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Left modular and extremal lattices through brick-directed algebras

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Abstract. We study some properties of the semidistributive lattices that arise as the lattice of torsion classes of finite dimensional algebras. First, we give a new realization of the left modular elements in terms of brick-splitting torsion classes. Restricted to finite lattices, this yields a full characterization of those lattices of torsion classes that are left modular (equivalently, extremal), thus trim. Beyond the setting of finite lattices, we also treat completely semidistributive weakly atomic lattices and generalize the notions of left modularity and extremality. Finally, we give an explicit construction to create arbitrarily large left modular and extremal semidistributive lattices.

Résumé. Nous étudions certaines propriétés des treillis semi-distributifs qui apparaissent comme treillis des classes de torsion d'algèbres de dimension finie. Tout d'abord, nous proposons une nouvelle réalisation des éléments gauchers modulaires en termes de classes de torsion issues du brick-splitting. Lorsqu'on se restreint aux treillis finis, cela conduit à une caractérisation complète des treillis de classes de torsion qui sont à la fois gauchers modulaires et extrémaux, donc trim. Au-delà du cadre des treillis finis, nous considérons également les treillis complètement semi-distributifs et faiblement atomiques, et nous généralisons les notions de modularité gauche et d'extrémalité. Enfin, nous présentons une construction explicite permettant de produire des treillis semi-distributifs gauchers modulaires et extrémaux de taille arbitrairement grande.

Keywords: brick-splitting torsion pair, brick-directed algebra, semidistributive lattice, left modular lattice, extremal lattices, trim lattice.

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1 Introduction

A rich collection of (finite and infinite) semidistributive lattices naturally arises from the representation theory of algebras, particularly in connection with the recent studies of τ-tilting theory, torsion classes, and related domains. The systematic treatment of such lattices has led to many fundamental results at the intersection of representation theory and lattice theory (see, for instance, [7] and [11], and references therein). In this work, we are mainly concerned with some families of semidistributive lattices, where our setting is originally inspired by the lattice theory of torsion classes in the module category of finite dimensional algebras. More specifically, we treat those completely semidistributive lattices which are weakly atomic. We now outline the content of this manuscript.

Section 2 focuses on various generalizations of distributive lattices. After a brief recollection of some background, in Theorem 1, we recall some recent results on left modularity and extremality in the context of finite semidistributive lattices. Then, we move to the setting of infinite lattices and, for weakly atomic completely semidistributive lattices, in Theorem 2, we give a characterization of left modular elements. Moreover, for such lattices, in Theorem 3, we generalize the notions of left modularity and extremality beyond the finite case.

Section 3 is centered on the lattice of torsion classes of finite dimensional algebras. After a quick review of some important properties of such lattices, we introduce the brick-splitting torsion classes and, in Theorem 6, we give a new characterization of the left modular elements of lattices of torsion classes. Moreover, we introduce brick-directed algebras and, in Theorem 8, we give a complete characterization of those algebras whose lattice of torsion classes is trim (equivalently, left modular or extremal).

Finally, in Section 4, we present an explicit construction that generates brick-directed algebras of representation-finite, tame, and wild types; see Theorem 9. Consequently, for such algebras, the lattices of torsion classes give concrete examples of arbitrarily large lattices which satisfy the assumptions of our main results. Hence, we obtain an abundance of (finite and infinite) left modular and extremal semidistributive lattices.

2 Lattice theory

In this section, after a brief recollection of some standard materials from lattice theory, we primarily focus on left modularity and extremality in the setting of semidistributive lattices. For the more rudimentary materials in lattice theory, we refer to [5].

