[m+1]_q}{[m]_q}, q^{-m}\Big) \text{ and } \widetilde{i_Z}(m;q) = q^{(n-d)m}[m]_q^d T_M\Big(\frac{[m-1]_q}{[m]_q}, q^{-m}\Big),$$

where $T_M(x,y)$ is the Tutte polynomial of M and $[k]_q := \frac{1-q^k}{1-q}$ is the kth q-integer.

Our proof of Theorem 1 relies on the theory of zonotopal algebras. We explain zonotopal algebras and prove Theorem 1 in Section 6. From Theorem 1, we conclude numerous results about the behavior of $i_Z(m;q)$ and $\widetilde{i_Z}(m;q)$ as m grows. We summarize our results in the below theorem. For more details, see Section 4.

Theorem 2. *Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope. The following is true:*

1. *[Proposition 9] There is a polynomial $\mathrm{ehr}_Z(t;q) \in \mathbb{Q}(q)[t]$, which we call the graded Ehrhart polynomial, such that $\mathrm{ehr}_Z([m]_q;q) = i_Z(m;q)$. The polynomial $\mathrm{ehr}_Z(t;q)$ exhibits a q-analogue of Ehrhart–Macdonald reciprocity.*

2. *[Proposition 11] Conjecture 1.1 of [16] holds for unimodular zonotopes. Namely, the graded Ehrhart series*

$$E_Z(t,q) = \sum_{m \geq 0} i_Z(m;q)t^m \quad \text{and} \quad \widetilde{E_Z}(t,q) = \sum_{m \geq 1} \widetilde{i_Z}(m;q)t^m$$

are rational functions in $\mathbb{Q}(q,t)$ with a particularly nice form. The Ehrhart series $E_Z(t,q)$ and $\widetilde{E_Z}(t,q)$ exhibit a q-analogue of Ehrhart–Macdonald reciprocity.

1.2 Algebraic Results

For any lattice polytope P, there is a bigraded $\mathbb{C}$-algebra $\mathcal{H}_P$ and a homogeneous ideal $\widetilde{\mathcal{H}_P}$ of $\mathcal{H}_P$ whose bigraded Hilbert series are the graded Ehrhart series $E_P(t,q)$ and $\widetilde{E_P}(t,q)$, respectively. The algebra $\mathcal{H}_P$ is called the *harmonic algebra* of P and the ideal $\widetilde{\mathcal{H}_P}$ is called the *interior ideal* of P[1]. See Section 3.1 for the definitions of $\mathcal{H}_P$ and $\widetilde{\mathcal{H}}_P$.

Our main algebraic theorem gives an algebro-geometric interpretation of the harmonic algebra of a unimodular zonotope. If $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ is a unimodular zonotope, the *arrangement Schubert variety* Y_Z of Z is a complex d-dimensional subvariety of $(\mathbb{P}^1)^n$, defined as the closure of the rowspace of A. See Section 3.2 for more details.

Theorem 3. *Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope. The harmonic algebra $\mathcal{H}_Z$ is isomorphic to the homogeneous coordinate ring of Y_Z under the Segre embedding $Y_Z \subseteq \mathbb{P}^{2^n-1}$.*

We sketch a proof of Theorem 3 in Section 6. Theorem 3 lets us prove new facts about the harmonic algebras of unimodular zonotopes. The following theorem summarizes our results in this direction. For more details, see Section 5.

Theorem 4. *Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope. The following is true*[2]*:*

1. *[Section 5] Conjecture 5.5 of [16] holds for unimodular zonotopes. Namely, the harmonic algebra $\mathcal{H}_Z$ is a finitely generated, Cohen–Macaulay $\mathbb{C}$-algebra. After a degree shift by q-degree d, the canonical module $\Omega\mathcal{H}_Z$ is isomorphic to the interior ideal $\widetilde{H_Z}$.*

[1]The harmonic algebra was originally defined in [16] as an $\mathbb{R}$-algebra $\mathcal{H}_P^{\mathbb{R}}$ and the interior ideal as an ideal $\widetilde{\mathcal{H}_P^{\mathbb{R}}} \subseteq \mathcal{H}_P^{\mathbb{R}}$. In order to uniformly state our theorems, we work with the complexifications $\mathcal{H}_P := \mathcal{H}_P^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $\widetilde{\mathcal{H}_P} := \widetilde{\mathcal{H}_P^{\mathbb{R}}} \otimes_{\mathbb{R}} \mathbb{C}$.

[2]The real coefficient analogues of these results hold for $\mathcal{H}_P^{\mathbb{R}}$ and $\widetilde{\mathcal{H}_P^{\mathbb{R}}}$ as well.

