Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #21, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Shuffle Tableaux, Littlewood–Richardson Coefficients, and Schur Log-Concavity

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Abstract. We give a new formula for the Littlewood–Richardson coefficients in terms of peelable tableaux compatible with shuffle tableaux, in the same fashion as Remmel–Whitney rule. This gives an efficient way to compute generalized Littlewood–Richardson coefficients for Temperley–Lieb immanants of Jacobi–Trudi matrices. We will also show that our rule behaves well with Bender–Knuth involutions, recovering the symmetry of Littlewood–Richardson coefficients. As an application, we use our rule to prove a special case of a Schur log-concavity conjecture by Lam–Postnikov–Pylyavskyy.

1 Introduction

The Littlewood–Richardson coefficients $c^{\nu}_{\lambda,\nu}$ are some of the most important structure constants in algebraic combinatorics. Since the Littlewood–Richardson rule was introduced [18], several variants of the rule have been formulated, including Knutson–Tao honeycombs [15], Zelevinsky pictures [28], Berenstein–Zelevinsky triangles [1] and their more general rule for any Lie algebra $\mathfrak{g}$ [3]. Another rule is Remmel–Whitney rule [24], which was first formulated using *compatible tableaux*. For simplicity, this rule can also be phrased in terms of counting Reiner–Shimozono *peelable tableaux* [23] as follows. Let D be the skew diagram of shape $\lambda * \mu = (\lambda_1 + \mu_1, \dots, \lambda_k + \mu_1, \mu_1, \dots, \mu_\ell)/(\mu_1^k)$. Visually, D is obtained by putting the Young diagram of λ to the top right of that of μ. For $1 \le i \le \ell(\nu) - 1$, let a_i be the number of columns with squares on both rows i and $i+1$. A semistandard Young tableau (SSYT) T is a D-peelable tableau if for all $1 \le i \le \ell(\nu) - 1$, there are at least a_i disjoint pairs of i-square and $(i+1)$-square such that the i-square is Northeast of the $(i+1)$-square. Then, $c^{\nu}_{\lambda,\mu}$ is the number of D-peelable tableaux of shape ν.

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1.1 Shuffle diagram, peelable tableaux, and Littlewood–Richardson coefficients

In this paper, we give a new formula for Littlewood–Richardson coefficients in terms of peelable tableaux. The key difference is that our rule uses the *shuffle diagram* of λ and μ instead of the skew diagram $\lambda * \mu$. In fact, our formula can be used to compute the Schur expansion of any product of two skew-Schur functions.

Definition 1.1. Given two skew shapes λ/μ and ν/ρ, the *shuffle diagram* D of shape $(\lambda/\mu) \circledast (\nu/\rho)$ is constructed as follows.

- If λ/μ has a square in position (i, j), then add into D a square in position $(2i - 1, 2j - 1)$.
- If ν/ρ has a square in position (i, j), then add into D a square in position $(2i, 2j)$.

Example 1.2. Let $\lambda/\mu = (3,2)/(1,0)$ and $(\nu/\rho) = (3,3)/(1,0)$. Then the shuffle diagram of shape $(\lambda/\mu) \circledast (\nu/\rho)$ is the right diagram in Figure 1.

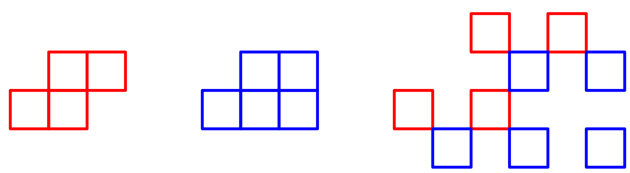


Figure 1: The Young diagram of $(3,2)/(1,0)$ (left) and $(3,3)/(1,0)$ (center) and their shuffle diagram (right)

Definition 1.3 (D-peelable tableaux)**.** Given two skew shape λ/μ and ν/ρ, we construct the shuffle diagram D of shape $(\lambda/\mu) \circledast (\nu/\rho)$ by interlacing the squares of λ/μ and ν/ρ (see Definition 1.1 and Example 1.2). Let a_i be the number of columns with squares on both rows i and $i + 2$. A SSYT T is a *D-peelable tableau* if for all i, there are at least a_i disjoint pairs of one i-square and one $(i + 2)$-square such that the i-square is Northeast of the $(i + 2)$-square. We call such pair a *NE matching of i and $i + 2$*.