For a (possibly infinite) partially ordered set $(P, \leq)$, recall that $x < z$ in P is said to be a *cover* relation, denoted by $x \lessdot z$, provided that for each $y \in P$, if $x \leq y \leq z$, we have either $x = y$ or $y = z$. The *Hasse quiver* of P, denoted by $\mathrm{Hasse}(P)$, is a directed graph which has P as the vertex set, and arrows $x \to z$ for each cover relation $x \lessdot z$. Recall that $(P, \leq)$ is a *complete lattice* if for any subset S in P, there exists a unique element of P, smallest with the property of being larger than or equal to all elements of S, called the *join* of S and denoted by $\bigvee S$; and a unique element of P, largest with the property of being smaller than or equal to all elements of S, called the *meet* of S and denoted by $\bigwedge S$.

In the following, $(L, \leq)$, or simply L, always denotes a complete lattice, and for x and y in L, the join and meet are respectively denoted by $x \vee y$ and $x \wedge y$. Moreover, $\hat{1}$ and $\hat{0}$, respectively, denote the greatest element and the least element of L. A subset C of L is called a *chain* if it is a torally ordered set with respect to $\leq$, and it is said to be a finite chain if C consists of only finite many elements. In particular, the *length* of $x_0 < x_1 < \cdots < x_r$ is defined to be r. The length of L is the maximum of the lengths of finite chains, if there exists such a maximum; otherwise, L is of infinite length. A chain C is *maximal* if it cannot be further refined to a chain by inserting more elements.

We say $x \in L$ is *join-irreducible* if, for each y and z in L with $y, z < x$, then $x \neq y \vee z$. By convention, $\hat{0}$ is not considered as a join-irreducible element. Moreover, $x \in L$ is *completely join-irreducible* if for any subset $S \subseteq \{y \in L \mid y < x\}$, we have $x \neq \bigvee_{y \in S} y$. Observe that x is completely join irreducible if and only if every object less than x is less than or equal to a unique element x_* which x covers. By $\mathrm{JI}(L)$ we denote the set of all join-irreducible elements in L, and $\mathrm{JI}^c(L)$ denotes the subset of $\mathrm{JI}(L)$ consisting of completely join-irreducible elements. The notion of (completely) meet-irreducible elements, and consequently the sets $\mathrm{MI}(L)$ and $\mathrm{MI}^c(L)$, are defined dually. By convention, $\hat{1}$ does not belong to $\mathrm{MI}(L)$. Similarly, m is completely meet irreducible if and only if every object greater than m is greater than or equal to a unique element m^* which covers m.

A lattice $(L, \leq)$ is said to be *distributive* if the two operations (join and meet) distribute over each other; namely, for any $x, y, z \in L$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and also $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. A natural generalization of distributive lattices are the *semidistributive* lattices, defined as following: For every three elements $x, y, z \in L$, if $x \vee y = x \vee z$, then $x \vee y = x \vee (y \wedge z)$, and, furthermore, if $x \wedge y = x \wedge z$, then $x \wedge y = x \wedge (y \vee z)$. Moreover, L is said to be *completely semidistributive* if, for $x \in L$ and every subset $S \subseteq L$, the following hold:

- If $x \wedge y = x \wedge z$ for all $y, z \in S$, then $x \wedge (\bigvee S) = x \wedge y$, for every $y \in S$;
- If $x \vee y = x \vee z$ for all $y, z \in S$, then $x \vee (\bigwedge S) = x \vee y$, for every $y \in S$.

Every finite semidistributive lattice is completely semidistributive. For each completely semidistributive lattice L, there is a bijection $\kappa : \mathrm{JI}^c(L) \simeq \mathrm{MI}^c(L)$ such that $m := \kappa(j)$ satisfies $m \wedge j = j_*$ and $m \vee j = m^*$. More explicitly, $\kappa(j) := \max\{x \in L \mid x \wedge j = j_*\}$.

2.1 Finite lattices

Among finite lattices, the distributive ones have rich combinatorial structures and are extensively studied; see [8] and the reference therein. However, many interesting finite lattices arising in different settings do not satisfy the distributivity laws. Hence, various generalizations of finite distributive lattices have appeared in the literature. The most natural one is probably the (finite) semidistributive lattices, already recalled above. In this subsection, we briefly recall some other generalizations of finite distributive lattices which are important in our work. Building upon some recent results recalled below, in Section 2.2 we give proper generalizations of these notions to some infinite lattices.

