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Factorizations for the Iwahori–Hecke algebra

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Abstract. We initiate the theory of factorizations in the Type A Iwahori–Hecke algebra, and prove q-analogues of the famous *Harer–Zagier* formula, as well as generalizations of classical factorization problems for the long-cycle due to Bédard, Goulden, Goupil, Hurwitz, Jackson, Matsumoto, Novak, and others. Our work reframes the notion of factorization in terms of certain structure constants in the Hecke algebra center, and provides a uniform method of approaching such problems. Our formulas reveal significant combinatorial structure in the Hecke algebra center, including the appearance of q-binomial, q-Catalan, and q-Narayana numbers.

Keywords: Hecke algebra, Harer–Zagier formula, Hurwitz numbers, Jucys–Murphy elements, symmetric functions, permutation factorization, q-analogues, long cycle

1 Introduction

Given a permutation σ in the symmetric group $\mathfrak{S}_n$, there is an extensive and active body of research (see e.g. [2, 3, 7, 10, 14]) that studies the question: how many different ways can σ be written as a product of permutations satisfying a given condition? This is called a *factorization* of σ. Permutation factorizations have diverse applications, related to algebraic geometry, random matrix theory and mathematical physics [8, 9, 13].

We will be interested in four of the most classical and beautiful results in this area:

Theorem 1. *Write $\mathsf{c}_n := (1\,2\,\ldots\,n) \in \mathfrak{S}_n$ as long-cycle and C_n as the n-th Catalan number.*

1. *(Harer–Zagier [12]) Let $\epsilon_g(n)$ be the number of factorizations of $\mathsf{c}_{2n} \in \mathfrak{S}_{2n}$ into a product $\mathsf{c}_{2n} = \sigma\pi$, where σ is a fixed-point-free involution and π has $n+1-2g$ cycles. Then,*

$$\epsilon_0(n) = C_n \quad \text{and} \qquad \sum_{g\geq 0} \epsilon_g(n) x^{n+1-2g} = (2n-1)!! \sum_{k=1}^{n+1} \binom{n}{k-1} 2^{k-1} \binom{x}{k}. \tag{1.1}$$

2. *(Hurwitz [13]) Let $a(n;k)$ be the number of ways to write c_n as a product of k transpositions. Then, $a(n;n-1) = n^{n-2}$.*

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3. *(Matsumoto–Novak [18]) Let $b(n;k)$ be the number of ways to write c_n as a product of k* monotone *transpositions; that is, as a product*

$$\mathsf{c}_n = (i_1\ j_1)(i_2\ j_2)\cdots(i_k\ j_k) \quad \textit{where} \quad i_\ell < j_\ell \quad \textit{and} \quad j_1 \le j_2 \le \cdots \le j_k.$$

Then, $b(n;n-1) = C_{n-1}$.

4. *(Goulden–Jackson [7]) Let $c(n;p_1,\dots,p_m)$ be the number of ways to write c_n as a product $\mathsf{c}_n = \pi_1\pi_2\cdots\pi_m$, where π_i has $n-p_i$ cycles, and suppose $p_1+\cdots+p_m = n-1$. Then,*

$$c(n;p_1,\dots,p_m) = \frac{1}{n}\prod_{i=1}^{m}\binom{n}{p_i}.$$

In particular, $c(n;k,n-k-1)$ is the Narayana number $N_{n,k+1} = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}$.

Note that the results in Theorem 1 (2), (3) and (4) are the leading terms of generating functions for more general factorization questions (see Theorem 13 specialized to $q=1$).

Due to their importance and ubiquity in the world of permutation factorizations, we shall refer to Theorem 1 (1) as the *Harer–Zagier formula* and Theorem 1 parts (2), (3), and (4) as the *A-B-C's of permutation factorizations*. These results have served as the inspiration point for a robust factorization theory, and have far-reaching applications in other fields. For example, the Harer–Zagier formula in (1.1) was used to compute the orbifold Euler characteristic of the moduli space of curves, and $\epsilon_g(n)$ also enumerates certain unicellular maps on a genus g surface. Meanwhile, the Hurwitz formula $a(n;n-1) = n^{n-2}$ counts certain genus zero ramified coverings of an n-sphere [7, 13, 17].

In this extended abstract, we generalize each part of Theorem 1 to the *Type A Iwahori–Hecke algebra $\mathcal{H}_n(q)$*, and more broadly, initiate a theory of permutation factorizations for $\mathcal{H}_n(q)$. The Hecke algebra is an important tool in a diverse range of fields from geometric representation theory and knot theory to quantum computing, and our work is part of a larger movement to develop its importance in combinatorics, e.g. [11, 20].

The Hecke algebra $\mathcal{H}_n(q)$ is an algebra rather than a group, and multiplication is more complicated. Thus, our first task is to rephrase the notion of "factorization" so that it makes sense in $\mathcal{H}_n(q)$, which we achieve by studying *structure constants* of a natural basis of the *center* of $\mathcal{H}_n(q)$. Our approach has the benefit of unifying the techniques needed to prove (and generalize) Theorem 1, paving the way for an $\mathcal{H}_n(q)$-factorization theory. Furthermore, the fact that the formulas in Theorem 1 have geometric and topological incarnations suggests that our Hecke analogues may as well, related for instance to point counts in varieties or knot invariants; see e.g. [1, 5, 19].

