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Multipermutohedral Chow rings

Anastasia Nathanson*[1]

[1]*University of Minnesota*

Abstract. The multipermutohedral Chow ring was introduced in a series of papers by Clader, Damiolini, Eur, Huang, Li, and Ramadas to study moduli spaces with cyclic symmetry. It generalizes Chow rings of permutohedral varieties and type-B Coxeter arrangements. In the present work, we establish the combinatorial structure of the multipermutohedral Chow ring through an explicit Gröbner basis, yielding a Feichtner-Yuzvinsky-type monomial basis and a formula for the Hilbert series. Using this formula, we refine the palindromicity of the Hilbert series. From a representation-theoretic perspective, we also compute the equivariant Hilbert series under two natural group actions and construct combinatorial maps that establish equivariant unimodality and palindromicity.

Keywords: multipermutohedral, Chow ring, matroid, equivariant, unimodality

1 Introduction

The multipermutohedral Chow ring was recently introduced in a series of papers by Clader, Damiolini, Eur, Huang, Li, and Ramadas [3, 4, 2] to study the moduli space of rational curves with cyclic action. These rings generalize the permutohedral Chow ring and the cohomology ring of type-B Coxeter arrangements. This adds to the study of graded algebras arising from combinatorial structures such as permutohedral varieties, Coxeter complexes, and matroid Bergman fans, which have been extensively studied in algebraic combinatorics, revealing deep connections between geometry, representation theory, and combinatorics.

The multipermutohedral Chow ring is defined using the data of a tuple of integers $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ with each $\pi_i \geq 2$. This gives *color classes* $E_i := [\pi_i]$ that together form a *ground set* $E := \sqcup_{i=1}^{n} E_i$. A subset $S \subseteq E$ is *π-colored* if it contains at most one element from each color class. The collection $\mathcal{S}_\pi^+$ of π-colored sets forms a meet-semilattice under inclusion, playing the same role that the Boolean lattice does for the classical permutohedron. This data defines the *multipermutohedral Chow ring* (see Section 2.1):

$$A_\pi := \frac{k[x_S : S \in \mathcal{S}_\pi^+ \setminus \varnothing][y_1, \ldots, y_n]}{I + J}.$$

*natha129@umn.edu.

When $\pi_1 = \cdots = \pi_n = 2$, we obtain the Chow ring of the type-B Coxeter fan; when $\pi_1 = \cdots = \pi_n$, one recovers the Chow ring of the moduli space of rational pointed curves with cyclic symmetry (defined in [4]).

Main results

Our first result (Theorem 3) constructs an explicit Gröbner basis for the defining ideal $I + J$ of A_π. This yields a monomial basis FY_π for A_π resembling the monomial basis for Chow rings of matroids found by Feichtner and Yuzvinsky [5]. This basis leads to a closed formula for the Hilbert series $\mathrm{Hilb}(A_\pi, q)$, which turns out to be a symmetric polynomial in the variables $\pi_1, \pi_2, \ldots, \pi_n$ with coefficients in $\mathbb{Z}[q]$ (Theorem 7).

While the Kähler package guarantees that $\mathrm{Hilb}(A_\pi, q)$ is palindromic, we discover that the coefficient of the elementary symmetric polynomial $e_k(\pi_1, \ldots, \pi_n)$ in $\mathrm{Hilb}(A_\pi, q)$ is itself a reciprocal polynomial in q (Corollary 8). This finer palindromicity has a purely combinatorial origin in the structure of FY_π and provides a concrete explanation for Poincaré duality that goes beyond what geometric considerations alone predict.

Viewing this ring through the lens of representation theory also reveals some surprises. The *π-symmetry group*,

$$G_\pi := \mathfrak{S}_{m_2}[\mathfrak{S}_2] \times \mathfrak{S}_{m_3}[\mathfrak{S}_3] \times \cdots \qquad \text{where } m_k = \#\{\pi_i = k\},$$

acts on A_π by permuting subscripts S on variables x_S within color classes and permuting color classes of equal size. In the special case that $\pi = (p, p, \ldots, p)$, we obtain an explicit generating function for the $\mathfrak{S}_n[\mathfrak{S}_p]$-equivariant structure (Theorem 11), generalizing a result of Stembridge on type-B Coxeter fans [8].