Following [9], a finite lattice of length m is called *extremal* if $|\mathrm{JI}(L)| = |\mathrm{MI}(L)| = m$. Unlike distributive lattices, an extremal lattice may admit a maximal chain of length less than m. For an extremal lattice L, *spine* of L consists of all elements of L that lie on a chain of maximal length. The spine of an extremal lattice L is known to be a distributive sublattice of L; see [13]. Before considering another generalization of finite distributive lattices, let us recall that, for an arbitrary lattice $(L, \leq)$, an element x in L is called *left modular* if for every $y \leq z$ in L, we have $(y \vee x) \wedge z = y \vee (x \wedge z)$. Then, a finite lattice is called *left modular* if it has a maximal chain of left modular elements. These lattices were first introduced in [4]. More recently, in [12] the author considered those finite lattices which are simultaneously left modular and extremal; every such lattice is called *trim*. Trim lattices manifest many interesting properties (for more details, see [6, 10, 12, 13]).

The following theorem puts the above lattice theoretical notions into perspective and motivates some of our results in Section 2.2. Before we state this important result, let us emphasize that left modular and extremal lattices form two different subfamilies of finite lattices, whose intersection (i.e., trim lattices) properly contains the family of distributive lattices. In particular, for arbitrary finite lattices, the notions of left modularity and extremality do not imply each other (for more details, see [10], and references therein).

Theorem 1 ([13, 10])**.** *Let $(L, \leq)$ be a finite lattice. If L is semidistributive, the following are equivalent:*

1. *L is left modular;*
2. *L is extremal;*
3. *L is trim.*

2.2 Infinite lattices

We now turn our attention to infinite lattices. First, we recall that an arbitrary lattice $(L, \leq)$ is said to be *weakly atomic* if, for all $x < y$, there exist u and v with $x \leq u \lessdot v \leq y$, where $u \lessdot v$ denotes the cover relation. Every finite lattice is weakly atomic.

For each weakly atomic completely semidistributive lattice L, we give a characterization of the left modular elements in L. Below, $\mathcal{LM}(L)$ denotes the set of such elements.

Theorem 2. *Let $(L, \leq)$ be a weakly atomic completely semidistributive lattice. For $x \in L$, the following are equivalent:*

1. *x is left modular;*

2. *For each $j \in \mathrm{JI}^c(L)$, either $j \leq x$ or $x \leq \kappa(j)$ holds.*

Furthermore, $\mathcal{LM}(L)$ forms a complete sublattice of L which is completely distributive.

To prepare the setting for the main result of this section (Theorem 3), we need to recall some terminology and also introduce some new notions.

2.2.1 Well-separated κ-lattices

A complete lattice $(L, \leq)$ is called *spatial* if each $x \in L$ can be written as a join of completely join-irreducible elements. Moreover, L is called *dually spatial* if it satisfies the dual condition. Consequently, L is called *bispatial* if it is both spatial and dually spatial, that is, each $x \in L$ can be written as a join of completely join-irreducible elements, as well as a meet of completely meet-irreducible elements. We also recall that a complete lattice L is called a *κ-lattice* if L is bispatial and equipped with a bijection $\kappa : \mathrm{JI}^c(L) \to \mathrm{MI}^c(L)$ such that, for each $j \in \mathrm{JI}^c(L)$, we have $\kappa(j) := \max\{y \mid j \wedge y = j_*\}$, and, for every $m \in \mathrm{MI}^c(L)$, we have $\kappa^{-1}(m) := \min\{z \mid m \vee z = m^*\}$. Moreover, a κ-lattice L is *well-separated* if for any pair $y \not\leq z$, there is some $j \in \mathrm{JI}^c(L)$ with $j \leq y$ and $\kappa(j) \geq z$. We remark that a finite lattice is semidistributive if and only if it is a κ-lattice, in which case it is automatically well-separated (for more details, see [11]). In general, however, complete semidistributivity of an infinite lattice L does not imply that L is a κ-lattice (unless it is weakly-atomic), and being a well-separated κ-lattice does not imply complete semidistributivity of L (see [11, Examples 3.24 and 3.25]).