Theorem 1.4. Let D be the shuffle diagram of shape $(\lambda/\mu) \circledast (\nu/\rho)$. For any κ, the Littlewood–Richardson coefficient $c^{\kappa}_{\lambda/\mu,\nu/\rho}$ counts the number of D-peelable tableaux of shape κ.

Example 1.5. Continuing Example 1.2, let D be the shuffle diagram in Figure 1. For a SSYT T to be D-peelable, we need one compatible pair of 1 and 3, and two compatible pairs of 2 and 4. Let $\kappa = (4, 3, 2)$, Figure 2 shows all three possible D-peelable tableaux of shape κ. Thus, $c^{\kappa}_{\lambda/\mu,\nu/\rho} = 3$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array}$$

Figure 2: D-peelable tableaux

The main tool for proving Theorem 1.4 is a bijection (Definition 2.4) between D-peelable tableaux and *Yamanouchi shuffle tableaux* of shape D, which was first introduced to study Temperley–Lieb immanants of Jacobi–Trudi matrices.

1.2 Temperley–Lieb immanants

In his groundbreaking 1990 paper [19] Lusztig has introduced *canonical bases* of quantum groups. They inspired several major developments in mathematics, such as Kashiwara crystal bases [13, 14] and Fomin–Zelevinsky cluster algebras [11, 2]. In type A, explicit formulas for dual canonical bases were given by Du [8, 9] and by Rhoades–Skandera [26, 27], where the coefficients are given as evaluations of Kazhdan–Lusztig polynomials.

Another deep positivity property of dual canonical bases in type A is Schur positivity: all of their elements evaluate to Schur positive results on generalized Jacobi–Trudi matrices. This was proved by Rhoades–Skandera in [26, Proposition 3], relying on Haiman's result [12, Theorem 1.5], which in turn relies on the (proofs of) Kazhdan–Lusztig conjecture by Beilinson–Bernstein [5] and Brylinski–Kashiwara [6]. However, no direct combinatorial proof was known, and there is no general interpretation for the coefficients of Schur functions appearing in the evaluation.

Recently, for a subset of dual canonical bases, called *Temperley–Lieb immanants*, introduced by Rhoades and Skandera in [25], Nguyen–Pylyavskyy [20] gave a combinatorial interpretation in terms of Yamanouchi shuffle tableaux. These Temperley–Lieb immanants have been used by Lam–Postnikov–Pylyavskyy [16] to prove several deep Schur-positivity conjectures.

However, the rule given in [20] was incomplete since they defined Yamanouchi shuffle tableaux as those on which no lowering crystal operators (see Definition 2.2) can be applied. Our peelable tableaux rule completes their story by giving a better description of Yamanouchi shuffle tableaux (see Corollary 3.3).

1.3 Bender–Knuth involutions and symmetry of Littlewood–Richardson coefficients

The Littlewood-Richardson coefficients also have an obvious symmetry: $c^{\kappa}_{\lambda/\mu,\nu/\rho} = c^{\kappa}_{\nu/\rho,\lambda/\mu}$. Nevertheless, this symmetry is not straightforward from either the classical rule or Remmel-Whitney rule. With our new rule, this symmetry is surprisingly easy to see.

Observe that if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)$ is the content of any $(\lambda/\mu) \circledast (\nu/\rho)$-peelable tableau, then $(\alpha_2, \alpha_1, \alpha_4, \alpha_3, \ldots)$ is the content of any $(\nu/\rho) \circledast (\lambda/\mu)$-peelable tableau. Thus, if one wants to find a (shape-preserving) bijection between $(\lambda/\mu) \circledast (\nu/\rho)$-peelable tableaux and $(\nu/\rho) \circledast (\lambda/\mu)$-peelable tableaux, one would look for a bijection that swaps the content α_1 and α_2, α_3 and α_4, and so on. The most natural operations that swap adjacent contents are the Bender–Knuth involutions (see Definition 4.1). Indeed, this is the desired bijection.