Finally, by adapting symmetric chain decomposition techniques from [1], we construct G_π-equivariant maps

$$A^k \leftrightarrow A^{n-k} \quad \text{and} \quad A^k \hookrightarrow A^{k+1} \qquad \text{for } k \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

which establish equivariant palindromicity and equivariant unimodality (Theorems 15 and 16). In fact, the bijective maps between complementary degree pieces preserve the basis FY_π. If we restrict to the subgroup $H_\pi := \mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2} \times \cdots \times \mathfrak{S}_{\pi_n} \subseteq G_\pi$, we also construct H_π-equivariant injective degree-raising maps that preserve FY_π. However, we prove that it is impossible to build such maps that commute with the action of G_π even when $\pi = (2, 2)$ (Proposition 17).

2 Background

The permutohedral Chow ring is built from the Boolean lattice of subsets of $[n]$. The multipermutohedral construction generalizes this by allowing multiple copies of each $i \in [n]$.

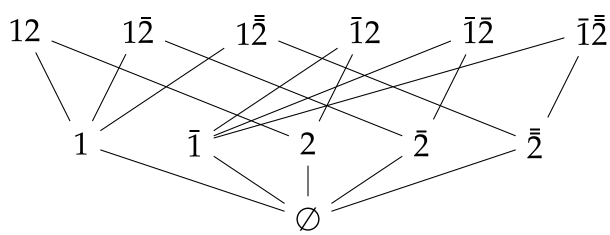
A tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{Z}^n_{\geq 2}$ defines *color classes* $E_i := \{i_1, i_2, \ldots, i_{\pi_i}\} \cong [\pi_i]$ for each $i \in [n]$. Their union is the *ground set* $E := E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$.

A subset $S \subseteq E$ is *π-colored* if S contains at most one element from E_i for each $i \in [n]$. Denote the collection of π-colored sets by $\mathcal{S}^+_\pi$ and the nonempty π-colored sets by $\mathcal{S}_\pi$. For each π-colored set S, define the following statistics:

- Let $\underline{S} :- \{i \colon E_i \cap S \neq \varnothing\} \subsetneq [n]$ be the "underlying" set of S.
- Let $\mathring{S} :- \{i \colon E_i \cap S = \varnothing\} = [n] \setminus \underline{S}$ be the "zero" set of S.

We represent each i_k by writing i with $(k-1)$ bars over it. A π-colored set S is then a decorated version of its underlying set $\underline{S}$, where each $i_k \in S$ is decorated with $(k-1)$ bars. For example, for $\pi = (3,2,4)$, the ground set is $E = \{1, \bar{1}, \bar{\bar{1}}, 2, \bar{2}, 3, \bar{3}, \bar{\bar{3}}, \bar{\bar{\bar{3}}}\}$, and $S = \{2, \bar{\bar{3}}\}$ is a π-colored set.

Example 1. For a smaller example, consider the tuple $\pi = (2,3)$. The collection of π-colored sets $\mathcal{S}^+_\pi$ forms a poset when ordered by inclusion. The Hasse diagram of $\mathcal{S}^+_{(2,3)}$ is below.



Note that the poset $\mathcal{S}^+_\pi$ pictured above is a meet-semilattice, where the meet of two sets is their intersection. The join of two π-colored sets does not always exist, as their set union might not be π-colored.

2.1 Multipermutohedral Chow ring

Given a tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, the *multipermutohedral Chow ring* over a field k is the polynomial ring

$$A_\pi :- \frac{\mathbb{Z}[x_S \colon S \in \mathcal{S}_\pi][y_i \colon i \in [n]]}{I + J}$$

with ideals $I = \langle x_S x_T \colon S \text{ and } T \text{ are incomparable, i.e., } S \not\subset T \text{ and } T \not\subset S \rangle$ and

$$J = \left\langle \gamma_{i_k} \colon i_k \in E_i,\, i \in [n] \right\rangle, \qquad \text{where} \qquad \gamma_{i_k} :- y_i + \sum_{\substack{S \in \mathcal{S}_\pi \colon \\ S \ni i_k}} x_S.$$

Note that instead of $\mathbb{Z}$, we can use any field. It is easy to see that A_π is graded, so we write $A_\pi = \bigoplus_{k \geq 0} A^k_\pi$.

