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The Chromatic Symmetric Function for Unicyclic Graphs

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Abstract. Motivated by the question of which structural properties of a graph can be recovered from the chromatic symmetric function (CSF), we study the CSF of connected unicyclic graphs. While it is known that there can be non-isomorphic unicyclic graphs with the same CSF, we find that in many cases we can recover data such as the number of leaves, number of internal edges, cycle size, and number of attached non-trivial trees, by extending known results for trees to unicyclic graphs. These results are obtained by analyzing the CSF of a connected unicyclic graph in the *star basis* using the deletion-near-contraction (DNC) relation developed by Aliste-Prieto, Orellana and Zamora, and computing the "leading" partition, as well as coefficients indexed by hook partitions. We also show that the cycle sizes of bicyclic graphs can be recovered, and we obtain explicit formulas for star-expansions of specific families of graphs.

Keywords: chromatic symmetric functions, unicyclic graphs, star basis

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1 Introduction

In [14] Richard Stanley introduced the *chromatic symmetric function* (CSF) for any simple graph G as a symmetric function generalization of the chromatic polynomial of a graph (see Sec. 2.2). A significant body of research on $\mathbf{X}_G$ has focused on Stanley's "tree isomorphism problem," which asks whether the chromatic symmetric function distinguishes non-isomorphic trees. While this question remains open in general, the chromatic symmetric function has been shown to distinguish various subclasses of trees (see, e.g., [2, 6, 9, 10, 12, 16]), as well as all trees on up to 29 vertices [8].

Many key structural properties of a graph can be recovered from $\mathbf{X}_G$ by examining its coefficients when expanded with respect to various bases of symmetric functions. For example, we can recover the number of vertices, edges, girth and number of connected components [12], the number of triangles in G [13], and the degree sequence and number of leaves for trees [12]. In particular, $\mathbf{X}_G$ can tell us whether G is a connected unicyclic graph [12]. Recently, the chromatic symmetric function has been studied [1, 7] in terms of the *star basis*, $\{\mathfrak{st}_\lambda \mid \lambda \vdash n\}$, where $\lambda \vdash n$ means λ is an integer partition of n.

When G is a tree, the authors of [7] study the expansion $\mathbf{X}_G$ in the star basis, $\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$. They find the *leading partition* λ_{lead} of $\mathbf{X}_G$, defined as the smallest partition in lexicographic order appearing in $\mathbf{X}_G$ with non-zero coefficient. The authors additionally provided formulas for the leading coefficient $c_{\lambda_{\text{lead}}}$, hook coefficients, and show that trees of small diameter are distinguished by their chromatic symmetric function.

In this manuscript, we provide analysis of the chromatic symmetric function of connected unicyclic graphs in the star basis. We determine some structural features of these graphs from specific coefficients in their star basis expansion. Our first main result, Theorem 3.1, provides an explicit formula for the coefficients c_λ in the star basis expansion when λ is a hook partition. As an immediate application, we show in Corollary 3.2 that the size of the cycle in any unicyclic graph can be directly determined from the coefficient $c_{(n)}$. The number of non-trivial rooted trees attached to the cycle can also be detected from hook coefficients, with one exception. Next, in Theorem 5.3, we determine the leading partition λ_{lead} for unicyclic graphs. Using λ_{lead}, we show in Corollary 5.4 how to compute the number of leaves in the graph when the number of attached trees is known. We also give formulas for the leading coefficient $c_{\lambda_{\text{lead}}}$ for any unicyclic graph.

With our results on the hook coefficients and the leading partition we distinguish certain classes of unicyclic graphs based on their CSF. In Corollary 3.3, we show that unicyclic graphs with different cycle sizes must have differing CSFs. Finally, in Proposition 5.5 we prove that *cuttlefish* graphs, which are a certain subclass of *squids*, are distinguished from among all connected unicyclic graphs.