2.2.2 Left modularity and Extremality for infinite lattices

We now extend left modularity and extremality from the setting of finite lattices. More specifically, for an arbitrary lattice $(L, \leq)$, we say L is *left modular* if it has an inclusion-maximal chain of left modular elements. In contrast with left modularity, the generalization of extremality that we propose in this work is a bit more technical. In particular, we say that an arbitrary lattice $(L, \leq)$ is *extremal* if there exist a bijection $\lambda : \mathrm{JI}^c(L) \to \mathrm{MI}^c(L)$ and a maximal chain C in L such that, for each $x \in C$, if $j \in \mathrm{JI}^c(L)$ satisfies $j \not\leq x$, then $x \leq \lambda(j)$, and further, if $x \lessdot y$ for two elements x, y of C, then there is a unique completely join-irreducible j with $j \leq y$ and $j \not\leq x$, and a unique completely meet-irreducible m with $x \leq m$ and $y \not\leq m$, and $\lambda(j) = m$.

Provided that we restrict ourselves to the setting of finite lattices, the following lemma shows that the more general notion of extremality considered in this subsection actually coincides with the classical notion of extremality treated in the literature; see Section 2.1.

Lemma 1. *A finite lattice L is extremal in the above sense if and only if it is extremal in the usual sense.*

Now, we can state the main result of this subsection.

Theorem 3. *Let $(L, \leq)$ be a well-separated κ-lattice. Then, L is left modular if and only if L is extremal.*

The preceding theorem can be seen as a conceptual generalization of the setting and assertions of Theorem 1. As shown in the rest of this note, we give concrete examples of infinite lattices which satisfy the setting of Theorem 3, and give a representation-theoretical characterization of those lattices of torsion classes that admit such properties.

3 Lattice of torsion classes

In this section, we study some important properties of the lattice of torsion classes that arise in representation theory of finite dimensional algebras. We start with a brief recollection of some notions in representation theory needed below. For more rudimentary materials, we refer to [2].

Below, A is assumed to be a unital connected finite dimensional associative algebra over a field k, and $\operatorname{mod} A$ denotes the category of finitely generated left A-modules. Let $\operatorname{ind} A$ denote the set of all isomorpism classes of indecomposable modules in $\operatorname{mod} A$. We recall that a module M in $\operatorname{mod} A$ is said to be a *brick* if $\operatorname{End}_A(M)$ is a division algebra. Let $\operatorname{brick} A$ denote the set of all isoclasses of bricks in $\operatorname{mod} A$. Obviously, $\operatorname{brick} A \subseteq \operatorname{ind} A$. Then, A is called *representation-finite* (respectively, *brick-finite*) if $\operatorname{ind} A$ (respectively, $\operatorname{brick} A$) is a finite set. Each representation-finite algebra is brick-finite, but the converse is far from true, in general.

Throughout, by a subcategory $\mathcal{C}$ of $\operatorname{mod} A$, we always assume that $\mathcal{C}$ is full and closed under isomorphism classes. A subcategory $\mathcal{C}$ in $\operatorname{mod} A$ is called *extension-closed* provided that for each short exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} A$, if L and N belong to $\mathcal{C}$, then so does M. An extension-closed subcategory $\mathcal{T}$ in $\operatorname{mod} A$ is called a *torsion class* if $\mathcal{T}$ is also closed under quotients (i.e., if $X \in \mathcal{T}$, any quotient module of X is also in $\mathcal{T}$). Let $\operatorname{tors} A$ denote the set of all torsion classes in $\operatorname{mod} A$. It is known that each $\mathcal{T} \in \operatorname{tors} A$ is uniquely determined by the bricks that it contains.