This is the Chow ring of the multipermutohedral fan Σ_π, which is studied in [2] and, in more special cases, in [3] and [4]. Because the n-tuple π fully characterizes the ring, we disregard the fan structure in this extended abstract and study the ring only using the properties of π.

Remark 2. This ring was first introduced in [2] with a slightly different presentation. In that work, the Chow ring of the multipermutohedral fan is $A_\pi = \hat{R}/(I + \hat{J})$, where $\hat{R} = k[x_S \colon S \in \mathcal{S}_\pi]$. The ideal I is the same as above, while

$$\hat{J} = \left\langle \sum_{\substack{S\in\mathcal{S}_\pi: \\ S\ni i_j}} x_S - \sum_{\substack{S\in\mathcal{S}_\pi: \\ S\ni i_k}} x_S \colon i_j, i_k \in E_i,\, i \in [n] \right\rangle.$$

This presentation is equivalent to the one above via the map $R \to \hat{R}$ that sends

$$x_S \mapsto x_S \quad \text{for } S \in \mathcal{S}_\pi \qquad \text{and} \qquad y_i \mapsto -\sum_{\substack{S\in\mathcal{S}_\pi: \\ S\ni i_k}} x_S \quad \text{for some } i_k \in E_i, \text{ for every } i \in [n].$$

3 Gröbner basis and its consequences

To analyze the structure of the multipermutohedral Chow ring, we construct an explicit Gröbner basis for its defining ideals. The Gröbner basis and the resulting monomial basis are analogous to the Feichtner-Yuzvinsky monomials for the Chow ring of an atomic lattice defined in [5], which was recently studied in [6, 1]. At the end of this section, we use the basis to find an expression for the Hilbert series of A_π.

First, we recall some Gröbner basis theory for rings over $\mathbb{Z}$, as it is used in [5, 1]. Recall that a Gröbner basis of a polynomial ring R over $\mathbb{Z}$ is a particular generating set for the ideals of said ring with respect to a well-ordering $\prec$ of all monomials of R. For any polynomial $f \in R$, we use $\mathrm{in}_\prec(f)$ to denote the $\prec$-largest monomial of f. A polynomial f is *$\prec$-monic* if the coefficient of $\mathrm{in}_\prec(f)$is ± 1.

Given a monomial order $\prec$ of a ring R, we say that a subset $\{g_1, \ldots, g_p\}$ of an ideal $I \subseteq R$ is a *monic Gröbner basis* for I with respect to $\prec$, if each g_i is $\prec$-monic, and every polynomial f in I has $\mathrm{in}_\prec(f)$ divisible by at least one of the initial monomials $\{\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_p)\}$. A monomial is called a *standard monomial* for a Gröbner basis with respect to $\prec$ if it is divisible by none of the initial monomials.

3.1 A Gröbner basis

We define a partial order on the monomials of A_π by declaring, for all $i \in [n]$ and $S, T \in \mathcal{S}_\pi$, we have $y_i < x_S$ and $x_T x_S$ when $|T| > |S|$. Let $\prec$ be a linear extension of this partial order.

Theorem 3. *Let* $\mathbf{G} := \{g_{S,T}, g_{i,S}\}$ *be the family of polynomials shown below, indexed by distinct pairs* $S, T \in \mathcal{S}_\pi^+$ *and* $i \in [n]$. *Then* $\mathbf{G}$ *is a* $\prec$*-monic Gröbner basis for* $I + J$ *with respect to* $\prec$ *defined above.*

condition on $g_{i,S}$	polynomial $g_{i,S}$	$\mathrm{in}_\prec(g_{i,S})$
$S = \emptyset$	y_i^2	y_i^2
$S \in \mathcal{S}_\pi$ and $i \in \underline{S}$	$x_S y_i$	$x_S y_i$

condition on $g_{S,T}$	polynomial $g_{S,T}$	$\mathrm{in}_\prec(g_{S,T})$
S, T are incomparable	$x_S x_T$	$x_S x_T$
$S = \emptyset$ and $T \in \mathcal{S}_\pi$	$\left(\sum_{\substack{U \in \mathcal{S}_\pi: \\ U \supseteq T}} x_U \right)^{\lvert T \rvert} + \prod_{i \in \underline{T}} y_i$	$x_T^{\lvert T \rvert}$
$S, T \in \mathcal{S}_\pi$ and $S \subsetneq T$	$x_S \left(\sum_{\substack{U \in \mathcal{S}_\pi: \\ U \supseteq T}} x_U \right)^{\lvert T \rvert - \lvert S \rvert} + x_S \prod_{i \in (\underline{T} \setminus \underline{S})} y_i$	$x_S x_T^{\lvert T \rvert - \lvert S \rvert}$