In our full article [3], we also present computational data and examples of non-isomorphic graphs with a 4-cycle that share the same CSF. This demonstrates that for bipartite graphs, the CSF alone does not necessarily distinguish non-isomorphic graphs.

As a result, we establish that the tree isomorphism problem is independent of the bipartite property of trees.

Previous work has been done on computing the chromatic symmetric function of graphs with cycles. For example, see [5, 12, 13, 15]. We note that unicyclic graphs are not distinguished by their chromatic symmetric function [13, 14]. However, by analyzing the coefficients and partitions appearing in $\mathbf{X}_G$, we gain insight into when two unicyclic graphs can or cannot be isomorphic.

Towards a more general understanding of the CSF, in Section 6 we give an explicit formula for the coefficient $c_{(n)}$ in $\mathbf{X}_G$ when G is a connected bicyclic graph. This coefficient depends only on the sizes of the cycles in the graph and the number of edges shared by the two cycles. Using several results involving the sums of coefficients in $\mathbf{X}_G$ (see [3]) indexed by partitions of the same length, we can also recover the sizes of the cycles in connected bicyclic graphs.

2 Definitions and preliminaries

2.1 Graph theory

We will utilize standard definitions and fundamental concepts from graph theory. For an overview of graph theory background and notation, refer to, for example, [17].

We call a tree comprised of a single vertex a *trivial tree*, while any tree with more than one vertex is a *non-trivial tree*. A *unicyclic graph* is a graph that contains exactly one cycle. Connected unicyclic graphs can also be characterized as simple graphs whose number of edges equals the number of vertices. We consider connected unicyclic graphs G constructed from c rooted trees $T_1, \ldots, T_c$ by adding edges between the roots of consecutive trees T_i and T_{i+1} for $1 \leq i \leq c-1$, and connecting the root of T_c with the root of T_1, creating a cycle of size c in the graph G with vertices corresponding to the roots of the trees. We also call such a graph a *c-unicyclic* graph. We use r to denote the number of non-trivial rooted trees. For example, the graph on the left of Figure 1 is a unicyclic graph constructed from 4 rooted trees with roots v_1, v_2, v_3 and v_4 where v_4 is the root of a trivial tree. In this case, we have $r = 3$.

We next define specific types of vertices and edges in a graph that will be used throughout this manuscript. A *leaf* is a vertex of degree one, while an *internal vertex* has degree at least two. An *internal edge* is an edge such that both of its endpoints are internal vertices, while a *leaf-edge* is an edge such that at least one of its endpoints is a leaf. For a graph G, we denote by $I(G)$ the set of all internal edges of G. The connected components of the graph $G \setminus I(G)$ are referred to as the *leaf components*.

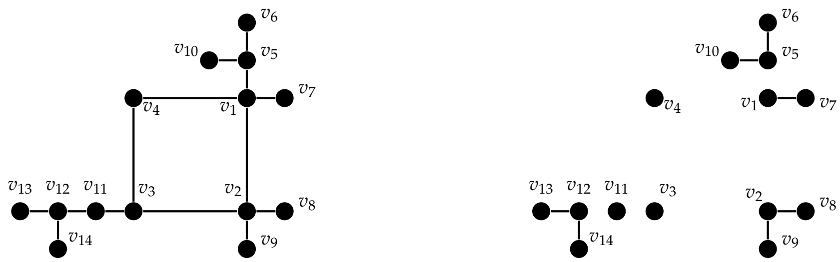


Figure 1: Left: A unicyclic graph G on 14 vertices; right: $G \setminus I(G)$.