For each subcategory $\mathcal{C}$ in $\operatorname{mod} A$, we set

$$\mathcal{C}^{\perp} := \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(C, X) = 0, \text{ for every } C \in \mathcal{C}\}.$$

Let $\mathcal{T} \in \operatorname{tors} A$. Then, $\mathcal{T}^{\perp}$ is a *torsion-free* class in $\operatorname{mod} A$, meaning that $\mathcal{T}^{\perp}$ is a subcategory of $\operatorname{mod} A$ which is closed under extensions and submodules. Hence, $(\mathcal{T}, \mathcal{T}^{\perp})$ is said to be a *torsion pair* in $\operatorname{mod} A$. Obviously, $(0, \operatorname{mod} A)$ and $(\operatorname{mod} A, 0)$ form torsion pairs in $\operatorname{mod} A$. We say $(\mathcal{T}, \mathcal{T}^{\perp})$ is a non-trivial torsion pair in $\operatorname{mod} A$ provided $\mathcal{T} \neq 0$ and $\mathcal{T} \neq \operatorname{mod} A$. Note that A admits no non-trivial torsion pair if and only if A is local. Recall that a $(\mathcal{T}, \mathcal{T}^{\perp})$ is said to be a *splitting* torsion pair if for any $X \in \operatorname{ind} A$ we have $X \in \mathcal{T}$ or $X \in \mathcal{F}$. An arbitrary (non-local) algebra may admit no non-trivial splitting torsion pairs (for more details, see [2, Chapter VI]).

In the following, we treat $\operatorname{tors} A$ as a complete lattice. More specifically, for $\mathcal{T}$ and $\mathcal{T}'$ in $\operatorname{tors} A$, we put $\mathcal{T} \leq \mathcal{T}'$ if $\mathcal{T} \subseteq \mathcal{T}'$. Moreover, for a family of torsion classes $\{\mathcal{T}_i\}_{i \in I}$ in $\operatorname{tors} A$, we have $\bigwedge_{i \in I} \mathcal{T}_i = \bigcap_{i \in I} \mathcal{T}_i$, while the join $\bigvee_{i \in I} \mathcal{T}_i$ is defined as the intersection of all torsion classes $\mathcal{T} \in \operatorname{tors} A$ that contain $\bigcup_{i \in I} \mathcal{T}_i$. Henceforth, by $(\operatorname{tors} A, \leq)$, we denote the lattice of torsion classes in $\operatorname{mod} A$. We often suppress $\leq$ from the notation.

The next theorem lists some important properties of the lattice of torsion classes.

Theorem 4 ([7, 11])**.** *Let A be a finite-dimensional algebra. Then,*

1. $(\operatorname{tors}(A), \leq)$ *is weakly atomic and completely semidistributive.*
2. $(\operatorname{tors}(A), \leq)$ *is a well-separated κ-lattice.*

The quiver $\operatorname{Hasse}(\operatorname{tors} A)$ is known to be equipped with an important brick-labeling, that is, each arrow is uniquely labeled by an element of $\operatorname{brick} A$; see [3, 7]. However, a single element of $\operatorname{brick} A$ may appear as the label of more than one arrow in $\operatorname{Hasse}(\operatorname{tors} A)$. More precisely, for $\mathcal{U} \subseteq \mathcal{T}$, if $\operatorname{brick}[\mathcal{U}, \mathcal{T}] := \{X \in \operatorname{brick} A \mid X \in \mathcal{T} \cap \mathcal{U}^{\perp}\}$, then there is an arrow $\mathcal{T} \to \mathcal{U}$ in $\operatorname{Hasse}(\operatorname{tors} A)$ if and only if $\operatorname{brick}[\mathcal{U}, \mathcal{T}]$ has a unique element. This brick is the *label* of $\mathcal{T} \to \mathcal{U}$. The following theorem further highlights the significance of bricks in the study and characterization of some lattice theoretical features of $\operatorname{tors} A$.

Theorem 5 ([7])**.** *Let A be an algebra and let $\mathcal{U} \subseteq \mathcal{T}$ in* $\operatorname{tors} A$.