The proof of this theorem is a fairly straightforward adaptation of the proof of Theorem 1 of Feichtner and Yuzvinsky [5]. Similarly, this Gröbner basis characterization immediately yields a combinatorial description of a monomial basis for A_π that will be useful to us in the remainder of this work.

Corollary 4. *The multipermutohedral Chow ring* A_π *is a free* $\mathbb{Z}$*-module with* $\mathbb{Z}$*-basis given by*

$$\mathrm{FY}_\pi :- \left\{ \prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j \in Z} y_j \right\},$$

where the indexing set for the monomials runs through all flags $\emptyset :- S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_\ell$ *of* π*-colored sets, exponents* $f_i \leq \lvert S_i \setminus S_{i-1} \rvert - 1$, *and subsets* $Z \subseteq \mathring{S}_\ell$. *Denote the set of degree* k *basis monomials with* FY_π^k.

Example 5. The tuple $\pi = (2,2)$ defines the ring A_π, with the following monomial basis shown in the table below.

degree	FY_π basis
2	$y_1 y_2$
1	$x_{12}, x_{1\bar{2}}, x_{1\bar{\bar{2}}}, x_{\bar{1}2}, x_{\bar{1}\bar{2}}, y_1, y_2$
0	1

The degrees of these graded pieces are the type-B Eulerian sequence $(1,6,1)$. This follows from the fact that $A_{(2,2,\ldots,2)}$ is the Chow ring of the type-B Coxeter fan.

Note that the ring in the example vanishes for all degrees above $k > 2 = n$. This is the case for all multipermutohedral Chow rings as a result of Corollary 4.

Corollary 6. *Given $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, the multipermutohedral Chow ring vanishes in degrees strictly above n; that is, $A_\pi = \bigoplus_{i=0}^{n} A^i$.*

3.2 Hilbert series

The Hilbert series of A_π can be computed as $\mathrm{Hilb}(A_\pi, q) = \sum_{a \in \mathrm{FY}_\pi} q^{\deg(a)}$ using the monomials $a \in \mathrm{FY}_\pi$ defined in Corollary 4. We apply the approach used by Liao in [6], associating a composition to each monomial $a \in \mathrm{FY}_\pi$ to obtain the following result.

Theorem 7. *The Hilbert series of the multipermutohedral Chow ring is the following symmetric polynomial in the variables $\pi_1, \pi_2, \ldots, \pi_n$ with coefficients in $\mathbb{Z}[q]$:*

$$\mathrm{Hilb}(A_\pi, q) = \sum_{k=0}^{n} e_k(\pi_1, \pi_2, \ldots, \pi_n)(1+q)^{n-k} \left[\frac{z^k}{k!}\right] \left(\frac{1-q}{e^{qz} - qe^z}\right),$$

where e_k denotes the kth elementary symmetric polynomial and $\left([x^i] f(x)\right)$ denotes the coefficient of x^i in a power series $f(x) \in \mathbb{Q}[[x]]$.

To analyze this further, we name the coefficient of $e_k(\pi_1, \pi_2, \ldots, \pi_n)$ in $\mathrm{Hilb}(A_\pi, q)$:

$$a(k,n) := \left(\left[\frac{z^k}{k!}\right] \frac{1-q}{e^{qz} - qe^z}\right)(1+q)^{n-k}.$$

We define a generating function for $a(k,n)$, with a variable z recording the value of k and a variable u tracking the value of $n-k$:

$$F(z,u;q) :- \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a(k,n) \frac{z^k}{k!} u^{n-k}.$$