2.2 The chromatic symmetric function

We refer to [11] for general background on symmetric functions. Let G be a finite graph with vertex set $V = \{v_1, \ldots, v_n\}$. A *proper coloring* of G is a function $\kappa : V \to \mathbb{N}$ such that $\kappa(v_i) \neq \kappa(v_j)$ whenever v_iv_j is an edge of G with $1 \leq i < j \leq n$. The *chromatic symmetric function* (CSF) [14] of G is defined by

$$\mathbf{X}_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)},$$

where the sum is over all proper colorings κ of G. The function $\mathbf{X}_G$ is a homogeneous symmetric function of degree n, where n is the order of G. In the case that G is a disjoint union of two subgraphs, $G = H_1 \sqcup H_2$, the chromatic symmetric function of G is $\mathbf{X}_G = \mathbf{X}_{H_1} \cdot \mathbf{X}_{H_2}$. In their work [4], Cho and van Willigenburg proved the following result, giving new bases for symmetric functions.

Theorem 2.1 ([4, Theorem 5])**.** *For any positive integer k, let G_k denote a connected graph with k vertices and let $\{G_k\}_{k\geq 1}$ be a family of such graphs. Given a partition $\lambda \vdash n$, of length ℓ, define $G_\lambda = G_{\lambda_1} \sqcup G_{\lambda_2} \sqcup \cdots \sqcup G_{\lambda_\ell}$. Then $\{\mathbf{X}_{G_\lambda} \mid \lambda \vdash n\}$ is a basis for Λ^n.*

We focus on the star basis of symmetric functions, which is constructed from the chromatic symmetric functions of star graphs. A *star graph* on k vertices, denoted by St_k, is a tree with $k-1$ vertices of degree 1 and one central vertex of degree $k-1$. We denote the chromatic symmetric function of St_k by $\mathfrak{st}_k := \mathbf{X}_{\mathrm{St}_k}$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, the chromatic symmetric function of the star forest $\mathrm{St}_{\lambda_1} \sqcup \mathrm{St}_{\lambda_2} \sqcup \cdots \sqcup \mathrm{St}_{\lambda_\ell}$ is denoted by

$$\mathfrak{st}_\lambda := \mathfrak{st}_{\lambda_1} \mathfrak{st}_{\lambda_2} \cdots \mathfrak{st}_{\lambda_\ell}.$$

By Theorem 2.1, the set $\{\mathfrak{st}_\lambda \mid \lambda \vdash n\}$ is a basis, called the *star basis*, for the space Λ^n. Hence, we may express $\mathbf{X}_G$ as:

$$\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda.$$

Definition 2.2. Order the partitions of n lexicographically in the order inherited from $\mathbb{N}^n$. Given a graph G, we define the *leading partition* to be the *smallest* partition in lexicographic order such that $c_{\lambda_{\text{lead}}} \neq 0$, where $\mathbf{X}_G = \sum_\lambda c_\lambda \mathfrak{st}_\lambda$ is the star-expansion of G. It will be denoted $\lambda_{\text{lead}}(\mathbf{X}_G)$ or simply λ_{lead} when the context is clear.

2.3 Deletion-near-contraction relation

We now recall the deletion-near-contraction relation of [1], which provides an algorithm for expressing the chromatic symmetric function of a graph in the star basis. Denote by $G \setminus e$ the graph obtained from G by deleting edge e. The *leaf-contraction* operation is denoted $G \odot e$ and yields the graph obtained by contracting edge e and attaching a new leaf to the vertex that results from this contraction, connected by a new leaf-edge ℓ_e. The *dot-contraction* $(G \odot e) \setminus \ell_e$ amounts to contracting edge e and then adding an isolated vertex. We note that if G contains a 3-cycle and e is an edge on the cycle, then $G \odot e$ and $(G \odot e) \setminus \ell_e$ will produce graphs with multiple edges. To ensure that all graphs are simple, we delete any duplicate edges in these cases.