1. *There is a bijection $\psi : \mathrm{JI}^c([\mathcal{U}, \mathcal{T}]) \to \operatorname{brick}[\mathcal{U}, \mathcal{T}]$. In particular, ψ sends $\mathcal{T}'$ to the brick that labels the unique arrow of* $\operatorname{Hasse}[\mathcal{U}, \mathcal{T}]$ *that starts at $\mathcal{T}'$.*
2. *There is a bijection $\phi : \mathrm{MI}^c([\mathcal{U}, \mathcal{T}]) \to \operatorname{brick}[\mathcal{U}, \mathcal{T}]$. In particular, ϕ sends $\mathcal{U}'$ to the brick that labels the unique arrow of* $\operatorname{Hasse}[\mathcal{U}, \mathcal{T}]$ *that ends at $\mathcal{U}'$.*

Consequently, the three sets $\operatorname{brick} A$, $\mathrm{JI}^c(\operatorname{tors} A)$, *and* $\mathrm{MI}^c(\operatorname{tors} A)$ *are in bijection.*

3.1 Brick-splitting torsion classes

For an algebra A, we use the set of bricks and their properties to introduce a modern brick-analogue of the classical splitting torsion pairs. In particular, for $\mathcal{T} \in \operatorname{tors} A$, we say $(\mathcal{T}, \mathcal{T}^{\perp})$ is a *brick-splitting* torsion pair if every $B \in \operatorname{brick} A$ belongs either to $\mathcal{T}$ or $\mathcal{T}^{\perp}$. That being the case, $\mathcal{T}$ is called a *brick-splitting* torsion class. This gives a generalization of the classical notion. Each splitting torsion pair in $\operatorname{mod} A$ is obviously brick-splitting, but the converse is not necessarily true; see the example following Theorem 6.

Now we state our first main result of this section.

Theorem 6. *For any torsion class $\mathcal{T} \in \operatorname{tors} A$, the following are equivalent:*

1. *$\mathcal{T}$ is a left modular element of the lattice $\operatorname{tors} A$;*
2. *$(\mathcal{T}, \mathcal{T}^{\perp})$ is a brick-splitting torsion pair;*
3. *Every B in $\operatorname{brick} A$ appears as an arrow in $\operatorname{Hasse}[0, \mathcal{T}]$ or $\operatorname{Hasse}[\mathcal{T}, \operatorname{mod} A]$.*

Before we present an example to clarify the above notions, let us highlight an interesting consequence of the preceding theorem for brick-finite algebras. In particular, by Theorems 1, 4, and 6, for a brick-finite algebra A, if $\operatorname{Hasse}(\operatorname{tors} A)$ has a path of length $|\operatorname{brick} A|$, then every torsion class on that path is brick-splitting.

Example: Let Q be the following quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

and define $A := kQ/I$, where I is generated by all paths of length 2. In particular, A is a radical square zero string algebra. Thus, we can describe the indecomposable A-modules in terms of the strings in (Q, I), and we have

$$\operatorname{ind} A = \{e_1, e_2, \alpha, \beta, \gamma, \beta\alpha^{-1}, \gamma^{-1}\beta, \gamma^{-1}\beta\alpha^{-1}\}.$$

Presented in terms of their radical filtration, we obtain

$$\operatorname{ind} A = \left\{ 1, 2, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 2 & & 1 \end{smallmatrix}, \begin{smallmatrix} 2 & & 1 \\ & 2 & \end{smallmatrix}, \begin{smallmatrix} 2 & & 1 & \\ & 2 & & 1 \end{smallmatrix} \right\}.$$

Here, the simple modules S_1 and S_2 correspond to the strings of zero length e_1 and e_2, respectively associated to vertices 1 and 2, and the indecomposable projective modules P_1 and P_2 respectively correspond to the strings $\beta\alpha^{-1}$ and γ. One can easily check that bricks in $\operatorname{mod} A$ correspond to the strings e_1, e_2, and β. That is, $\operatorname{brick} A = \left\{ 1, 2, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}$.