The remainder of the abstract proceeds as follows: Section 2 reviews the pertinent features of the center of the symmetric group algebra and of the Hecke algebra; Section 3 gives a Hecke analogue of the Harer–Zagier formula (Theorem 8), and Section 4 describes the A-B-C's of permutation factorizations for the Hecke algebra (Theorem 13), including generating functions for q-deformations of $a(n;k)$, $b(n;k)$ and $c(n;p_1,\cdots,p_n)$.

2 Centers of the symmetric group and Hecke algebra

The *Type A Iwahori–Hecke algebra* $\mathcal{H}_n(q)$ is a q-deformation of $\mathbb{Q}[\mathfrak{S}_n]$. Explicitly, it is the $\mathbb{Q}[q^{\pm}]$-algebra generated by $T_1, \ldots, T_{n-1}$ subject to the relations

$$(a)\ T_i^2 = (q-1)T_i + q, \quad (b)\ T_iT_j = T_jT_i \text{ for } |i-j| \geq 2, \quad (c)\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

Setting $q = 1$ recovers the presentation of $\mathbb{Q}[\mathfrak{S}_n]$ in terms of adjacent transpositions $s_1, \ldots, s_{n-1}$. Moreover, if $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ is a reduced (i.e. minimal) expression for w, then $T_w := T_{i_1} \cdots T_{i_k}$ is well-defined, and $\{T_w\}_{w \in \mathfrak{S}_n}$ forms a linear basis for $\mathcal{H}_n(q)$.

Note the relation (a) above is the only one that differs from the usual presentation of $\mathbb{Q}[\mathfrak{S}_n]$; this relation makes $\mathcal{H}_n(q)$ richer, yet more complicated. For instance, by Theorem 1 (2), the long cycle $\mathsf{c}_3 = (123)$ can be written as a product of two transpositions in $a(3,2) = 3^1$ possible ways. However, the corresponding products in $\mathcal{H}_n(q)$ give *linear combinations* of elements whenever the factorization is not a reduced expression for c_3:

$$\begin{array}{ccc} (23)(12) = \mathsf{c}_3 & (12)(13) = \mathsf{c}_3 & (13)(23) = \mathsf{c}_3 \\ T_{(23)}T_{(12)} = T_{\mathsf{c}_3} & T_{(12)}T_{(13)} = qT_{\mathsf{c}_3} + (q-1)T_{(13)} & T_{(13)}T_{(23)} = qT_{\mathsf{c}_3} + (q-1)T_{(13)}. \end{array}$$

This highlights the necessity of understanding factorization as an *algebraic*, rather than purely combinatorial, entity. To this end, we will study the *center* of the symmetric group algebra and of the Hecke algebra, denoted $Z_{\mathfrak{S}_n} := Z(\mathbb{Q}[\mathfrak{S}_n])$ and $Z_{\mathcal{H}_n(q)} := Z(\mathcal{H}_n(q))$.

In what follows, we will see that each part of Theorem 1 can be rephrased algebraically in terms of $Z_{\mathfrak{S}_n}$, and then deformed to $\mathcal{H}_n(q)$. Our proof methods will rely heavily on the properties of $Z_{\mathcal{H}_n(q)}$ established below. Throughout, we make use of the following standard q-analogues of the integers. For $j \in \mathbb{Z}$, and $n, k \in \mathbb{Z}_{\geq 0}$, define

$$[j]_q := \frac{q^j - 1}{q - 1}, \qquad [n]!_q := [n]_q\,[n-1]_q \cdots [1]_q\,, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q\ [n-k]!_q},$$

as well as the *q-Catalan numbers* $C_n(q)$ and the *q-Narayana numbers* $N_{n,k}(q)$ defined as

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \qquad \text{and} \qquad N_{n,k}(q) := \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

2.1 Center of the symmetric group algebra

The center of the symmetric group algebra, $Z_{\mathfrak{S}_n} := Z(\mathbb{Q}[\mathfrak{S}_n])$ is intimately connected to its representation theory. We review here the most pertinent aspects of this theory.

First, $Z_{\mathfrak{S}_n}$ has a natural *conjugacy class basis* $\{\mathfrak{C}_\mu\}_{\mu \vdash n}$, defined as follows. Let $\mathrm{cyc}(\sigma)$ be the cycle type of a permutation $\sigma \in \mathfrak{S}_n$. Then,

$$\mathfrak{C}_\mu := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{cyc}(\sigma) = \mu}} \sigma.$$

Second, $Z_{\mathfrak{S}_n}$ has a beautiful characterization due to Jucys in terms of *the ring of symmetric functions* and the *Jucys–Murphy elements*. Denote the former by Λ, with coefficients in $\mathbb{Q}$, and its well-known bases $e_\lambda, h_\lambda, p_\lambda$, and s_λ called the *elementary, homogeneous, power sum*, and *Schur basis*, respectively. The latter are elements $J_1, \ldots, J_n$ in $\mathbb{Q}[\mathfrak{S}_n]$ defined as

$$J_1 := 0 \quad \text{and} \quad J_k := (1\ k) + (2\ k) + \cdots + (k{-}1\ \ k) \text{ for } 1 < k \leq n.$$