Proposition 2.3 ([1])**.** *[The deletion-near-contraction (DNC) relation] Let G be a simple graph and e be any edge in G. Then*

$$\mathbf{X}_G = \mathbf{X}_{G\setminus e} - \mathbf{X}_{(G\odot e)\setminus \ell_e} + \mathbf{X}_{G\odot e}.$$

Observe that if e is a leaf-edge, then the DNC relation does not provide any new information for $\mathbf{X}_G$, as $G \setminus e \cong (G \odot e) \setminus \ell_e$ and $G \odot e \cong G$. Thus, we only apply the DNC relation to internal edges of G. Furthermore if e is an internal edge, then the graphs $G \setminus e$, $(G \odot e) \setminus \ell_e$, and $G \odot e$ each have fewer internal edges than G. Note that a simple graph with no internal edges is a star forest. Therefore, by repeatedly applying the DNC relation to internal edges, we can express $\mathbf{X}_G$ as a linear combination of $\mathfrak{st}_\lambda$, where λ is a partition of the number of vertices of G.

The star-expansion algorithm introduced in [1] formalizes the recursive construction of $\mathbf{X}_G$ in the star basis described above. It builds the so-called *DNC-tree* which is used to write the star-expansion of $\mathbf{X}_G$, as we recall now. Let $\iota(H)$ and $\iota(G)$ denote the number of isolated vertices of the graphs H and G, respectively. Let $\lambda(H)$ be the partition whose parts correspond to the orders of the connected components of H.

Theorem 2.4 ([1])**.** *For every simple graph G, let $\mathcal{T}(G)$ be a DNC-tree obtained from the star-expansion algorithm, and let $L(\mathcal{T}(G))$ be the multiset of leaf labels of $\mathcal{T}(G)$. Then*

$$\mathbf{X}_G = \sum_{H \in L(\mathcal{T}(G))} (-1)^{\iota(H)-\iota(G)} \mathfrak{st}_{\lambda(H)}.$$

Example 2.5. We give an example of the star-expansion algorithm in Figure 2 for the graph G at the top of the DNC-tree. Red indicates to which edge the DNC relation is applied at each step. The resulting CSF in the star basis is

$$\mathbf{X}_G = 2\mathfrak{st}_{(4)} - 2\mathfrak{st}_{(3,1)} + \mathfrak{st}_{(2,2)}.$$

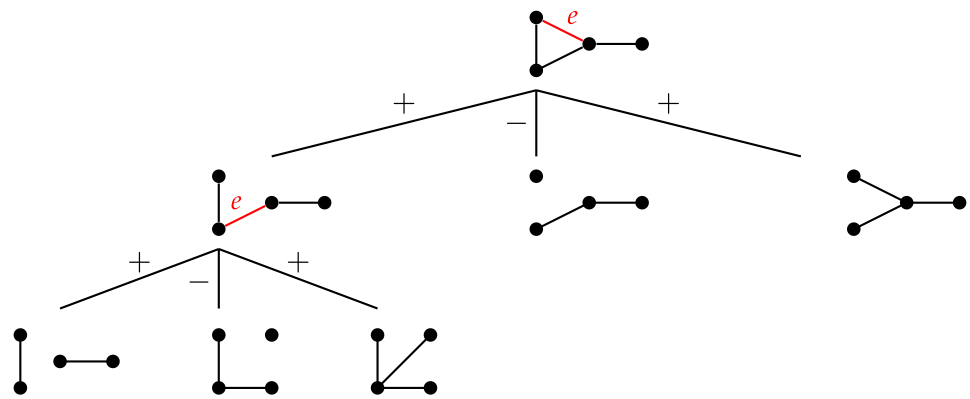


Figure 2: The DNC-tree $\mathcal{T}(G)$.

3 Coefficients for hook partitions

We give the coefficients for hook partitions for all connected unicyclic graphs. Our result relies on the formula of [7, Proposition 3.8] which gives $c_{(n-m_1,1^{m_1})} = (-1)^{m_1}\binom{k}{m_1}$ for T an n-vertex tree. Note that a unicyclic graph with cycle size c always has at least c internal edges. The following result is proved by induction on the size of the cycle.