Theorem 1.6. Let T be a $(\lambda/\mu) \circledast (\nu/\rho)$-peelable tableau, and let $\ell = \ell(\lambda)$. Let

$$T' = \mathrm{BK}_1 \circ \mathrm{BK}_3 \circ \cdots \circ \mathrm{BK}_{2\ell-1}(T),$$

then T' is a $(\nu/\rho) \circledast (\lambda/\mu)$-peelable tableau. Note that T and T' have the same shape.

One application of Theorem 1.6 is that instead of using the shuffle diagram, one can use any "partial shuffle diagram" as in Figure 3. Thus, we have $2\ell(\lambda)$ sets of peelable tableaux giving the same Littlewood–Richardson coefficients. These $2\ell(\lambda)$ sets are all related to each other by a sequence of Bender–Knuth involutions. Furthermore, we recover Remmel–Whitney rule by using the rightmost shuffle diagram in Figure 3.

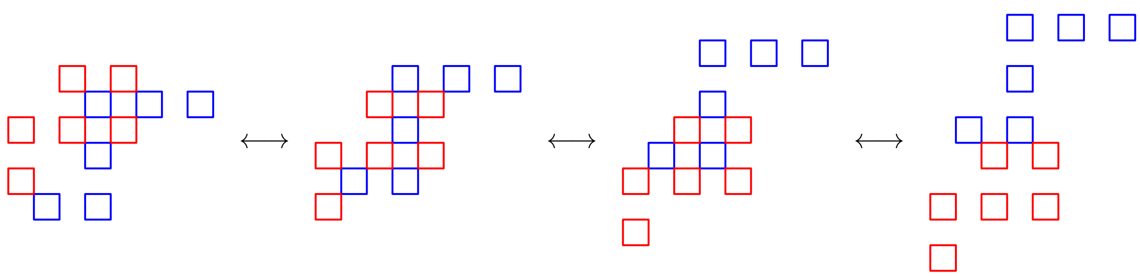


Figure 3: Partial shuffle diagrams

1.4 Schur log-concavity

Our last application is to prove a special case of Lam–Postnikov–Pylyavskyy's Schur log-concavity conjecture. Recall that the dominant weights of $\mathfrak{sl}_n$ can be viewed as partitions λ with $\lambda_n = 0$. Let V_λ be the highest weight $\mathfrak{sl}_n$-module corresponding to λ. A natural question is when a tensor product $V_\lambda \otimes V_\mu$ is contained in $V_\nu \otimes V_\rho$. Since the multiplicity of V_κ in $V_\lambda \otimes V_\mu$ is exactly $c^\kappa_{\lambda,\mu}$, this question can be phrased as when one has $c^\kappa_{\lambda,\mu} < c^\kappa_{\nu,\rho}$ for all κ. Combinatorially, this is equivalent to when the difference $s_\nu s_\rho - s_\lambda s_\mu$ is Schur positive. Several conjectures and results of this form can be found in [4, 10, 17, 21].

In [16], Lam–Postnikov–Pylyavskyy proved several Schur positivity results, and they conjectured a sufficient condition for this question. Let $\alpha_{ij} = e_i - e_j$ be the roots of the type A root system. An *alcoved polytope* is the intersection of some half-spaces bounded by the hyperplanes $\{\tau \mid \langle \alpha_{ij}, \tau \rangle = m\}$. Given two weights λ, μ, the *parallelepiped* $P_{\lambda,\mu}$ is the minimal alcoved polytope containing λ and μ. We have the following conjecture.

Conjecture 1.7 ([22, Conjecture 15.1]). If $\lambda + \mu = \nu + \rho$ and $\nu, \rho \in P_{\lambda,\mu}$, then $s_\nu s_\rho - s_\lambda s_\mu$ is Schur positive.

A weaker version of Conjecture 1.7 was proved in [7], where they showed that the Schur support of $s_\lambda s_\mu$ is contained in that of $s_\nu s_\rho$. Their approach used Horn—Klyachko inequalities, which can only tell when $c^\kappa_{\lambda\mu}$ is positive.

A naive approach to the conjecture is to find an injection from some set of objects counting $c^\kappa_{\lambda,\nu}$ to those counting $c^\kappa_{\nu,\rho}$. The challenge of this approach is that $\ell(\lambda)$ may be arbitrary large while ν may differ from λ on only some rows. Most known rules do not accommodate these local changes. For example, if $\ell(\lambda) = 1000$, but ν is obtained from λ by removing a single box on row 1, then using Remmel-Whitney rule to find an injection would require replacing a 1 with a 1001 in the peelable tableaux. This is no easy task.