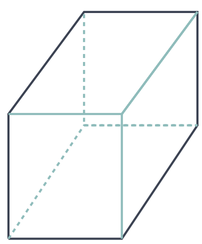
2. *[Proposition 12] The harmonic algebra $\mathcal{H}_Z$ has a presentation*

$$\mathcal{H}_Z \simeq \frac{\mathbb{C}[z_S : S \subseteq \{1,\dots,n\}]}{I_Z}$$

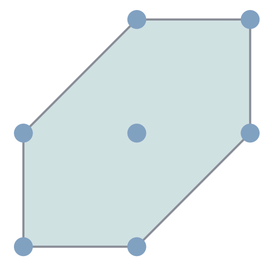
where $\deg(z_S) = (1, |S|)$ *and I_Z is a homogeneous ideal constructed from the circuits of M.*

2 Illustration of results

We now illustrate our results by working out the graded Ehrhart theory of a hexagon in the plane. Let $Z \subseteq \mathbb{R}^2$ be the zonotope defined by the matrix A in the following figure.



$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow$$



From the matrix A defining Z, we can see that the matroid M of Z is the uniform matroid $U_{2,3}$ with bases $\{12, 13, 23\}$. We now calculate $i_Z(1;q)$ in two different ways. The polynomial $i_Z(1;q)$ is the Hilbert series of the ring $\mathrm{Orb}(Z \cap \mathbb{Z}^2)$ (Definition 5). In Proposition 14, we prove that $\mathrm{Orb}(Z \cap \mathbb{Z}^2)$ is isomorphic to the external zonotopal algebra of Z (Definition 13). Through direct calculation, we can compute

$$\mathrm{Orb}(Z \cap \mathbb{Z}^2) \simeq R_Z^{\text{ext}} = \frac{\mathbb{C}[r_1, r_2]}{\langle r_1^3, r_2^3, (r_1 + r_2)^4 \rangle} \quad \text{and} \quad i_Z(1;q) = 1 + 2q + 3q^2 + q^3.$$

We can also compute $i_Z(1;q)$ using Theorem 1:

$$i_Z(1;q) = qT_{U_{2,3}}(1+q, q^{-1}) = q((q+1)^2 + (1+q) + q^{-1}) = 1 + 2q + 3q^2 + q^3.$$

The graded Ehrhart polynomials $\mathrm{ehr}_Z(t;q)$ and $\widetilde{\mathrm{ehr}}_Z(t,q)$ in $\mathcal{R}_q \subseteq \mathbb{Q}(q)[t]$ of Z are

$$\mathrm{ehr}_Z(t;q) = (q^3 - q)t^3 + 3q^2t^2 + 3qt + 1$$
$$\widetilde{\mathrm{ehr}}_Z(t;q) = (1 - q^{-2})t^3 + 3q^{-2}t^2 - 3q^{-2}t + q^{-2}.$$

Recall that $\mathrm{ehr}_Z(t;q)$ and $\widetilde{\mathrm{ehr}}_Z(t;q)$ are polynomials such that $\mathrm{ehr}_Z([m]_q;q) = i_Z(m;q)$ and $\widetilde{\mathrm{ehr}}_Z([m]_q;q) = \widetilde{i}_Z(m;q)$ for $m \geq 1$ (Proposition 9). Even though the coefficients of $\widetilde{\mathrm{ehr}}_Z(t;q)$ are Laurent polynomials in q, after evaluating $\widetilde{\mathrm{ehr}}_Z(t;q)$ at $t = [m]_q$ for any positive integer m, we will obtain a polynomial in q. For example, plugging in $t = [1]_q = 1$ into the above equations yields our earlier calculation of $i_Z(1;q)$ and that $\widetilde{\mathrm{ehr}}_Z([1]_q;q) = \widetilde{i_Z}(1;q) = 1$.