Theorem 3.1. *Let G be a connected unicyclic graph on n vertices with a single cycle of size $c \geq 3$, let $\lambda = (n-m_1, 1^{m_1})$ be a hook partition, k be the number of internal edges of G, and $r \geq 0$ be the number of non-trivial rooted trees attached to the cycle in G. If $\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$, then*

$$c_{(n-m_1,1^{m_1})} = (-1)^{m_1}\left[(r-1)\binom{k-2}{m_1-1} + (c-1)\binom{k-2}{m_1}\right]. \tag{3.1}$$

We immediately obtain the following corollaries.

Corollary 3.2. *Let G be a connected unicyclic graph of order n with a single cycle of size $c \geq 3$. If $\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$, then*

$$c_{(n)} = c - 1.$$

Corollary 3.3. *If G_1 and G_2 are connected unicyclic graphs with cycles of different sizes, then $\mathbf{X}_{G_1} \neq \mathbf{X}_{G_2}$.*

In particular, the previous corollary shows that the star-expansion of the chromatic symmetric function of a unicyclic graph can be used to distinguish the cycle C_n from other connected unicyclic graphs of order n. The cycle will be the only connected unicyclic graph with $c_{(n)} = n - 1$.

The next proposition tells us that the longest hook partition encodes the relationship between the number of internal edges and rooted trees.

Proposition 3.4. *Let G be a connected unicyclic graph of order n, k internal edges, r rooted trees, and a cycle of size $c \geq 3$. If $\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$ and $\tilde{\lambda} = (n - m_1, 1^{m_1})$ is the longest hook partition such that $c_{\tilde{\lambda}} \neq 0$, then*

$$k = \begin{cases} m_1 + 2 & \text{if } r = 0 \text{ or } 1, \\ m_1 + 1 & \text{if } r \geq 2. \end{cases}$$

Furthermore, the coefficient $c_{\tilde{\lambda}}$ corresponding to the longest hook partition is

$$c_{\tilde{\lambda}} = \begin{cases} (-1)^{k-2} & \text{if } r = 0, \\ (-1)^{k-2}(c-1) & \text{if } r = 1, \\ (-1)^{k-1}(r-1) & \text{if } r \geq 2. \end{cases}$$

Example 3.5. Let G be the graph in Figure 3. We have $\mathbf{X}_G = 3\,\mathfrak{st}_{(8)} - 10\,\mathfrak{st}_{(7,1)} + 4\,\mathfrak{st}_{(6,2)} + 12\,\mathfrak{st}_{(6,1,1)} + 2\mathfrak{st}_{(5,3)} - 10\,\mathfrak{st}_{(5,2,1)} - 6\,\mathfrak{st}_{(5,1,1,1)} + 1\,\mathfrak{st}_{(4,4)} - 5\,\mathfrak{st}_{(4,3,1)} + 2\,\mathfrak{st}_{(4,2,2)} + 8\,\mathfrak{st}_{(4,2,1,1)} + 1\,\mathfrak{st}_{(4,1,1,1,1)} + 2\,\mathfrak{st}_{(3,3,2)} + 2\,\mathfrak{st}_{(3,3,1,1)} - 4\,\mathfrak{st}_{(3,2,2,1)} - 2\,\mathfrak{st}_{(3,2,1,1,1)} + 1\,\mathfrak{st}_{(2,2,2,1,1)}$. The hook coefficients are highlighted in red and match Equation 3.1 as $n = 8$, $c = 4$, $k = 5$, and $r = 2$. Additionally, the longest hook partition is $\tilde{\lambda} = (4,1,1,1,1)$ and the corresponding coefficient is $c_{\tilde{\lambda}} = (-1)^{k-1}(r-1) = 1$.

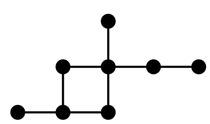


Figure 3: A unicyclic graph G on 8 vertices.