Theorem 2 (Jucys [16])**.** *For a symmetric function $f \in \Lambda$, write $f(\Xi_n) := f(J_1, \ldots, J_n)$. Then an element $z \in \mathbb{Q}[\mathfrak{S}_n]$ is in $Z_{\mathfrak{S}_n}$ if and only if there exists an $f \in \Lambda$ such that $z = f(\Xi_n)$.*

Example 3. We have $e_1(\Xi_n) = \sum_{1 \leq i < j \leq n} (i\,j) = \mathfrak{C}_{2,1^{n-2}}$, and more generally by Jucys [16],

$$e_k(\Xi_n) = \sum_{\substack{\lambda \vdash n \\ k = n - \ell(\lambda)}} \mathfrak{C}_\lambda. \tag{2.1}$$

Using the expression $E_n(x) := \prod_{i=1}^{\infty}(t + x_i) = \sum_k e_k(x) t^k$, one can equivalently write

$$E_n(\Xi_n) = \prod_{k=1}^{n}(t + J_k) = \sum_{\lambda \vdash n} \mathfrak{C}_\lambda \; t^{n-\ell(\lambda)}. \tag{2.2}$$

In addition to characterizing the center, the Jucys–Murphy elements play an important role in constructing the irreducible representations of $\mathfrak{S}_n$, called Specht modules. In particular, the J_k pairwise commute, and their simultaneous eigenspaces decompose $\mathbb{Q}[\mathfrak{S}_n]$. The projectors onto these eigenspaces are called *Young's seminormal forms*, and can be used to construct elegant Specht module bases.

The seminormal forms can be defined using Lagrange interpolation. While we omit their precise definition, we note several important properties. First, the seminormal forms are indexed by standard Young tableaux $T \in \mathsf{SYT}(n)$, and can thus be written as $\{\pi_T\}_{T \in \mathsf{SYT}(n)}$. Second, they are mutually orthogonal and idempotent, meaning that $\pi_T \pi_S = \delta_{S,T}\ \pi_T$. Third, the eigenvalue of J_k acting on π_T is the *content* of the box labeled k in T, defined as $\mathrm{cont}_k(T) := j - i$ if k in the (i,j)-th cell of T (in English notation).

Example 4. We have $J_1 = 0$, and $\mathrm{cont}_1(T) = 0$ for any $T \in \mathsf{SYT}(\lambda)$, agreeing with $J_1 \cdot \pi_T = 0$ for any T. Consider the tableaux of shape $(2,1)$. Letting

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \text{and} \qquad Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \qquad \text{we have}$$

$$\begin{aligned} J_2 \cdot \pi_T &= \mathrm{cont}_2(T) \cdot \pi_T = \mathbf{1} \cdot \pi_T & \qquad J_2 \cdot \pi_Q &= \mathrm{cont}_2(Q) \cdot \pi_Q = -\mathbf{1} \cdot \pi_Q \\ J_3 \cdot \pi_T &= \mathrm{cont}_3(T) \cdot \pi_T = -\mathbf{1} \cdot \pi_T & \qquad J_3 \cdot \pi_Q &= \mathrm{cont}_3(Q) \cdot \pi_Q = \mathbf{1} \cdot \pi_Q. \end{aligned}$$

Note that for both T and Q, the set of possible contents is $\{-1, 0, 1\}$. In general, any tableaux of the same shape will have the same (multi)set of possible contents.

Finally, the seminormal forms have the further useful property that

$$\sum_{T \in \mathsf{SYT}(n)} \pi_T = 1 \qquad \text{and} \qquad \sum_{T \in \mathsf{SYT}(\lambda)} \pi_T = \pi_\lambda, \tag{2.3}$$

where $\pi_\lambda \in Z_{\mathfrak{S}_n}$ is the canonical *isotypic projector* sending any $\mathfrak{S}_n$-representation V to the irreducible representations indexed by λ inside V. The π_λ are a basis for $Z_{\mathfrak{S}_n}$.

The relationship between $\mathfrak{C}_\mu$ and π_λ is encapsulated by the *Frobenius characteristic map* Ψ. Let $Z_{\mathfrak{S}} := \bigoplus_n Z_{\mathfrak{S}_n}$. There is an algebra isomorphism[1] $\Psi : Z_{\mathfrak{S}} \longrightarrow \Lambda$ sending

$$\begin{aligned} \Psi : \pi_\lambda &\mapsto \frac{f^\lambda}{n!}\, s_\lambda(x) \\ \mathfrak{C}_\mu &\mapsto z_\mu^{-1}\, p_\mu(x), \end{aligned}$$

where z_μ is the size of the conjugacy class indexed by μ and $f^\lambda = \#\mathsf{SYT}(\lambda)$.