The q-binomial coefficient polynomials $\begin{bmatrix} t \\ k \end{bmatrix} \in \mathcal{R}_q \subseteq \mathbb{Q}(q)[t]$ are polynomials such that, after setting $t = [m]_q$, we obtain the q-binomial coefficients $\binom{m}{k}_q$. We can expand $\mathrm{ehr}_Z(t;q)$ into the q-binomial coefficient polynomial basis as

$$\mathrm{ehr}_Z(t;q) = \begin{bmatrix} t \\ 0 \end{bmatrix} + (q^3+3q^2+2q)\begin{bmatrix} t \\ 1 \end{bmatrix} + (q^6+3q^5+4q^4-2q^2)\begin{bmatrix} t \\ 2 \end{bmatrix} + (q^9+2q^8+q^7-q^6-2q^5-q^4)\begin{bmatrix} t \\ 3 \end{bmatrix}.$$

As the generating functions of q-binomial coefficients are well understood, we can use our expansion of $\mathrm{ehr}_Z(t;q)$ into binomial coefficients to compute

$$E_Z(t,q) = \sum_{m\geq 0} i_Z(m;q)t^m = \frac{-q^4t^3 - (q^3-2q^2)t^2 + (2q^2+q)t + 1}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}.$$

Note that, unlike in classical Ehrhart theory, our numerator can have negative coefficients! Doing a similar process for the generating function of $\widetilde{i_Z}(t;q)$, we obtain

$$\widetilde{E_Z}(t,q) = \sum_{m\geq 1} \widetilde{i_Z}(m;q)t^m = \frac{(-q^4t^3 - (q^3-2q^2)t^2 + (2q^2+q)t + 1)t}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}.$$

With these explicit presentations, one can compute that $q^2\widetilde{E_Z}(t,q) = (-1)^3 E_Z(t^{-1},q^{-1})$. In Proposition 11, we prove that such a reciprocity holds for all unimodular zonotopes.

We now describe a presentation of the harmonic algebra $\mathcal{H}_Z$ following Proposition 12. The only circuit C of $U_{2,3}$ is $C = 123$. The compliment of C is the empty set. Thus

$$\mathcal{H}_Z \simeq \frac{\mathbb{C}[z_\emptyset, z_1, z_2, z_3, z_{12}, z_{13}, z_{23}, z_{123}]}{\langle z_1 + z_2 - z_3, z_S z_T - z_{S\cup T} z_{S\cap T} : S, T \subseteq [3]\rangle}.$$

Using the single linear relation, we can verify that $(\mathcal{H}_Z)_{(1,*)}$ has $\mathbb{C}$-basis

$$(\mathcal{H}_Z)_{(1,*)} = \mathrm{span}_{\mathbb{C}}\{z_\emptyset, z_1, z_2, z_{12}, z_{13}, z_{23}, z_{123}\}. \tag{2.1}$$

As each variable z_S has degree $(1, |S|)$, we can use our $\mathbb{C}$-basis to confirm that $i_Z(1;q)$ is equal to $\mathrm{Hilb}((\mathcal{H}_Z)_{(1,*)};q)$. Proposition 12 tells us that this equality continues to hold upon replacing 1 with any non-negative integer m.

The internal ideal $\widetilde{\mathcal{H}_Z}$ can be identified with the ideal $\langle z_\emptyset\rangle$ of our presentation of $\mathcal{H}_Z$. As an example of this, note that in t-degree 2, $\widetilde{\mathcal{H}}_{Z(2,*)} = \langle z_\emptyset\rangle_{(2,*)}$ has $\mathbb{C}$-basis

$$\widetilde{\mathcal{H}}_{Z(2,*)} = \mathrm{span}_{\mathbb{C}}\{z_\emptyset^2, z_1z_\emptyset, z_2z_\emptyset, z_{12}z_\emptyset, z_{13}z_\emptyset, z_{23}z_\emptyset, z_{123}z_\emptyset\}. \tag{2.2}$$

Note that our basis (2.2) of $(\widetilde{\mathcal{H}_Z})_{(2,*)}$ is simply our basis (2.1) of $(\mathcal{H}_Z)_{(1,*)}$ multiplied by $z_\emptyset$. This is a relic of the fact that $\langle z_\emptyset \rangle \simeq \mathcal{H}_Z(1,0)$ as bigraded $\mathcal{H}_Z$ modules. Combined with Theorem 4, this observation tells us that $\mathcal{H}_Z$ is a Gorenstein algebra and that $\Omega(\mathcal{H}_Z) \simeq \mathcal{H}_Z(1,2)$ as bigraded $\mathcal{H}_Z$ modules. In [7], we we classify which unimodular zonotopes have Gorenstein harmonic algebras.

3 Background

A *zonotope* $Z \subseteq \mathbb{R}^d$ is a polytope that can be written as the image of the unit cube $[0,1]^n \subseteq \mathbb{R}^n$ under a linear projection $A : \mathbb{R}^n \to \mathbb{R}^d$. We often take A as part of the data of Z and write $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$. If A can be written as a $d \times n$ full rank integer matrix which is totally unimodular, i.e. all minors of A are contained in the set $\{-1,0,1\}$, then we say that Z is a *unimodular zonotope*. We now explain graded Ehrhart theory and give a brief introduction to arrangement Schubert varieties.