Our rule accommodates these local changes, since the parts of λ and μ are interlacing. Thus, in the same example above, using our rule would require replacing a 1 with a 2 in the peelable tableaux regardless of $\ell(\lambda)$. This is significantly more feasible. We will demonstrate this by proving a special case of the conjecture. Our injection (Definition 5.1) is relatively simple since it amounts to locally changing $2i - 1$ to $2i$ for some i. This illustrates the potential of our new rule in proving this conjecture.

Theorem 1.8. Suppose $\lambda = (a^k, 1^{n-k-1}, 0)$, and $\nu = \lambda - (e_m + e_{m+1} + \ldots + e_k)$, and λ, μ, ν, ρ satisfy the conditions of Conjecture 1.7. Then, $s_\nu s_\rho - s_\lambda s_\mu$ is Schur positive.

Acknowledgement

We thank Alex Postnikov and Pavlo Pylyavskyy for telling us about their Schur log-concavity conjecture. We thank Vic Reiner for telling us about peelable tableaux. We thank Daniel Soskin for helpful conversations.

2 Peelable tableaux and Littlewood–Richardson coefficients

Definition 2.1. A *shuffle tableau* of shape $(\lambda/\mu) \circledast (\nu/\rho)$ is a filling of the shuffle diagram of shape $(\lambda/\mu) \circledast (\nu/\rho)$ such that the entries are weakly increasing along the rows and strictly increasing along the columns. The *weight* of a shuffle tableau T, denoted $\omega(T)$, is $\prod_{i \geq 1} x_i^{c_i}$, where c_i is the number of i-entries in T.

Observe that the squares in the same row or column of D are exactly those on the same row or column of the Young diagram of either λ/μ or ν/ρ. Thus, one can think of a shuffle tableau of shape $(\lambda/\mu) \circledast (\nu/\rho)$ as interlacing two SSYTs or shape λ/μ and ν/ρ. In particular, we have

$$\sum_{\substack{T \text{ of shape} \\ (\lambda/\mu)\circledast(\nu/\rho)}} \omega(T) = s_{\lambda/\mu} s_{\nu/\rho}.$$

On the set of shuffle tableaux, Nguyen–Pylyavskyy gave a crystal structure. Their crystal operators can be described briefly as follows.

Definition 2.2. Given a shuffle tableau T, the i-reading word w_i of T is obtained by first ignoring the columns containing both i and $i+1$, then iteratively reading the remaining i and $i+1$ from bottom to top, left to right. The *crystal operators* E_i and F_i act on $w_i(T)$ in the following (standard) way.

- View each i as a close parenthesis ")" and each $i+1$ as an open parenthesis "(".
- Match the parentheses in the usual way.
- E_i changes the leftmost unmatched $i+1$ to i (if exists), and F_i changes the rightmost unmatched i to $i+1$ (if exists).

The connected components of the graph on shuffle tableaux formed by the above crystal operators are called *Temperley–Lieb crystals*. In fact, they are type A Kashiwara crystals.

Theorem 2.3 ([20, Theorem 5.1])**.** Temperley–Lieb crystals are type A Kashiwara crystals.

Recall that a Yamanouchi shuffle tableau is a tableau on which no E_i's can be applied. Temperley–Lieb crystals are type A Kashiwara crystals, so counting Yamanouchi shuffle tableau gives the Littlewood–Richardson coefficients. Thus, to prove Theorem 1.4, we need to find a bijection between peelable tableaux and Yamanouchi shuffle tableaux. This bijection is as follows.

Definition 2.4. Given a D-peelable tableau T, $\psi(T)$ is constructed as follows: if there are k i's in row r of T, then in $\psi(T)$, put k r's in row i. Sort all the rows in increasing order.

Theorem 1.4 follows from the following result on ψ.

Proposition 2.5. ψ is a bijection between D-peelable tableaux and Yamanouchi shuffle tableaux on D such that for any D-peelable tableau T, the shape of T is the same as the content of $\psi(T)$.