Corollary 3.6. *Let G_1 and G_2 be connected unicyclic graphs on n vertices with k_1 and k_2 internal edges and r_1 and r_2 rooted trees respectively. If $k_1 = k_2$ and $r_1 \neq r_2$ then $\mathbf{X}_{G_1} \neq \mathbf{X}_{G_2}$.*

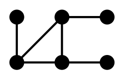
The coefficient $c_{(n-1,1)}$ can be used to write a quadratic equation for r or k, yielding two possible solutions. We note however, that hook coefficients alone are insufficient for recovering these values exactly. For example, see Figures 4 and 5.

Figure 4: Two graphs with the same hook coefficients: $c_{(8)} = 3$, $c_{(7,1)} = -9$, $c_{(6,1,1)} = 9$, $c_{(5,1^3)} = -3$, and $c_\lambda = 0$ for all other hooks λ.

On the other hand, there is an algorithm to obtain r from the hook coefficients when r is not 1 or c.

Corollary 3.7. *We can recover r, the number of non-trivial rooted trees attached to the cycle, from the CSF of a connected unicyclic graph whenever $1 < r < c$.*

Remark 3.8. Corollary 3.7 is the best we can do among unicyclic graphs, since there are unicyclic graphs with only one rooted tree attached to a cycle that have the same chromatic symmetric function as another that has c rooted trees attached to the cycle. The smallest example from an infinite family with this behavior appears below [13].



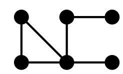


Figure 5: Two graphs with the same chromatic symmetric function where one has one nontrivial rooted tree and the other has three.

4 Cycles

We are able to explicitly compute the coefficients of all partitions in the star-expansions of paths, cycles, and cycles with a single leaf (also called *pan graphs*). These results are shown by analyzing words corresponding to paths in the DNC tree. For brevity, we only state the result on cycles here and refer to [3] for our full results.

Proposition 4.1. *Let C_n be the cycle on n vertices for $n \geq 3$, and let $\mathbf{X}_{C_n} = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$. Then for $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$ we have*

$$c_\lambda = (-1)^{m_1} \begin{cases} -\binom{n-2}{m_1-1} + (n-1)\binom{n-2}{m_1} & \text{if } \lambda = (n-m_1, 1^{m_1}),\ m_1 \neq n-1, \\ \\ \frac{n}{\ell(\lambda)-m_1}\binom{m_2+\cdots+m_n}{m_2,\dots,m_n}\binom{n-\ell(\lambda)+m_1-1}{m_1} & \text{else.} \end{cases}$$

Example 4.2. The CSF of the 5-cycle C_5 is

$$4\,\mathfrak{st}_{(5)} - 11\,\mathfrak{st}_{(4,1)} + 5\,\mathfrak{st}_{(3,2)} + 9\,\mathfrak{st}_{(3,1,1)} - 5\,\mathfrak{st}_{(2,2,1)} - 1\,\mathfrak{st}_{(2,1,1,1)}.$$

We see that the leading partition of $\mathbf{X}_{C_n}$ is $(2, 1^{n-2})$, since the only smaller partition is (1^n) which is seen to have coefficient 0 by our formula. The coefficient of $(2, 1^{n-2})$ is always $(-1)^{n-2}$.

5 Leading partitions for unicyclic graphs

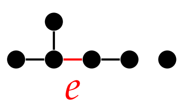
In this section, we describe the leading partition of the chromatic symmetric function of any connected unicyclic graph G on n vertices, extending a result of Gonzalez, Orellana, and Tomba [7, Theorem 4.15] that $\lambda_{\text{lead}}(\mathbf{X}_F) = \lambda_{\text{LC}}(F)$ for forests F. In [3], we also compute the coefficient of this leading partition in terms of the degrees of certain vertices, extending another result of [7].