3.1 Graded Ehrhart theory

Recently, Reiner and Rhoades have introduced a new q-analogue of Ehrhart theory [16]. For every lattice polytope P and non-negative integer m, they construct polynomials $i_P(m;q)$ and $\widetilde{i_P}(m;q)$ in $\mathbb{Z}[q]$ with non-negative coefficients such that, after setting q to one, $i_P(m;1) = |mP \cap \mathbb{Z}^d|$ and $\widetilde{i_P}(m;1) = |\mathrm{int}(mP) \cap \mathbb{Z}^d|$ where $\mathrm{int}(mP)$ is the interior of the mth dilate of P. Reiner and Rhoades construct these polynomials using the methods of *orbit harmonics*. The orbit harmonics ring $\mathrm{Orb}(\mathcal{Z})$ of a finite subset $\mathcal{Z} \subseteq \mathbb{C}^d$ is the ring

$$\mathrm{Orb}(\mathcal{Z}) = \frac{\mathbb{C}[x_1, x_2, \ldots, x_d]}{\mathrm{gr}\, I(\mathcal{Z})}$$

where $\mathrm{gr}\, I(\mathcal{Z})$ is the ideal generated by all of the top degree homogeneous components f_k of polynomials $f = f_k + f_{k-1} + \ldots + f_0$ vanishing on $\mathcal{Z}$. The ring $\mathrm{Orb}(\mathcal{Z})$ is graded by degree and has total dimension equal to the cardinality of $\mathcal{Z}$. In this paper our loci $\mathcal{Z}$ will be the lattice point of real polytopes. We will view $\mathcal{Z}$ as a point locus in $\mathbb{C}^d$ via the canonical inclusion $\mathcal{Z} \subseteq \mathbb{Z}^d \hookrightarrow \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^d$.

For a graded $\mathbb{C}$-algebra $R = \oplus_{i \geq 0} R_i$, the Hilbert series of R is the generating function $\mathrm{Hilb}(R;q) := \sum_{i \geq 0} \dim_{\mathbb{C}}(R_i) q^i$. If R is finite dimensional as a $\mathbb{C}$-vector space, then $\mathrm{Hilb}(R;q)$ is a polynomial in $\mathbb{Z}[q]$.

Definition 5. *The graded lattice point counts $i_P(m;q)$ and $\widetilde{i_P}(m;q)$ are the Hilbert series*

$$i_P(m;q) := \mathrm{Hilb}(Orb(mP \cap \mathbb{Z}^d);q) \quad \textit{and} \quad \widetilde{i_P}(m;q) := \mathrm{Hilb}(Orb(int(mP) \cap \mathbb{Z}^d);q).$$

The graded Ehrhart series $E_P(m;q)$ and $\widetilde{E_P}(m;q)$ are the bivariate generating functions

$$E_P(t,q) = \sum_{m \geq 0} i_P(m;q) t^m \quad \textit{and} \quad \widetilde{E_P}(t,q) = \sum_{m \geq 1} \widetilde{i_P}(m;q) t^m .$$

There is a bigraded $\mathbb{C}$-algebra $\mathcal{H}_P$ and a homogeneous ideal $\widetilde{\mathcal{H}_P} \subseteq \mathcal{H}_P$ whose bigraded Hilbert series are the graded Ehrhart series $E_P(t,q)$ and $\widetilde{E_P}(t,q)$, respectively. The algebra $\mathcal{H}_P$ is called the *harmonic algebra* of P and the ideal $\widetilde{\mathcal{H}_P}$ is called the *interior ideal* of P. We now define these objects.

Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, write $f(D)$ for the partial differential operator $f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Given an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, let $I^\perp$ denote the Macaulay inverse system:

$$I^\perp := \{f \in \mathbb{C}[x_1, \ldots, x_d] : (f(D) \cdot g)|_0 = 0 \text{ for all } g \in I\}.$$

For a lattice polytope $P \subseteq \mathbb{R}^d$, we set

$$V_P := (\operatorname{gr} I(P \cap \mathbb{Z}^d))^\perp \quad \text{and} \quad \widetilde{V_P} := (\operatorname{gr} I(\operatorname{int}(P) \cap \mathbb{Z}^d))^\perp.$$

If $I \subseteq \mathbb{C}[x_1, \ldots, x_d]$ is a graded ideal, then $I^\perp$ is a graded vector space which is naturally isomorphic to the linear dual of the quotient by I. This ensures that

$$i_P(m;q) = \sum_{j \geq 0} \dim(V_{mP})_j q^j \quad \text{and} \quad \widetilde{i_P}(m;q) = \sum_{j \geq 0} \dim(\widetilde{V_{mP}})_j q^j.$$