Proof idea of Proposition 2.5. We first prove a bijection φ from Yamanouchi shuffle tableaux to *D-compatible tableaux*, which are standardizations of D-peelable tableaux. Then, ψ is simply the composition of φ and *inverse standardization*. To prove that φ is a bijection, we carefully check the definitions of crystal operators and D-compatibility. □

Example 2.6. ψ maps the three D-peelable tableaux in Figure 2 to the three shuffle tableaux in Figure 4. We encourage the readers to check that they are Yamanouchi.

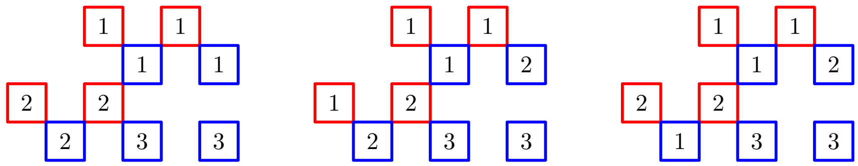


Figure 4: Yamanouchi shuffle tableaux

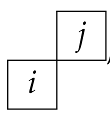
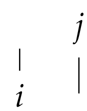
3 Temperley–Lieb immanants

Given partitions $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$, the *generalized Jacobi–Trudi matrix* $A_{\mu,\nu}$ is the finite matrix whose i, j entry is $h_{\mu_i - \nu_j}$. By convention, we have $h_0 = 1$ and $h_m = 0$ for all $m < 0$.

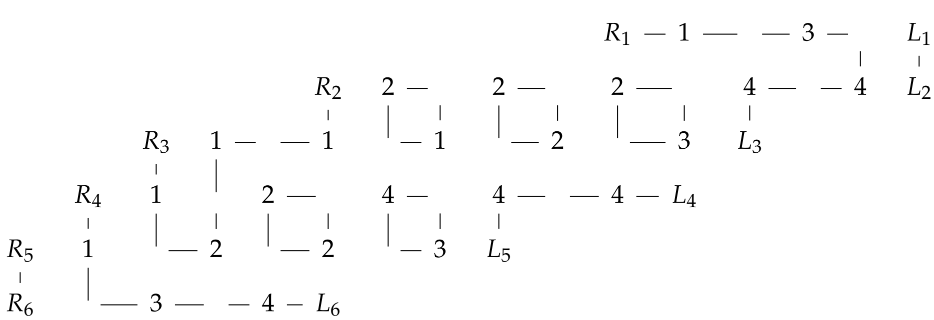
Given μ and ν, we can define two skew shapes μ_R/ν_R and μ_B/ν_B where

$$\mu_R = (\mu_{2i-1} + i - k)_{i=1}^{k}, \nu_R = (\nu_{2i-1} + i - k)_{i=1}^{k},$$

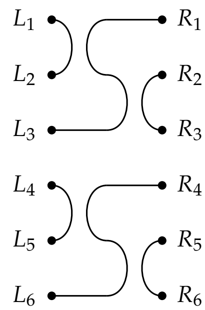
$$\mu_B = \left(\mu_{2i} + i - \left\lfloor \frac{n+1}{2} \right\rfloor\right)_{i=1}^{k}, \nu_B = \left(\nu_{2i} + i - \left\lfloor \frac{n+1}{2} \right\rfloor\right)_{i=1}^{k}.$$

For a shuffle tableau of shape $(\mu_R/\nu_R) \circledast (\mu_B/\nu_B)$, we obtain the Temperley–Lieb type as follows: for each pair , if $i \leq j$, then draw two lines ; if $i > j$, then draw two lines . By convention, the squares North and West of the tableau have value $-\infty$, and those South and East of the tableau have value ∞. Finally, we put L_i at the end of the ith row and R_i at the beginning of the ith row.

Example 3.1. Let $\mu = (9, 9, 7, 6, 4, 2)$ and $\nu = (7, 4, 2, 1, 0, 0)$, then $\mu_R/\nu_R = (7, 6, 4)/(5, 1, 0)$ and $\mu_B/\nu_B = (7, 5, 2)/(2, 0, 0)$. A shuffle tableau of shape $(\mu_R/\nu_R) \circledast (\mu_B/\nu_B)$ can be seen below, together with the lines drawn.



This means that the Temperley–Lieb type is



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The Littlewood–Richardson rule for Temperley–Lieb immanants of $A_{\mu,\nu}$ is as follows.