2.2 Hecke algebra center

The center of $\mathcal{H}_n(q)$ has a parallel, though more complicated, story. The analog of the conjugacy class sum basis $\mathfrak{C}_\mu$ in $Z_{\mathfrak{S}_n}$ is called the *Geck–Rouquier basis* $\{\Gamma_\mu\}_{\mu\vdash n}$. Unlike the conjugacy class sums $\mathfrak{C}_\mu$ in $\mathbb{Q}[\mathfrak{S}_n]$, the Γ_μ are not simply sums over T_w for w of cycle type μ. Instead, [6] shows they are the unique family of elements in $Z_{\mathcal{H}_n(q)}$ such that

1. specializing Γ_μ to $q = 1$ gives $\Gamma_\mu|_{q=1} = \mathfrak{C}_\mu \in \mathbb{Q}[\mathfrak{S}_n]$;
2. if w has minimal length in its conjugacy class, the coefficient of T_w in Γ_μ is 1 if w has cycle type μ, and 0 if not.

Note that the above says nothing about the coefficient of w if it is *not* of minimal length. In fact, there is no known general expression for Γ_μ in the $\{T_w\}$ basis of $\mathcal{H}_n(q)$.

Example 5. Consider $\Gamma_{(3,1)}$ below, with minimal length elements colored in blue:

$$\begin{aligned} \Gamma_{(3,1)} = (1-q^{-1})\Big((q^{-1}-q^{-2})T_{(14)(23)} + q^{-1}T_{(1423)} + q^{-1}T_{(1324)} + 2q^{-1}T_{(14)} + T_{(13)} + T_{(24)}\Big) \\ + q^{-1}T_{(142)} + q^{-1}T_{(143)} + q^{-1}T_{(134)} + q^{-1}T_{(124)} + T_{(132)} + T_{(123)} + T_{(243)} + T_{(234)}. \end{aligned}$$

Despite a more mysterious basis, $Z_{\mathcal{H}_n(q)}$ still has a beautiful characterization mirroring Theorem 2, this time in terms of the *q-Jucys–Murphy elements*

$$J_1(q) := 0 \quad \text{and} \quad J_k(q) := q^{1-k}T_{(1\,k)} + q^{2-k}T_{(2\,k)} + \cdots + q^{-1}T_{(k-1\,k)} \text{ for } 1 < k \le n.$$

Theorem 6 (Francis–Wang [4]). *For $f \in \Lambda$, write $f(\Xi_n(q)) := f(J_1(q), J_2(q), \ldots, J_n(q))$. Then, an element $z \in \mathcal{H}_n(q)$ is in $Z_{\mathcal{H}_n(q)}$ if and only if there is an $f \in \Lambda$ such that*

$$z = f(\Xi_n(q)).$$

One now has a description of the elementary symmetric function $E_n(\Xi_n(q))$ due to [4]:

$$E_n(\Xi_n(q)) = \prod_{k=1}^{n}\big(t + J_k(q)\big) = \sum_{\lambda\vdash n} \Gamma_\lambda\, q^{-(n-\ell(\lambda))}\, t^{n-\ell(\lambda)}. \tag{2.4}$$

[1]Note that Ψ is an algebra isomorphism with respect to *outer* multiplication, rather than inner (Kronecker) multiplication, meaning that for $z_i \in Z_{\mathfrak{S}_{n_i}}$, we have $z_1 z_2 \in Z_{\mathfrak{S}_{n_1+n_2}}$.

As in the symmetric group algebra, the $J_k(q)$ pairwise commute. The projectors onto their simultaneous eigenspaces in $\mathcal{H}_n(q)$ yield an analog of the Young seminormal forms $\{\pi_T(q)\}_{T\in\mathsf{SYT}(n)}$, which can be used to construct bases of irreducible $\mathcal{H}_n(q)$ representations. The $\pi_T(q)$ are orthogonal idempotents, and the $J_k(q)$ act on them by the q-analogues of content:

$$J_k(q)\ \pi_T(q) = [\mathsf{cont}_k(T)]_q\ \pi_T. \tag{2.5}$$

One can again sum the $\pi_T(q)$ to obtain the identities

$$\sum_{T\in\mathsf{SYT}(n)} \pi_T(q) = 1 \qquad \text{and} \qquad \sum_{T\in\mathsf{SYT}(\lambda)} \pi_T(q) = \pi_\lambda(q), \tag{2.6}$$

where the $\pi_\lambda(q)$ are $\mathcal{H}_n(q)$-isotypic projectors, and once more form a basis for $Z_{\mathcal{H}_n(q)}$.

The relationship between the Γ_μ and $\pi_\lambda(q)$ can be formalized using the *q-Frobenius map* $\Psi_q : Z_{\mathcal{H}(q)} \to \Lambda[q^{\pm}]$, where $Z_{\mathcal{H}(q)} := \bigoplus_n Z_{\mathcal{H}_n(q)}$; see Wan–Wang [21]. This map is an algebra isomorphism defined by:

$$\Psi_q : \pi_\lambda(q) \mapsto \frac{f^\lambda(q)}{[n]!_q}\ s_\lambda(x) \tag{2.7}$$

$$\Gamma_\mu \mapsto (q-1)^{\ell(\mu)}\ q^{n-\ell(\mu)}\ m_\mu\left[\tfrac{x}{q-1}\right], \tag{2.8}$$

where $m_\mu\left[\frac{x}{q-1}\right]$ is *plethystic notation* and $f^\lambda(q) = \dfrac{q^{b(\lambda)}[n]!_q}{\prod_{(i,j)\in\lambda}[h(i,j)]_q}$. Here, $h(i,j)$ is the hook length of the cell (i,j) in λ and $b(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$. Though not obvious, specializing (2.8) to $q=1$ recovers $\Psi(\mathfrak{C}_\mu) = z_\mu^{-1} p_\mu$ [21].