By a standard generating function argument, we get the closed formula

$$F(z,u;q) = \frac{1-q}{(e^{qz} - qe^{z})(1-u(1+q))}. \tag{3.1}$$

The fact that the multipermutohedral Chow ring has the Kähler package was established in [4, Theorem 2.7], where the ring is shown to be equivalent to the Chow ring of the product of projective varieties. Even further, they provide a combinatorial way to see the palindromicity of the coefficients of $\mathrm{Hilb}(A_\pi, q)$ in [4, Theorem 4.13]. The generating function $F(z,u;q)$ in Equation (3.1) shows a stronger symmetry: each coefficient $a(k,n)$ of $e_k(\pi_1, \pi_2, \ldots, \pi_n)$ is itself palindromic.

Corollary 8. *The generating function $F(z,u;q)$ for the coefficient of $a(k,n)$ of $e_k(\pi_1, \pi_2 \ldots, \pi_n)$ in $\mathrm{Hilb}(A_\pi, q)$ satisfies $F(z,u;q) = F(qz, qu; 1/q)$. This implies that each coefficient $a(k,n)$ is a reciprocal polynomial in q, with center of symmetry around the power $q^{\frac{n}{2}}$.*

The generating function $F(z,u;q)$ proves to be a useful tool for calculating the Hilbert series of A_π. We can calculate the value of $a(k,n)$ for any k by taking the limit as $z \to 0$ of $\frac{d^k}{dz^k} F(z,u;q)$. Doing so, we obtain:

k	0	1	2	3	$\cdots$	n
$a(k,n)$	$(1+q)^n$	0	$q(1+q)^{n-2}$	$(q^2+q)(1+q)^{n-3}$	$\cdots$	$\dfrac{q-q^n}{1-q}$

Note that $a(1,n) = 0$ corresponds with the fact that there exists no singleton S such that $x_S \in \mathrm{FY}_\pi$, which follows from the standard monomial definition.

4 Group action

For $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, we now study the action of two natural symmetry groups on A_π. Denote multiplicities $m_k := \#\{\pi_i = k\}$, which allow us to write $\pi = 2^{m_2} 3^{m_3} \cdots$. The associated *$\pi$-symmetry group* is the product of wreath products

$$G_\pi := \mathfrak{S}_{m_2}[\mathfrak{S}_2] \times \mathfrak{S}_{m_3}[\mathfrak{S}_3] \times \cdots .$$

We also consider the action of a subgroup of G_π which is a product of symmetric groups for each color class

$$H_\pi := \mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2} \times \cdots \times \mathfrak{S}_{\pi_n}.$$

Both G_π and H_π act on the ground set $E = E_1 \sqcup \cdots \sqcup E_n$ by permuting elements within each color class, while G_π also swaps color classes of equal size. Note that $G_\pi = H_\pi$

exactly when $\pi_i \neq \pi_j$ for all $i, j \in [n]$. On the other hand, we say that π is *uniform* if $\pi_1 = \cdots = \pi_n$. Each group has an induced action on $\mathcal{S}_\pi$ and hence on the variables $\{x_S : S \in \mathcal{S}_\pi\} \cup \{y_1, \ldots, y_n\}$ of A_π. Since $g \in G_\pi$ preserves incomparability of π-colored sets and membership relations $a_i \in S$, we obtain the following lemma.

Lemma 9. *The action of G_π permutes the monomial basis FY_π and each graded subset FY_π^k. Thus, the $\mathbb{Z}G_\pi$-module structure on A_π and each of its homogeneous components A_π^k lift to G_π-permutation representations.*

4.1 H_π-equivariant Hilbert series

For each $i \in [n]$, we define $V_i = \mathbb{C}^{\pi_i}$ to be the π_i-dimensional defining permutation representation of $\mathfrak{S}_{\pi_i}$ over $\mathbb{C}$. For each $I \subseteq [n]$, let $V_I = \bigotimes_{i \in I} V_i$ be a representation of $\prod_{i \in I} \mathfrak{S}_{\pi_i}$. We also regard V_I as a representation of H_π via inflation through the group surjection $H_\pi = \prod_{i=1}^n \mathfrak{S}_{\pi_i} \twoheadrightarrow \prod_{i \in I} \mathfrak{S}_{\pi_i}$ that forgets the other factors $\mathfrak{S}_{\pi_j}$ for $i \notin I$. For $k \geq 0$, we set