In the case where there are no non-trivial rooted trees attached to the cycle (i.e., $r = 0$), the leading partition is $(2, 1^{n-2})$ as discussed in the previous section. Here we consider the case where $r \geq 1$. Recall that $I(G)$ is the set of internal edges of the graph G and the connected components of $G \setminus I(G)$ are called the *leaf components* of G.

Definition 5.1. Notice that $G \setminus I(G)$ is a star forest since every remaining edge is incident with a leaf. We define $\lambda_{\text{LC}}(G)$ to be the partition whose parts are the orders of the connected components of the resulting star forest in $G \setminus I(G)$.

Also recall that λ_{lead} was defined in Definition 2.2 as the smallest partition, λ, in lexicographic order such that $c_\lambda \neq 0$ in $\mathbf{X}_G$.

Example 5.2. Let F be the forest in Figure 6. After removing the only internal edge e indicated in red, we get $\lambda_{\text{lead}}(\mathbf{X}_F) = \lambda_{\text{LC}}(F) = (3, 2, 1)$.



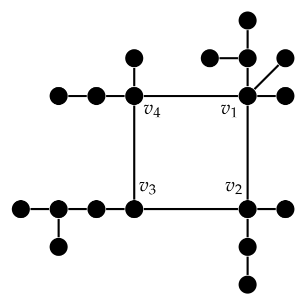


Figure 6: Left: a forest on 6 vertices; right: a graph with a 4-cycle.

Our main result for this section is the following.

Theorem 5.3. *Let G be a unicyclic graph and $r \geq 1$ denote the number of non-trivial rooted trees attached to the cycle in G. Let $\lambda_{\text{lead}} = \lambda_{\text{lead}}(\mathbf{X}_G)$.*

1. *If $r = 1$ then*
$$\lambda_{\text{lead}} = \lambda_{\text{LC}}(G \setminus e)$$
where e is chosen to be an edge in the cycle incident to the root of the non-trivial tree.

2. *If $r \geq 2$ then*
$$\lambda_{\text{lead}} = \lambda_{\text{LC}}(G).$$

The above result is true for all unicyclic graphs, though the proof proceeds by induction on the cycle size of a connected unicyclic graph. The analysis also allows us to deduce the following.

Corollary 5.4. *Let λ_{lead} be the leading partition in the star-expansion of $\mathbf{X}_G$ where G is a connected unicyclic graph with a cycle of size c. If G has exactly 1 non-trivial rooted tree, then the number of leaves in G is equal to $|\lambda_{\text{lead}}| - \ell(\lambda_{\text{lead}}) - 1$; if G has at least 2 non-trivial rooted trees, then the number of leaves in G is equal to $|\lambda_{\text{lead}}| - \ell(\lambda_{\text{lead}})$.*

5.1 Application to cuttlefish graphs

A *squid* is a connected unicyclic graph with exactly one vertex of degree greater than 2. Alternatively one can think of a squid as a unicyclic graph with 1 non-trivial rooted tree which consists of some number of paths (called *tentacles*) emanating from the root. We define *cuttlefish* to be squids such that all tentacles are of length 1.

Proposition 5.5. *The chromatic symmetric function can distinguish cuttlefish from among all connected unicyclic graphs on n vertices.*

Remark 5.6. Martin, Morin, and Wagner show in [12, Theorem 12] that no two squids have the same CSF. Note that while the class of squids is larger than that of cuttlefish, their result differs from the proposition above in that they do not distinguish squids from among all unicyclic graphs. On the other hand, both results implicitly give a procedure for reconstructing the graph from the CSF when the ambient class of graphs is known.

6 Bicyclic graphs

A simple connected graph G is said to be *bicyclic* if the number of edges is equal to the number of vertices plus one. Let $\mathcal{B}_n$ denote the set of connected bicyclic graphs of order n. For any $G \in \mathcal{B}_n$, the unique minimal bicyclic subgraph $\tilde{G}$ falls into one of two cases:

1. $\tilde{G}$ is constructed from two disjoint cycles C_s and C_t by adding a path P_h (with $h \geq 1$) from a vertex in C_s to any vertex in C_t. In the case where $h = 1$, we have two cycles that share a single vertex.