For us, one of the primary uses of the $\pi_T(q)$ is their interaction with symmetric functions. Given a partition λ, let $\mathsf{qcont}(\lambda)$ be the multi-set consisting of the q-contents of the boxes in λ; for instance, the hook $\lambda = (n-k, 1^k)$ has $\mathsf{qcont}(n-k,1^k) = \{\!\{[j]_q\}\!\}_{j\in[-k,n-k-1]}$.

Using the properties of the $\pi_T(q)$ in (2.5) and (2.6), one can show that for $f \in \Lambda$,

$$\pi_\lambda(q)\ f(\Xi_n(q)) = f(\mathsf{qcont}(\lambda)). \tag{2.9}$$

In what follows, we consider the case $f = E_n(x)$ and $\lambda = (n-k, 1^k)$, which yields

$$\pi_{(n-k,1^k)}(q)\ E_n\big(\Xi_n(q)\big) = \prod_{j\in[-k,n-k-1]} (t + [j]_q). \tag{2.10}$$

3 The Harer–Zagier formula for the Hecke algebra

Let us now revisit the *Harer–Zagier formula* in (1.1), which gives a generating function for $\epsilon_g(n)$. To obtain a deformation of (1.1), we will rephrase the original factorization problem as a computation in $Z_{\mathfrak{S}_n}$. Recall that $\epsilon_g(n)$ counts the number of ways to obtain

c_{2n} as a product of $\sigma\pi$ where $\mathrm{cyc}(\pi)$ has length $n+1-2g$ and $\mathrm{cyc}(\sigma) = 2^n$. In fact, this is equivalent to determining the coefficient of c_{2n} in the product:

$$[\mathsf{c}_{2n}]\ \mathfrak{C}_{2^n}\left(\sum_{\ell(\lambda)=n+1-2g}\mathfrak{C}_\lambda\right) = [\mathfrak{C}_{2n}]\mathfrak{C}_{2^n}\left(\sum_{\ell(\lambda)=n+1-2g}\mathfrak{C}_\lambda\right),$$

where $[y]x$ indicates the coefficient of y in x. Hence, one obtains the generating function

$$\sum_{h\geq 0}\epsilon_{h/2}(n)\ t^{n+1-h} = [\mathfrak{C}_{2n}]\ \mathfrak{C}_{2^n}\cdot\left(\sum_{\lambda\vdash 2n}\mathfrak{C}_\lambda\ t^{2n-\ell(\lambda)}\right) = [\mathsf{c}_{2n}]\ \mathfrak{C}_{2^n}\ E_{2n}(\Xi_{2n}). \tag{3.1}$$

Equation (3.1) is readily generalizable to $\mathcal{H}_n(q)$, where we substitute Γ_λ for $\mathfrak{C}_\lambda$, and consider the coefficient of the element $T_{\mathsf{c}_{2n}} := T_{2n-1}\cdots T_2T_1 = T_{(1\,2\,\ldots\,2n)}$. Note that since c_{2n} has minimal length in its conjugacy class, the following are equivalent:

$$\sum_{h\geq 0}\epsilon_{h/2}(n;q)t^{n+1-h} := [\Gamma_{2n}]\Gamma_{2^n}\left(\sum_{\lambda\vdash 2n}\Gamma_\lambda t^{2n-\ell(\lambda)}q^{\ell(\lambda)-2n}\right) = [T_{\mathsf{c}_{2n}}]\ \Gamma_{2^n}E_{2n}\big(\Xi_{2n}(q)\big). \tag{3.2}$$

For simplicity, write $P_n(t;q) := \sum_{h\geq 0}\epsilon_{h/2}(n;q)t^{n+1-h}$.

Example 7. Consider the values of $P_n(t)$ and $P_n(t,q)$ for $n = 2,3$:

n	$P_n(t)$	$P_n(t,q)$
2	$2t^3+t$	$\frac{(q^2+1)}{q^2}t^3 + \frac{(q-1)(q^2+1)(q+1)^2}{q^4}t^2 + \frac{(q^6-q^3+1)}{q^5}t$
3	$5t^4+10t^2$	$\frac{[5]_q(q^2-q+1)}{q^5}t^4 + \frac{(q-1)[5]_q(q^4+2q^3+4q^2+2q+1)(q^2-q+1)}{q^8}t^3$ $+\frac{[5]_q(q^2-q+1)(q^8+2q^7+q^6-q^5-4q^4-q^3+q^2+2q+1)}{q^{10}}t^2 + \frac{(q-1)[5]_q[3]_q(q^8+1)}{q^{11}}t$