Definition 6. *For a lattice polytope $P \subseteq \mathbb{R}^d$, consider the bigraded polynomial ring $\mathbb{C}[x_0, \ldots, x_d]$ where $\deg(x_0) = (1,0)$ and $\deg(x_i) = (0,1)$ for all $i \geq 1$. The harmonic algebra $\mathcal{H}_P$ and the interior ideal $\widetilde{H_P}$ are the bigraded subspaces of $\mathbb{C}[x_0, \ldots, x_d]$ defined as*

$$\mathcal{H}_P := \bigoplus_{m \geq 0} \mathbb{C} \cdot x_0^m \otimes_{\mathbb{C}} V_{mP} \quad \text{and} \quad \widetilde{\mathcal{H}_P} := \bigoplus_{m \geq 0} \mathbb{C} \cdot x_0^m \otimes_{\mathbb{C}} \widetilde{V_{mP}}.$$

Our construction ensures that

$$\operatorname{Hilb}(\mathcal{H}_P; t, q) := \sum_{m,j} \dim(\mathcal{H}_P)_{m,j} t^m q^j = E_P(t,q)$$

and similarly that $\operatorname{Hilb}(\widetilde{\mathcal{H}_P}; t, q) = \widetilde{E_P}(t,q)$. One of the main contributions of Rhoades and Reiner is to show that $\mathcal{H}_P$ admits the structure of a $\mathbb{C}$-algebra and that $\widetilde{H_P}$ includes as an ideal under this structure.

Theorem 7 ([16]). *The harmonic algebra $\mathcal{H}_P$ has the structure of a ring, where the multiplication is $(x_0^i \otimes f, x_0^j \otimes g) \mapsto x_0^{i+j} \otimes fg$. Moreover, the interior ideal $\widetilde{\mathcal{H}}_P$ is naturally a subspace of $\mathcal{H}_P$ and forms an ideal under this product structure.*

3.2 Arrangement Schubert varieties

Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope. After tensoring with $\mathbb{C}$, the rowspace of the matrix A is a d-dimensional linear subspace $L := \operatorname{rowspace}(A) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathbb{C}^n$.

Definition 8. *The arrangement Schubert variety Y_Z of Z is the closure of L under the embedding*

$$L \hookrightarrow \mathbb{C}^n \hookrightarrow (\mathbb{P}^1)^n$$

where the second map is defined by sending a point $(x_1, \ldots, x_n) \in \mathbb{C}^n$ to the point $([x_1 : 1], \ldots, [x_n : 1]) \in (\mathbb{P}^1)^n$.

The arrangement Schubert variety is an important projective variety whose geometry is largely dictated by the matroid M of A. It was first studied concurrently by [1] and [14], and was later used by [12] to prove the top heavy conjecture for realizable matroids.

Consider the Segre embedding $\Phi : (\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1}$ defined by sending a point $([x_1 : y_1], [x_2 : y_2], \ldots, [x_n : y_n])$ to the point $[z_\emptyset : z_{\{1\}} : z_{\{1,2\}} : \ldots : z_{[n]}]$ where the coordinates range over all subsets $S \subseteq [n]$ and $z_S = \prod_{i \in S} x_i \prod_{j \in [n] \setminus S} y_j$. By composing the embeddings $Y_Z \hookrightarrow (\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$, we can consider Y_Z as a projective subvariety of $\mathbb{P}^{2^n-1}$. It turns out that Y_Z is closed under the $\mathbb{C}^*$ action on $\mathbb{P}^{2^n-1}$ defined by $t \cdot [z_S]_{S \subseteq [n]} = [t^{|S|} z_S]_{S \subseteq [n]}$. Such an action gives the coordinate ring of Y_Z the structure of a bigraded algebra. We use this bigrading in our statement of Theorem 3.