$$\mathcal{V}_k(\pi) = \bigoplus_{\substack{I \subseteq [n] \\ |I| = k}} V_I.$$

Note that the dimension of $\mathcal{V}_k(\pi)$ can be computed by the evaluation of the elementary symmetric polynomial $e_k(\pi_1, \pi_2, \ldots, \pi_n)$. Hence, $\mathcal{V}_k(\pi)$ is an H_π-representation "lifting" $e_k(\pi_1, \pi_2, \ldots, \pi_n)$. With this, we generalize the Hilbert series found in Theorem 7 to an H_π-equivariant form.

Corollary 10. *The H_π-equivariant Hilbert series of the multipermutohedral Chow ring is*

$$\mathrm{Hilb}_{H_\pi}(A_\pi, q) = \sum_{k=0}^{n} \mathcal{V}_k(\pi)(1+q)^{n-k} \left[\frac{z^k}{k!}\right] \left(\frac{1-q}{e^{qz} - qe^z}\right).$$

4.2 G_π-equivariant Hilbert series for uniform π

In the special case that π is uniform, so $\pi = (p, \ldots, p) =: (p^n)$, the π-symmetry group G_π is the wreath group $\mathfrak{S}_n[\mathfrak{S}_p] =: G_n^{(p)}$. This has a tower of groups

$$1 = G_0^{(p)} < G_1^{(p)} < G_2^{(p)} < \cdots$$

where $G_i^{(p)}$ is the wreath product $\mathfrak{S}_i[\mathfrak{S}_p]$.

Consider the ring of graded representations of this tower of groups, with grading variable q, denoted $\Lambda^{(p)}[q] := \bigoplus_{n=0}^{\infty} \mathcal{R}\left(G_n^{(p)}\right)[q]$ over $\mathbb{C}$. This ring is endowed with an external induction product

$$\mathcal{R}\left(G_{n_1}^{(p)}\right) \otimes \mathcal{R}\left(G_{n_2}^{(p)}\right) \to \mathcal{R}\left(G_{n_1+n_2}^{(p)}\right)$$
$$\chi_1 \otimes \chi_2 \mapsto \chi_1 \otimes \chi_2 \uparrow_{G_{n_1}^{(p)} \times G_{n_2}^{(p)}}^{G_{n_1+n_2}^{(p)}}.$$

We denote the image of this product with $\chi_1 \star \chi_2$. The action of $G_n^{(p)}$ extends to monomials of the ring $A_{(p^n)}$. We study the graded character of this action, which we write as $\mathrm{Hilb}_{G_n^{(p)}}(A_{(p^n)}, q)$. Note that this is an element of $\mathcal{R}\left(G_n^{(p)}\right)$ and hence the generating function $\sum_{n=0}^{\infty} \mathrm{Hilb}_{G_n^{(p)}}(A_{(p^n)}, q) z^n$ lies in $\Lambda^{(p)}[q][[z]]$.

We name some elements in $\mathcal{R}\left(G_n^{(p)}\right)$, and their generating functions. First, denote the trivial character of $G_n^{(p)}$ by τ_n and write $T(z) := \sum_{n=0}^{\infty} \tau_n z^n$. Also define the character $\omega_n := \mathbb{C}[G_n^{(p)}/\mathfrak{S}_n]$, a coset action which can be thought of as $G_n^{(p)}$ permuting π-colored subsets S where $\pi = (p^n)$. Then denote $\Omega(z) := \sum_{n=0}^{\infty} \omega_n z^n$.

Theorem 11. *The graded equivariant Hilbert series of $A_{(p^n)}$ under the action of $G_n^{(p)}$ is given by*

$$\sum_{n=0}^{\infty} \mathrm{Hilb}_{G_n^{(p)}}(A_{(p^n)}, q) z^n = \frac{(1-q)T(qz)T(z)}{\Omega(pqz) - q\Omega(pz)}.$$

When $p = 2$, this recovers Stembridge's formula for the cohomology of the Chow ring of the toric variety for the type-B Coxeter arrangement [8]. The generating function structure makes these characters computable via standard techniques in symmetric function theory. For non-uniform π, the G_π-equivariant Hilbert series would be a product of the result above for all distinct values π_i.