2. $\tilde{G}$ is constructed from three paths whose edge sets are disjoint, $P_{k+1}, P_{\ell+1}, P_{m+1}$, where $k, \ell, m \geq 1$ and at most one of them is equal to 1. The initial vertices of all three paths are identified as a single vertex u, and the terminal vertices are identified as a single vertex v. In other words, $\tilde{G}$ can be regarded as consisting of two cycles $C_{\ell+k}$ and $C_{\ell+m}$ which share a common path of $P_{\ell+1}$.

For any $G \in \mathcal{B}_n$, we say that G is a *type-I* (respectively, *type-II*) bicyclic graph if its unique minimal bicyclic subgraph is of the form of case (1) (respectively, case (2)) above. We illustrate these two types in Figure 7.

An inductive argument allows us to show the following.

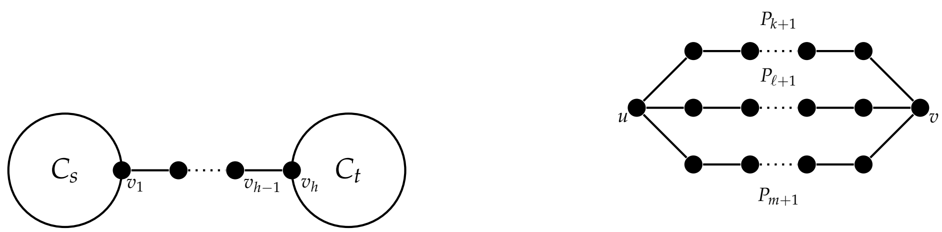


Figure 7: Left: a type-I bicyclic graph; right: a type-II bicyclic graph.

Proposition 6.1. *Let G be a bicyclic graph on n vertices.*

1. *If G is a type-I bicyclic graph formed by cycles of length $s, t \geq 3$, then*
$$c_{(n)} = (s-1)(t-1).$$
2. *If G is a type-II bicyclic graph formed by two cycles of length $s, t \geq 3$ which meet along a path $P_{\ell+1}$ for $\ell \geq 1$, then*
$$c_{(n)} = (s-1)(t-1) - 2\binom{\ell}{2}.$$
Note that when $\ell = 1$, the formula reduces to $c_{(n)} = (s-1)(t-1)$.

We are also able to recover the sizes of the two cycles in a bicyclic graph. In order to do this we need another equation in addition to $(s-1)(t-1) = c_{(n)}$ in type-I, and we need two additional equations in type-II as we have three unknowns. These come from looking at sums of coefficients corresponding to partitions of fixed length; full details are available in [3].

Theorem 6.2. *For any bicyclic graph G with cycle sizes $s \leq t$ meeting along $\ell < s/2$ edges, we can recover s and t from $\mathbf{X}_G$. If $\ell \geq 2$, we can also detect that G is of type-II and recover ℓ.*

We remark that it is impossible to distinguish between type-I bicyclic graphs and the $\ell = 1$ case of type-II bicyclic graphs. Indeed, the first example provided by Stanley of non-isomorphic graphs with the same chromatic symmetric function exhibits this difficulty [14] (see Figure 8). On the other hand, we have the following corollary.

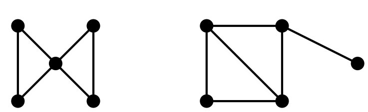


Figure 8: Two bicyclic graphs with identical chromatic symmetric functions

Corollary 6.3. *No bicyclic graph of type-I or of type-II with $\min\{k, \ell, m\} = 1$ can have the same CSF as a type-II bicyclic graph with $\min\{k, \ell, m\} \geq 2$ where k, ℓ, m are as in our definition of bicyclic graphs.*

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