4 Graded Ehrhart polynomials and quantum reciprocity

Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope. A consequence of Theorem 1 is that there are *quantum integer-valued polynomials* $\mathrm{ehr}_Z(t;q), \widetilde{\mathrm{ehr}}_Z(t;q) \in \mathbb{Q}(q)[t]$ such that for all integers $m \geq 1$, $\mathrm{ehr}_Z([m]_q;q) = i_Z(m;q)$ and $\widetilde{\mathrm{ehr}}_Z([m]_q;q) = \widetilde{i}_Z(m;q)$. A polynomial $f(t;q) \in \mathbb{Q}(q)[t]$ is quantum integer-valued, as defined by Harman and Hopkins [10], if $f([m]_q;q) \in \mathbb{Z}[q,q^{-1}]$ for all $m \in \mathbb{Z}$. The set of quantum integer-valued polynomials forms a subring $\mathcal{R}_q \subseteq \mathbb{Q}(q)[t]$. The ring $\mathcal{R}_q$ comes equipped with a bar involution $f \mapsto \overline{f}$ defined by the property that $\overline{f}([m]_q;q) = f([-m]_q;q^{-1})$ for all $m \in \mathbb{Z}$. This involution, after setting $q = 1$, reduces to the involution which maps a polynomial $f(t) \in \mathbb{Q}[t]$ to the polynomial $f(-t) \in \mathbb{Q}[t]$. In analogue to Ehrhart–Macdonald reciprocity, we prove that the polynomials $\mathrm{ehr}_Z(t;q)$ and $\widetilde{\mathrm{ehr}}_Z(t;q)$ are related by the bar involution.

Proposition 9 (Part 1 of Theorem 4)**.** *There are polynomials $\mathrm{ehr}_Z(t;q)$ and $\widetilde{\mathrm{ehr}}_Z(t;q)$ in $\mathcal{R}_q$ of t-degree n such that for all integers $m \geq 1$, $\mathrm{ehr}_Z([m]_q;q) = i_Z(m;q)$ and $\widetilde{\mathrm{ehr}}_Z([m]_q;q) = \widetilde{i}_Z(m;q)$. These polynomials satisfy a q-analogue of Ehrhart–Macdonald reciprocity:*

$$(-1)^d \overline{\mathrm{ehr}_Z}([m]_q;q) = (-1)^d \mathrm{ehr}_Z([-m]_q;q^{-1}) = q^d \widetilde{\mathrm{ehr}}_Z([m]_q;q).$$

A proof of Proposition 9 follows from careful algebraic manipulations of Theorem 1 that we omit from this extended abstract. Proposition 9 lets us apply a formal q-reciprocity identity (Proposition 10) in order to understand the graded Ehrhart series $E_Z(t,q)$ and $\widetilde{E}_Z(t,q)$.

Proposition 10. *Let $f(t;q) \in \mathcal{R}_q \subseteq \mathbb{Q}(q)[t]$ be a polynomial of t-degree n. The generating functions*

$$E(t,q) = \sum_{m\geq 0} f([m]_q;q)t^m \quad \text{and} \quad \overline{E}(t,q) = \sum_{m\geq 1} \overline{f}([m]_q;q)t^m = \sum_{m\geq 1} f([-m]_q;q^{-1})t^n$$

are both rational functions in $\mathbb{Q}(q,t)$. These rational functions can be written as

$$E(t,q) = \frac{N(t,q)}{(1-t)(1-tq)\cdots(1-tq^n)} \quad \text{and} \quad \overline{E}(t,q) = \frac{\overline{N}(t,q)}{(1-t)(1-tq)\cdots(1-tq^n)}$$

where $N(t,q), \overline{N}(t,q) \in \mathbb{Z}[t,q]$ and they are related by $\overline{E}(t,q) = -E(t^{-1},q^{-1})$.

Similar to proofs of ordinary formal reciprocity, our proof of Proposition 10 relies heavily on the fact that the ring $\mathcal{R}_q$ has a basis given by the q-binomial coefficient polynomials [10, Proposition 1.2]. Combining Proposition 9 and Proposition 10, we obtain Proposition 11.

Proposition 11 (Part 2 of Theorem 2)**.** *The Ehrhart series $E_Z(t,q)$ and $\widetilde{E_Z}(t,q)$ are rational function in $\mathbb{Q}(t,q)$ and can be written as*

$$E_Z(t,q) = \frac{N_Z(t,q)}{(1-t)(1-tq)\cdots(1-tq^n)} \quad \text{and} \quad \widetilde{E_Z}(t,q) = \frac{\widetilde{N_Z}(t,q)}{(1-t)(1-tq)\cdots(1-tq^n)}$$

where $N_Z(t,q), \widetilde{N_Z}(t,q) \in \mathbb{Z}[t,q]$. These functions exhibit "q-Ehrhart–Macdonald reciprocity", namely,

$$q^d \widetilde{E_Z}(t,q) = (-1)^{d+1} E_Z(t^{-1},q^{-1}).$$

5 Harmonic algebras and arrangement Schubert varieties

Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope and let $L = \text{rowspace}(A) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathbb{C}^n$. Our proof of Part 1 of Theorem 4 is entirely algebro-geometric. To save space in this extended abstract, we only highlight the broad strokes of our proof. Ardila and Boocher prove that Y_Z is a *multiplicity-free* subvariety of $(\mathbb{P}^1)^n$ [1, Theorem 1.3.c]. Work of Brion on multiplicity-free subvarieties [5] and Theorem 3 then implies that $\mathcal{H}_Z$ is finitely generated and Cohen–Macaulay. The key input to our identification of the canonical module is [8, Theorem 1.14].