4.3 Equivariant symmetry and unimodality

Our goal in this section is to build G_π- and H_π-equivariant maps between the graded pieces $(A_\pi^0, A_\pi^1, \ldots, A_\pi^n)$ of A_π viewed as G_π- and H_π-representations. These maps will exhibit equivariant palindromicity and unimodality. The H_π-equivariant maps can be constructed by adapting the maps in [1], but maps built in that manner do not commute with the action of a wreath product.

4.3.1 H_π-equivariant maps

First, we define the collection of flags of π-colored sets,

$$\mathcal{F} := \{\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_\ell : S_i \in \mathcal{S}_\pi, \text{ for } i = 1, \ldots, \ell\}.$$

For a monomial $a \in \mathrm{FY}_\pi$, we define the truncated support map supp : $\mathrm{FY}_\pi \to \mathcal{F}$ where $\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j\in Z} y_j \mapsto (S_1, \ldots, S_\ell)$. We order each fiber $\mathrm{supp}^{-1}(S_1, S_2, \ldots, S_\ell)$ by setting $a < b$ if the monomial a divides b.

Lemma 12. *For any flag of π-colored sets $F := \{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_\ell\} \in \mathcal{F}$, the fiber $\mathrm{supp}^{-1}(F)$ is the set of monomials $\left\{\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{i\in Z} y_i\right\}$ such that $1 \leq f_i \leq |S_i \setminus S_{i-1}| - 1$ and $Z \subseteq \mathring{S}_\ell$. Thus, there is a bijection between the fiber of $\mathrm{supp}^{-1}(F)$ and the Cartesian product of chains:*

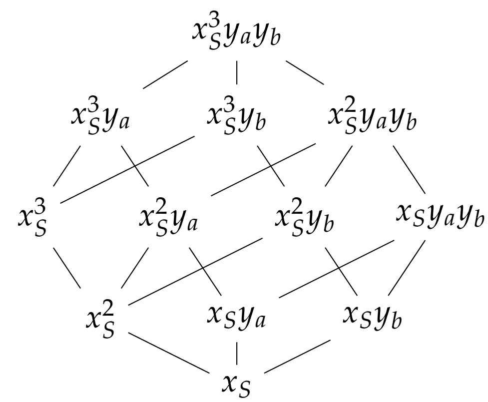
$$\mathrm{supp}^{-1}(S_1, S_2, \ldots, S_\ell) \longleftrightarrow \left(\prod_{i=1}^{\ell} [1, |S_i \setminus S_{i-1}| - 1]\right) \times \prod_{k \in \mathring{S}_\ell} [0, 1]$$

$$\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j\in Z} y_j \longleftrightarrow ((f_1, f_2, \ldots, f_\ell), Z)\,.$$

Example 13. Let $|\pi| = 6$. We consider a one element chain $F = \{S\}$, where $|S| = 4$. Let $\mathring{S} = \{a, b\}$. Then the poset $\mathrm{supp}^{-1}(F)$ is the Cartesian product of the following chains

$$[x_S, x_S^3] \times [1, y_a] \times [1, y_b],$$

whose corresponding Hasse diagram is on the right.



Observe that, given a monomial $\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j\in Z} y_j \in \mathrm{FY}_\pi$, its degree is computed by $|Z| + \sum_{i=1}^{\ell} f_i$. This implies the following lemma.

Lemma 14. *For any flag $F = \{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_\ell\}$, the degrees of the monomials in the fiber $\mathrm{supp}^{-1}(F)$ form a symmetric interval $[\ell, n - \ell]$ within the degree range $[0, n]$ of the ring A_π.*

Recall that all products of chains have a *symmetric chain decomposition* (SCD). An SCD partitions a poset $P = \bigsqcup_{i=1}^{t} C_i$ into chains C_i symmetric about the midpoint of the poset. We use this observation and Lemma 14 to prove the following theorem by pulling the decomposition back to each fiber $\mathrm{supp}^{-1}(F)$.