Theorem 3.2 ([20, Theorem 6.2])**.** For any partitions μ, ν, any Temperley–Lieb type τ, and any partition λ, the coefficient of the Schur function s_λ in $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$ is the number of Yamanouchi shuffle tableaux of shape $(\mu_R/\nu_R) \circledast (\mu_B/\nu_B)$ with Temperley–Lieb type τ and content λ.

Combining with Theorem 1.4, we have the following corollary.

Corollary 3.3. For any partitions μ, ν, any Temperley–Lieb type τ, and any partition λ, the coefficient of the Schur function s_λ in $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$ is the number of $(\mu_R/\nu_R) \circledast (\mu_B/\nu_B)$-peelable tableaux T of shape λ such that $\psi(T)$ has Temperley–Lieb type τ.

4 Bender–Knuth involutions and symmetry of Littlewood–Richardson coefficients

Definition 4.1. The *Bender–Knuth involution* BK_i for $1 \leq i \leq n-1$ is an involution of the set of SSYT of shape λ that sends a SSYT T to the SSYT obtained from the following procedure:

1. Let S be the skew tableau obtained by taking only the squares of T with entry equal i and $i+1$;

2. Each row of S contains a entries i that are directly above an $i+1$, b entries i that are alone in their columns, c entries $i+1$ that are alone in their columns, and d entries $i+1$ that are directly below an i for some $a, b, c, d \geq 0$;

3. Construct a skew tableau S' by swapping b and c in each row of S;

4. Define $\mathrm{BK}_i(T)$ to be the tableau obtained by replacing S with S' in T.

If $\alpha = (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_\ell)$ is the content of T, then $\alpha = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_\ell)$ is the content of $\mathrm{BK}_i(T)$.

Assume for simplicity that $\ell(\lambda) = \ell(\nu)$ (if $\ell(\lambda) > \ell(\mu)$, we can append 0's at the end of μ to get $\ell(\lambda) = \ell(\nu)$), we have the following theorem.

Theorem 1.6. Let T be a $(\lambda/\mu) \circledast (\nu/\rho)$-peelable tableau, and let $\ell = \ell(\lambda)$. Let

$$T' = \mathrm{BK}_1 \circ \mathrm{BK}_3 \circ \cdots \circ \mathrm{BK}_{2\ell-1}(T),$$

then T' is a $(\nu/\rho) \circledast (\lambda/\mu)$-peelable tableau. Note that T and T' have the same shape.

Theorem 1.6 can be proved by carefully checking how Bender–Knuth involutions act on pairs of i-squares and $(i+2)$-squares.

Example 4.2. Applying $\mathrm{BK}_1 \circ \mathrm{BK}_3$ to the peelable tableaux in Figure 2, we get the tableaux in Figure 5. Every tableaux in Figure 5 has at least two compatible pairs of 1 and 3 and one compatible pair of 2 and 4. Hence, they are all $(\nu/\rho) \circledast (\lambda/\mu)$-peelable, for λ, μ, ν, ρ defined in Example 1.2.

1	1	2	2
3	3	3	
4	4		

1	1	2	4
2	3	3	
3	4		

1	1	2	3
2	3	4	
3	4		

Figure 5: Image under $\mathrm{BK}_1 \circ \mathrm{BK}_3$ of the peelable tableaux in Figure 2

5 Schur log-concavity

Here, we solve Theorem 1.8 by constructing an injection from $\lambda \circledast \mu$-peelable tableaux to $\nu \circledast \rho$-peelable tableaux. We say a $(2i-1)$-square of a SSYT T is changeable if it can be increased to $2i$. This means that it is the rightmost $(2i-1)$-square on its row and the square directly below it does not contain $2i$. Similarly, a $2i$-square is changeable if it can be decreased to $2i-1$. We are ready to state our injection.

Definition 5.1 (Injection θ)**.** Let T be a $\lambda \circledast \mu$-peelable tableau, $\theta(T)$ is obtained from the following procedure.

1. Locate the leftmost changeable $(2m-1)$-square, call this square s_m and change this $2m-1$ to $2m$.

2. Locate the leftmost changeable $(2m+1)$-square that is weakly East of s_m, call this square s_{m+1} and change this $2m+1$ to $2m+2$, repeat for $2m+3, \ldots, 2k-1$.