Algebraically, Theorem 3 tells us that $\mathcal{H}_Z$ can be written as the bigraded quotient of the ring $\mathbb{C}[z_S : S \subseteq [n]]$ where $\deg(z_S) = (1, |S|)$. Recall from Section 3.2 that the bigrading comes from a special $\mathbb{C}^*$ action on Y_Z. We now give a presentation of this bigraded quotient. For each circuit C of M, there is a unique (up to scaling) linear function $f_C(z) = \sum_{i\in C} \alpha_i x_i$ which vanishes on L. Given a circuit C of M and subset $A \subseteq [n] \setminus C$, define the polynomial $f_C^A(z)$ in $\mathbb{C}[z_S : S \subseteq [n]]$ as $f_C^A(z) = \sum_{i\in C} \alpha_i z_{A\cup\{i\}}$.

Proposition 12 (Part 2 of Theorem 4)**.** *The harmonic algebra $\mathcal{H}_Z$ has the presentation*

$$\mathcal{H}_Z \simeq \frac{\mathbb{C}[z_S | S \subseteq [n]]}{\langle z_S z_T - z_{S\cup T} z_{S\cap T}, f_C^A(z) : S, T \subseteq [n],\, C \text{ a circuit},\ A \subset [n] \setminus C \rangle}. \tag{5.1}$$

Our proof of Proposition 12 is based on an inductive dimension counting argument. We now sketch a proof of Proposition 12 using Theorem 3.

Proof sketch of Proposition 12. Let $\mathcal{H}'_Z$ be the ring on the right hand side of Equation (5.1). It is not too hard to show that the binomials $z_S z_T - z_{S\cup T} z_{S\cap T}$ and linear functions $f_C^A(z)$ vanish on $Y_Z \subseteq \mathbb{P}^{2^n-1}$. By Theorem 3, this implies that the harmonic algebra $\mathcal{H}_Z$ is a quotient of $\mathcal{H}'_Z$. To show that $\mathcal{H}_Z$ is in fact isomorphic to $\mathcal{H}'_Z$, we prove that, for all $i \geq 0$,

$$\dim_{\mathbb{C}}(\mathcal{H}'_Z)_{(i,*)} = m^d T_M\left(\frac{m+1}{m}, 1\right) = i_Z(m;1) = \dim_{\mathbb{C}}(\mathcal{H}_Z)_{(i,*)} \tag{5.2}$$

where $(\mathcal{H}_Z)_{(i,*)}$ and $(\mathcal{H}'_Z)_{(i,*)}$ are the subspaces of $\mathcal{H}_Z$ and $\mathcal{H}'_Z$ consisting of all graded pieces of t-degree i and arbitrary q-degree. We prove the first equality of Equation (5.2) by first reducing to the $m = 1$ case and then applying an intricate deletion-contraction argument. The second equality of Equation (5.2) is the $q = 1$ evaluation of Theorem 1 and the third equality follows from the definition of the harmonic algebra. □

6 Zonotopal algebras and Theorems 1 and 3

Let $Z = A \cdot [0,1]^n \subseteq \mathbb{R}^d$ be a unimodular zonotope and let $L = \mathrm{rowspace}(A) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathbb{C}^n$. We now define the internal and external zonotopal algebra of Z. Given an element $v \in L$, write $m(v)$ for the number of coordinate functions of $\mathbb{C}^n$ which are nonzero on v. Let $R = \mathbb{C}[r_1, \ldots, r_d]$ be the polynomial ring over $\mathbb{C}$ whose generators are the rows of A. Any element $v \in L$ can be thought of as a degree one homogeneous polynomial in R. In the following, we make no distinction between v and its associated polynomial in R.

Definition 13. *Define the ideals J_Z^{ext} and J_Z^{int} of R as*

$$J_Z^{ext} := J_{Z,1} = \langle v^{m(v)+1} | 0 \neq v \in L \rangle \quad \text{and} \quad J_Z^{int} := J_{Z,-1} = \langle v^{m(v)-1} | 0 \neq v \in L \rangle.$$

The external and internal zonotopal algebras are the quotients $R_Z^{ext} = R/J_Z^{ext}$ and $R_Z^{int} = R/J_Z^{int}$.