Theorem 15. *For every π, there exists a family of H_π-equivariant injections*

$$\lambda_H : \mathrm{FY}_\pi^k \hookrightarrow \mathrm{FY}_\pi^{k+1} \quad \textit{for } k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

To define these maps, we follow the construction defined in [1]. For a degree k monomial a with $\mathrm{supp}(a) = F$, we pick a symmetric chain decomposition $\mathrm{supp}^{-1}(F) = \bigsqcup_{i\geq 0} C_i$. Then a lies in one unique chain C_i. We define $\lambda_H(a) = a'$ where a' is the unique element covering a in C_i.

4.3.2 G_π-equivariant maps

Theorem 16. *For every π, there exist:*

- *G_π-equivariant bijections $\beta : \mathrm{FY}_\pi^k \xrightarrow{\sim} \mathrm{FY}_\pi^{n-k}$ for $k \leq \lfloor \frac{n}{2} \rfloor$, and*
- *G_π-equivariant injections $\lambda_G : A_\pi^k \hookrightarrow A_\pi^{k+1}$ for $k \leq \lfloor \frac{n}{2} \rfloor$.*

The bijections $\beta : \mathrm{FY}_\pi^k \to \mathrm{FY}_\pi^{n-k}$ are defined by sending

$$\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j \in Z} y_j \mapsto \prod_{i=1}^{\ell} x_{S_i}^{f_i'} \prod_{j \in Z'} y_j \qquad \text{where } f_i' = |S_i \setminus S_{i-1}| - f_i \text{ and } Z' = \mathring{S}_\ell \setminus Z.$$

For $a \in \mathrm{FY}_\pi^k$, one can check that, indeed, $\deg(\beta(a)) = |\mathring{S}_\ell| + |S_\ell| - (|Z| + \sum_{i=1}^{\ell} f_i) = n - k$.

To define $\lambda_G : A_\pi^k \to A_\pi^{k+1}$ we define the map on the basis elements $a \in \mathrm{FY}_\pi^k$ and extend linearly. This map uses a classical construction for equivariant maps; see [7, Theorem 4.7].

$$\lambda_G(a) = \sum_{\substack{\mathrm{supp}(a') = \mathrm{supp}(a) \\ \deg(a'/a) = 1}} a'$$

One might hope to strengthen Theorem 15 and construct G_π-equivariant injections. However, we show that such maps cannot exist.

Proposition 17. *For π with $G_\pi \neq H_\pi$, there is no G_π-equivariant injection of $\mathrm{FY}_\pi^0 \to \mathrm{FY}_\pi^1$.*

While the monomial basis FY_π is determined by our choice of the Gröbner basis, we can show that Proposition 17 is independent of the choice of permutation basis of A_π by checking the irreducible representations of G_π. In this extended abstract, we show that the assertion is true for this choice of Gröbner basis.

Proof. We show this with the simplest example by considering $\pi = (2,2)$. We analyze this action through the Burnside ring $B(G)$, where $G = G_\pi = \mathfrak{S}_2[\mathfrak{S}_2]$ is the hyperoctahedral group, i.e. the type-B and type-C Coxeter group of rank 2. The Burnside ring $B(G)$ consists of formal linear combinations of equivalence classes of finite sets with a transitive G action. This ring has basis $[G/G_i]$ where G_i run through representatives of the G-conjugacy classes of subgroups of G. The multipermutohedral Chow ring $A_{(2^2)}$ expanded in the basis of the Burnside ring is shown below.

deg	FY_π basis	$B(G)$ basis
2	$y_1 y_2$	$[G/G]$
1	$x_{12}, x_{1\bar{2}}, x_{\bar{1}2}, x_{\bar{1}\bar{2}}, y_1, y_2$	$[G/\langle g \rangle] + [G/C_2^2]$
0	1	$[G/G]$

Here, the x variables form one orbit that is stabilized by the elements of G that swap color classes E_1 and E_2, which we label with $\langle g \rangle$. The y variables are fixed by the action of the product of cyclic groups $C_2 \times C_2$; one can think of this as the type-B action that changes the sign of a signed permutation. For a G-equivariant injection $\mathrm{FY}^0_\pi \to \mathrm{FY}^1_\pi$ to exist, the Burnside basis of FY^0_π must be a subset of the basis for FY^1_π. However, FY^1_π does not contain a copy of the trivial representation $[G/G]$. $\square$

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