3. If the result of Step 2 is a $\nu \circledast \rho$-peelable tableau, stop. Else, this means that there is no NE matching of $2k-1$ and $2k+1$. Let s_{k+1} be the (unique) $(2k+1)$-square. Locate the rightmost changeable $(2k+2)$-square and call this square t_{k+1}. Increase s_{k+1} to $2k+2$ and decrease t_{k+1} to $2k+1$.

4. If the result of Step 3 is a $\nu \circledast \rho$-peelable tableau, stop. Else, repeat the process with $2k+3$ and $2k+4$, $2k+5$ and $2k+6$, etc. until the result is a $\nu \circledast \rho$-peelable tableau. Call the involved squares $s_{k+2}, t_{k+2}, s_{k+3}, t_{k+2}, \ldots$. The step has to stop eventually since $\ell(\lambda) < \infty$.

Theorem 5.2. Given λ, μ, ν, ρ satisfying the conditions of Conjecture 1.7. The map θ in Definition 5.1 is well-defined and is an injection from $\lambda \circledast \mu$-peelable tableaux to $\nu \circledast \rho$-peelable tableaux. In particular, this implies Theorem 1.8.

Proof idea of Theorem 5.2. The non-trivial part of the proof is to verify that we can always carry out Step 3. We define the *path of* θ consisting of squares in consecutive columns, starting from column 1, one square for each column. If at some point we cannot carry out Step 3, we show that the path will have length λ_m, which implies $\mu_m \geq \lambda_m$. This in fact contradicts the condition $\nu, \rho \in P_{\lambda,\mu}$. $\square$

1	1	1	1	1	1	1	2	2	2	2	2
2	2	2	2	3	3	3	4	4			
3	3	3	3	4	4	5	6				
4	4	5	5	5	5	6					
5	5	6	6	7	7	7					
6	6	7	7	8	8	9					
7	7	8	8	10	10	11					
8	10	12	12	13							
10	12	14									
14	14										

(a) Start

1	1	1	1	1	1	1	2	2	2	2	2
2	2	2	2	3	3	3	4	4			
3	3	3	4	4	4	5	6				
4	4	5	5	5	6	6					
5	5	6	6	7	7	8					
6	6	7	7	8	8	9					
7	7	8	8	10	10	11					
8	10	12	12	13							
10	12	14									
14	14										

(b) After Step 1 and 2

1	1	1	1	1	1	1	2	2	2	2	2
2	2	2	2	3	3	3	4	4			
3	3	3	4	4	4	5	6				
4	4	5	5	5	6	6					
5	5	6	6	7	7	8					
6	6	7	7	8	8	10					
7	7	8	8	9	10	11					
8	10	12	12	13							
10	12	14									
14	14										

(c) After Step 3

1	1	1	1	1	1	1	2	2	2	2	2
2	2	2	2	3	3	3	4	4			
3	3	3	4	4	4	5	6				
4	4	5	5	5	6	6					
5	5	6	6	7	7	8					
6	6	7	7	8	8	10					
7	7	8	8	9	10	12					
8	10	11	12	14							
10	12	13									
14	14										

(d) Result

Figure 6: Stages of the injection

Example 5.3. Let $\lambda = (7,7,7,7,1,1,1,0)$, $\mu = (9,6,6,5,4,3,3,0)$, $\nu = \lambda - (e_2 + e_3 + e_4) = (7,6,6,6,1,1,1,0)$, and $\rho = (9,7,7,6,4,3,3,0)$. Figure 6a shows a $\lambda \circledast \mu$-peelable tableau.

The leftmost changeable 3 is colored yellow. The leftmost changeable 5 weakly East of the yellow 3 is colored yellow, and so is the next changeable 7.

After Step 1 and 2, we get the tableau in Figure 6b. This is, however, not $\nu \circledast \rho$-peelable, so we proceed to Step 3, which swaps 9 and the rightmost changeable 10 West of it. This gives us the tableau in Figure 6c. This is still not $\nu \circledast \rho$-peelable, so we repeat Step 4, swapping 11 and 12, then 13 and 14. The resulting tableau in Figure 6d is finally $\nu \circledast \rho$-peelable.

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