We remark to the reader that, although it does not appear in our present work, the analogously defined quotient $R/J_{Z,0}$ plays a prominent role in the theory of zonotopal algebras and is known as the *central zonotopal algebra* [11, 8, 15].

Zonotopal algebras are well studied rings originating from approximation theory [9] and with rich connections to geometry and combinatorics [11, 2, 15]. The following proposition shows that zonotopal algebras naturally arise as orbit harmonics rings.

Proposition 14. *The external zonotopal algebra R_Z^{ext} is isomorphic to $Orb(Z \cap \mathbb{Z}^d)$. The internal zonotopal algebra R_Z^{int} is isomorphic to $Orb(int(Z) \cap \mathbb{Z}^d)$.*

Proposition 14 was first proved in [15] for graphical zonotopes, in [6] for the internal case and is implicit in the main theorems of [11]. In our upcoming preprint [7], we give a short proof of Proposition 14 in its full generality.

Proposition 14 is one of our two main inputs to the proofs of Theorems 1 and 3. The second input is the observation that the dilation of a unimodular zonotope is also a unimodular zonotope. For any integer $m \geq 1$, let $A(m)$ be the $d \times (m \cdot n)$ matrix formed by repeating the columns of A m times. We call $A(m)$ the m-thickening of A. The dilate mZ is equal to the unimodular zonotope $A(m) \cdot [0,1]^{m \cdot n} \subseteq \mathbb{R}^d$.

Proof of Theorem 1. The Hilbert series of zonotopal algebras are well known. For example, it is shown in [2, 11] that the Hilbert series of R_Z^{ext} and R_Z^{int} are

$$\mathrm{Hilb}(R_Z^{\text{ext}}; q) = q^{n-d} T_M(1+q, q^{-1}) \quad \text{and} \quad \mathrm{Hilb}(R_Z^{\text{int}}; q) = q^{n-d} T_M(0, q^{-1}).$$

If $M(m)$ is the matroid of $A(m)$, then Proposition 14 tells us that

$$i_Z(m;q) = q^{nm-d} T_{M(m)}(1+q, q^{-1}) \quad \text{and} \quad \widetilde{i_Z}(m;q) = q^{nm-d} T_{M(m)}(0, q^{-1}). \tag{6.1}$$

In work of Jaeger, Vertigan and Welsh [13], it is shown that

$$T_{M(m)}(x,y) = [m]_y^d T_M\left(\frac{y[m-1]_y + x}{[m]_y}, y^m\right). \tag{6.2}$$

Combining Equations 6.1 and 6.2 yields the desired equality. □

Proof sketch of Theorem 3. We start with a geometric observation. As Y_Z is a multiplicity free subvariety [1, Theorem 1.3.c], it is projectively normal under any embedding of $(\mathbb{P}^1)^n$ into a projective space [5, Theorem 1]. This means that the homogeneous coordinate ring of Y_Z under the Segre embedding is the section ring of the sheaf $\mathcal{O}_{Y_Z}(1,\ldots,1)$ obtained by restricting $\mathcal{O}_{(\mathbb{P}^1)^n}(1,\ldots,1)$ to Y_Z [5, Theorem 1]. For any integer $m \geq 0$, after de-homogenizing, we can identify $H^0((\mathbb{P}^1)^n, \mathcal{O}(m,\ldots,m))$ with the vector subspace $C_m \subseteq \mathbb{C}[x_1,\ldots,x_n]$ spanned by all monomials whose multidegree is less than $(m,\ldots,m)$. Furthermore, the surjection $H^0((\mathbb{P}^1)^n, \mathcal{O}(m,\ldots,m)) \to H^0(Y_Z, \mathcal{O}(m,\ldots,m))$ can be identified with the restriction of the quotient $\mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}[x_1,\ldots,x_n]/I_L$ to C_m. Call the image of C_m under this quotient C_{mZ}. Work of Ardila–Postnikov tells us that $C_{1 \cdot Z}$ is isomorphic to $(J_Z^{\text{ext}})^{\perp}$[2, Proposition 4.5(1)]. With some careful manipulation, we extend their result and show that C_{mZ} is $(J_{mZ}^{\text{ext}})^{\perp}$. By combining the identification $C_{mZ} \simeq (J_{mZ}^{\text{ext}})^{\perp}$ with Proposition 14 and checking that the product structures agree, we obtain our result. □

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