While the expressions for $P_n(t,q)$ specialize to $P_n(t)$ at $q=1$, it is not clear that they have an elegant closed form expression. Miraculously, as in the case of the symmetric group, the secret to finding the q-Harer–Zagier formula lies in changing bases from $\{t^k\}_k$ in $\mathbb{C}[t,q^{\pm}]$ to a q-analogue of the binomial basis $\binom{t}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(t-i)$, written as

$$\binom{t}{k}_q := \frac{1}{[k]!_q}\prod_{i=0}^{k-1}\left(t-q^{-i}\,[i]_q\right) = \frac{1}{[k]!_q}\prod_{i=0}^{k-1}\left(t+[-i]_q\right).$$

Theorem 8 (q-Harer–Zagier formula)**.** *We have* $\epsilon_0(n;q) = q^{-(n-1)(n+2)/2}C_n(q)$ *and*

$$P_n(t;q) = [2n-1]!!_q\sum_{k=1}^{n+1}q^{3\binom{k-1}{2}-(n-1)(2k-1)}\begin{bmatrix}n\\k-1\end{bmatrix}_q\prod_{j=0}^{k-2}(1+q^{n-j})\binom{t}{k}_q. \tag{3.3}$$

Example 9. We can rewrite $P_3(t;q)$ above using Theorem 8:

$$P_3(t,q) = \frac{q^2+1}{q^2}\binom{t}{3}_q + \frac{(q^2+1)\ (q^2+q+1)}{q^3}\binom{t}{2}_q + \frac{q^2+q+1}{q}\binom{t}{1}_q.$$

3.1 Key ideas in the proof of Theorem 8

3.1.1 Tool 1: Jucys–Murphy elements and the seminormal forms

Our first step uses the properties of seminormal forms, as well as known character formulas for T_{c_n} by Ram [20]. It will be important in proving both Theorems 8 and 13.

Lemma 10. *Let $f \in \Lambda$ and $z \in Z_{\mathcal{H}_n(q)}$ written $z = \sum_\lambda c_\lambda \pi_\lambda(q)$ for scalars $c_\lambda \in \mathbb{Q}[q^{\pm}]$. Then*

$$[T_{\mathsf{c}_n}]\left(z \cdot f\big(\Xi_n(q)\big)\right) = \sum_{k=0}^{n-1} \frac{(-1)^k q^{\binom{k}{2}}}{[n]!_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q c_{(n-k,1^k)}\, f\big(\mathsf{qcont}_{(n-k,1^k)}\big).$$

We apply Theorem 10 by setting $z = \Gamma_{a^n}$ and $f = E_{an}(x)$. Write any Geck–Rouquier basis element as $\Gamma_\mu = \sum_\lambda \mathbb{B}_{\mu,\lambda}\pi_\lambda(q)$ for $\mathbb{B}_{\mu,\lambda} \in \mathbb{Q}[q^{\pm}]$. Then by Theorem 10 and (2.10),

$$[T_{(12\cdots n)}]\ \Gamma_{a^n}\ E_{an}\big(\Xi_{an}(q)\big) = \sum_{k=0}^{an-1} \frac{(-1)^k q^{\binom{k}{2}}}{[an]!_q} \begin{bmatrix} an-1 \\ k \end{bmatrix}_q \mathbb{B}_{(a^n),(an-k,1^k)} \Bigg(\prod_{j \in [-k, an-k]} (t + [j]_q) \Bigg).$$

It remains to understand the "unknown" term $\mathbb{B}_{(a^n),(an-k,1^k)}$.

3.1.2 Tool 2: Plethysm and the q-Frobenius map Ψ_q

The next step is to study the change-of-basis coefficients $\mathbb{B}_{\mu,\lambda}$ between the isotypic projector and the Geck–Rouquier bases of $Z_{\mathcal{H}_n(q)}$. In general, this is a difficult problem, equivalent to computing the coefficient of $s_\lambda(x)$ in a plethysm. More precisely,

$$\mathbb{B}_{\mu,\lambda} = [s_\lambda(x)] \frac{[n]!_q}{f^\lambda(q)} q^{n-\ell(\mu)} (q-1)^{\ell(\mu)} m_\mu \left[\frac{x}{q-1}\right].$$

However, these coefficients are surprisingly elegant in the case relevant for Theorem 8:

Lemma 11. *Let $\Gamma_\mu = \sum_\lambda \mathbb{B}_{\mu,\lambda}\pi_\lambda(q)$ for $\mathbb{B}_{\mu,\lambda} \in \mathbb{Q}[q^{\pm}]$. Then for $a \geq 1$ and $an \geq k$,*

$$\mathbb{B}_{(a^n),(an-k,1^k)} = q^{(a-1)n} \prod_{\substack{j \in [-k, an-k] \\ j \not\equiv 0 \bmod a}} [j]_q.$$

Proof idea. Using the map Ψ_q from (2.8), the crucial step is to prove the following:

$$s_{(k+1,1^{an-k-1})} * m_{a^n} = (-1)^{(a-1)n+k+\lfloor k/a \rfloor} s_{(n-\lfloor k/a \rfloor, 1^{\lfloor k/a \rfloor})}[p_a], \tag{3.4}$$

where the left-hand-side is the Kronecker product and the right-hand-side is plethysm of symmetric functions. Next, study the plethysm with $\frac{1}{q-1}$ on the right-hand-side. □

3.1.3 Tool 3: Hypergeometric series

Theorems 10 and 11 reduce the computation of $[T_{\mathsf{c}_{2n}}]\Gamma_{a^n}E_{an}(\Xi_{an}(q))$ to an expression in $\mathbb{Q}[q^{\pm}, t]$, but it still looks quite different from the formula in Theorem 8. The last step of our argument specializes to $a = 2$ and uses hypergeometric series to prove the claim.

4 A-B-C's of factorization for the Hecke algebra

Having reinterpreted the Harer–Zagier formula as a computation in $Z_{\mathfrak{S}_n}$, we now wish to do the same with the A-B-C's of permutation factorizations.

A: Recall from Theorem 3 that $e_1(\Xi_n) = \mathfrak{C}_{2,1^{n-2}} = \sum_{1\le i<j\le n}(i\ j)$. Consider

$$e_1^k(\Xi_n) = \big(e_1(\Xi_n)\big)^k = \Big(\sum_{1\le i<j\le n}(i\ j)\Big)^k = \mathfrak{C}^k_{2,1^{n-2}}. \tag{4.1}$$

Since every term in (4.1) is a product of k transpositions, the coefficient of c_n is precisely number of ways to obtain c_n as a product of k transpositions—in other words, $a(n;k)$. Setting $k = n-1$ recovers $a(n;n-1)$ from Theorem 1 (2).

B: Now consider $f = h_k$. We have

$$h_k(\Xi_n) = \sum_{\substack{2\le t_1\le\cdots\le t_k\le n\\ s_1<t_1,\dots,s_k<t_k}} (s_1\, t_1)\cdots(s_k\, t_k). \tag{4.2}$$

Thus again the coefficient of c_n in Equation (4.2) is precisely the number of ways to write c_n as a product of k transpositions $\tau_1\cdots\tau_k$ such that if $\tau_i = (s_i\ t_i)$ with $s_i < t_i$, then $t_1 \le t_2 \le \cdots \le t_k$. In other words, this coefficient is $b(n;k)$, the number of monotone factorizations of c_n into k transpositions.

C: Recall from (2.1) that $e_k(\Xi_n) = \sum_{\lambda: k=n-\ell(\lambda)} \mathfrak{C}_\lambda$. It follows that

$$e_{(p_1,\dots,p_m)}(\Xi_n) = e_{p_1}(\Xi_n)\, e_{p_2}(\Xi_n)\cdots e_{p_m}(\Xi_n) \tag{4.3}$$
$$= \sum_{\substack{\mu^{(i)}\vdash n\\ \ell(\mu^{(i)})=n-p_i}} \mathfrak{C}_{\mu^{(1)}}\cdots\mathfrak{C}_{\mu^{(m)}}. \tag{4.4}$$

Since the sum is over all partitions $\mu^{(i)}$ of n of length $n-p_i$, the coefficient of c_n in Equation (4.4) is precisely $c(n;p_1,\dots,p_m)$, the number of ways to write c_n as a product of permutations $\pi_1\cdots\pi_m$ where π_i has $n-p_i$ cycles.

Once more, the above quantities are entirely algebraic, and thus have a natural analogue in $\mathcal{H}_n(q)$, where we again consider the coefficient of T_{c_n}, or equivalently, Γ_n:

$$a_q(n;k) := [T_{\mathsf{c}_n}]\ e_{1^k}\big(\Xi(q)\big) = [T_{\mathsf{c}_n}]\ \Big(\sum_{1\le i<j\le n} q^{i-j}T_{(i\,j)}\Big)^k = [\Gamma_n]\, q^{-k}(\Gamma_{2,1^{n-2}})^k,$$

$$b_q(n;k) := [T_{\mathsf{c}_n}]\ h_k\big(\Xi_n(q)\big) = [T_{\mathsf{c}_n}] \sum_{\substack{2\le t_1\le\cdots\le t_k\le n\\ s_1<t_1,\dots,s_k<t_k}} q^{\sum_i(s_i-t_i)}T_{(s_1\,t_1)}\cdots T_{(s_k\,t_k)},$$

$$c_q(n;p_1,\dots,p_m) := [T_{\mathsf{c}_n}]\ e_{(p_1,\dots,p_m)}\big(\Xi_n(q)\big) = [\Gamma_n] \sum_{\substack{\mu^{(i)}\vdash n\\ \ell(\mu^{(i)})=n-p_i}} q^{-(p_1+\cdots+p_m)}\,\Gamma_{\mu^{(1)}}\cdots\Gamma_{\mu^{(m)}}.$$

Example 12. While $f(\Xi_n(q))$ will specialize to $f(\Xi_n)$ at $q = 1$, it is again not clear a priori that it will have a positive or closed form expression. However if we consider

$$\begin{aligned} e_1(\Xi_3(q))^2 = 3q^{-1}T_{(1)(2)(3)} - (2q^{-2} - 2q^{-1})T_{(12)} - (q^{-4} + 2q^{-3} - 2q^{-2} - q^{-1})T_{(13)} \\ + (q^{-3} + q^{-2} + q^{-1})T_{(132)} + (q^{-3} + q^{-2} + q^{-1})T_{(123)}, \end{aligned}$$

we see the coefficient of $T_{(123)}$ in $e_1(\Xi_3(q))^2$ is $q^{-3}[3]_q$. Note also that there are unexpected contributions to this coefficient. For instance, $q^{-4}T^2_{(13)}$ contributes $q^{-3} - 2q^{-2} + q^{-1}$.

We thus obtain an *A-B-C Theorem of factorization* for $\mathcal{H}_n(q)$.

Theorem 13 (A-B-C's of $\mathcal{H}_n(q)$)**.** *For $n \geq 1$, the $a_q(n; j), b_q(n; j)$ and $c_q(n; p_1, \ldots, p_m)$ satisfy:*

A: $a_q(n; n-1) = q^{-\binom{n}{2}}\,[n]_q^{n-2}$, and more generally

$$\sum_{j\geq 0} a_q(n; j)\, t^j = \frac{q^{-\binom{n}{2}}\ [n]_q^{n-2}\ t^{n-1}}{\prod_{j=0}^{n-1}\left(1 - \left(\frac{n - q^{-j}[n]_q}{1-q}\right) t\right)};$$

B: $b_q(n; n-1) = q^{-\binom{n}{2}}\,C_{n-1}(q)$, and more generally,

$$\sum_{j\geq 0} b_q(n; j)\, t^j = \frac{q^{-\binom{n}{2}}\ C_{n-1}(q)\ t^{n-1}}{\prod_{j=-(n-1)}^{n-1}(1 - [j]_q\, t)};$$

C: $c_q(n; p_1, \ldots, p_m) = q^{\sum_i \binom{p_i}{2} - \binom{n}{2}} \dfrac{1}{[n]_q} \prod_i \begin{bmatrix} n \\ p_i \end{bmatrix}_q$ when $p_1 + \cdots + p_m = n - 1$, and thus

$$c_q(n; k, n-1-k) = q^{1-(k+1)(n-k)} N_{n,k+1}(q).$$

More generally

$$\sum_{0\leq p_1,\ldots,p_m\leq n-1} c_q(n; p_1, \ldots, p_m)\ t_1^{n-p_1} \cdots t_m^{n-p_m} =$$

$$= [n]!_q^{m-1} \sum_{0\leq r_1,\ldots,r_m\leq n-1} q^{-\sum_i (n-r_i) r_i}\ \mathsf{M}^{n-1}_{(r_1,\ldots,r_m)}(q) \binom{t_1}{n - r_1}_q \binom{t_2}{n - r_2}_q \cdots \binom{t_m}{n - r_m}_q,$$

where $\mathsf{M}^n_{\mathbf{r}}(q) \in \mathbb{Z}_{\geq 0}[q]$ counts certain tuples of subspaces of the finite vector space $\mathbb{F}_q^n$:

$$\mathsf{M}^n_{(r_1,\ldots,r_m)}(q) := \#\Big\{(V_1, \ldots, V_m) : V_i \subset \mathbb{F}_q^n \text{ with } \dim(V_i){=}r_i \text{ and } V_1{+}\cdots{+}V_m = \mathbb{F}_q^n\Big\}.$$

The generating functions in Theorem 13 generalize $q = 1$ results by Jackson [15, p. 368], Matsumoto–Novak [18, Thm. 2] and Jackson [15, Thm. 4.3], respectively.

Key ideas in proof. Again, the first key idea is to use Theorem 10, in the case that $z = 1$ and f is the relevant symmetric function from the definitions of $a_q(n; j), b_q(n; j)$ and $c_q(n; p_1, \ldots, p_m)$. One then uses clever q-manipulations to obtain the claims. $\square$

We use Theorem 13 (C) to provide an explicit link between the coefficient of T_{c_n} in $f(\Xi_n(q))$ and the q-principal specialization of f for *any* homogeneous symmetric function f. This extends a result of Irving from the symmetric group to the Hecke algebra.

Corollary 14. *Let f be a homogeneous symmetric function of degree $n-1$. Then*

$$[T_{\mathsf{c}_n}]\, f(\Xi_n(q)) \quad = \quad \frac{q^{-\binom{n}{2}}}{[n]_q} f(1, q, q^2, \ldots, q^{n-1}). \tag{4.5}$$

Thus, for any partition $\lambda \vdash n-1$, we have

$$[T_{\mathsf{c}_n}]\ h_\lambda(\Xi_n(q)) \quad = \quad \frac{q^{-\binom{n}{2}}}{[n]_q} \prod_{i=1}^{\ell(\lambda)} \begin{bmatrix} n+\lambda_i-1 \\ \lambda_i \end{bmatrix}_q, \tag{4.6}$$

$$[T_{\mathsf{c}_n}]\ p_\lambda(\Xi_n(q)) \quad = \quad \frac{q^{-\binom{n}{2}}}{[n]_q} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{n\lambda_i}}{1-q^{\lambda_i}}, \tag{4.7}$$

$$[T_{\mathsf{c}_n}]\ s_\lambda(\Xi_n(q)) \quad = \quad \frac{q^{b(\lambda)-\binom{n}{2}}}{[n]_q} \prod_{(i,j)\in\lambda} \frac{1-q^{n+j-i}}{1-q^{h(i,j)}}. \tag{4.8}$$

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