Since A is representation-finite, there are only finitely many torsion classes. As depicted in Figure 1, the lattice structure of $\operatorname{tors} A$ can be described in terms of a finite 2-regular directed graph. In particular, each vertex in Figure 1 denotes a torsion class $\mathcal{T}$ in $\operatorname{tors} A$, labeled by a module M that generates $\mathcal{T}$; more specifically, $\mathcal{T} = \operatorname{Fac}(M)$, where $\operatorname{Fac}(M) := \{X \in \operatorname{mod} A \mid X \textit{ is a quotient of } M^d\textit{, for some } d \in \mathbb{Z}_{>0}\}$. Moreover, each arrow in Figure 1 indicates a cover relation in $\operatorname{tors} A$ induced by the inclusion.

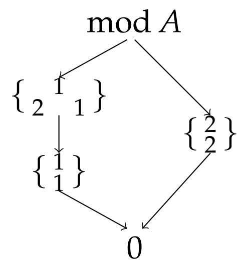

Figure 1: Lattice of torsion classes in the Example. A non-trivial torsion class is specified by the module that generates it.

Let $\mathcal{T}$ be the torsion class labeled by $\{{}_{2}{}^{1}{}_{1}\}$, that is, $\mathcal{T} = \mathrm{Fac}({}_{2}{}^{1}{}_{1})$. Then, the set of indecomposables in $\mathcal{T}$ is $\{1, \begin{smallmatrix}1\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, {}_{2}{}^{1}{}_{1}\}$. Moreover, observe that the set of indecomposables in $\mathcal{T}^{\perp}$ is $\{2, \begin{smallmatrix}2\\2\end{smallmatrix}\}$. Evidently, $(\mathcal{T}, \mathcal{T}^{\perp})$ is a brick-splitting torsion pair in mod A. However, $(\mathcal{T}, \mathcal{T}^{\perp})$ is not splitting, particularly because ${}^{2}{}_{2}{}^{1}$ and ${}^{2}{}_{2}{}^{1}{}_{1}$ do not belong to $\mathcal{T}$ nor $\mathcal{T}^{\perp}$.

If $\mathcal{T}' := \mathrm{Fac}(\begin{smallmatrix}1\\1\end{smallmatrix})$, one can similarly show that the torsion pair $(\mathcal{T}', \mathcal{T}'^{\perp})$ is brick-splitting, but not splitting. Moreover, unlike $\mathcal{T}$ and $\mathcal{T}'$, a simple computation shows that the torsion class labeled by $\{\begin{smallmatrix}2\\2\end{smallmatrix}\}$ is not brick-splitting. More precisely, if $\mathcal{T}'' := \mathrm{Fac}(\begin{smallmatrix}2\\2\end{smallmatrix})$, then the brick $\begin{smallmatrix}1\\2\end{smallmatrix}$ does not belong to $\mathcal{T}''$, nor $\mathcal{T}''^{\perp}$.

Finally, observe that mod A has no non-trivial splitting torsion pair.

3.2 Brick-directed algebras

To state the second main result of this section, some preparation is needed. Recall that, for an algebra A, a *path* in mod A is a sequence $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t$, where t is a positive integer, $X_0, \ldots, X_t$ are indecomposables, and for $1 \leq i \leq t$, $f_i : X_{i-1} \to X_i$ is a non-zero non-invertible morphism in mod A. Such a path is called a *cycle* if X_0 and X_t are isomorphic. Then, $X \in \mathrm{ind}\, A$ is said to be *directing* if it does not lie on any cycle in mod A, and an algebra A is called *representation-directed* if all indecomposables are directed. Observe that every representation-directed algebra is representation-finite, but one can easily find examples to show the converse is not true in general.

As another application of bricks, we give a generalization of representation-directed algebras as follows. First, we say $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m$ is a *brick-cycle* in mod A if each X_i is a brick, for all $0 \leq i \leq m$, with $X_0 \simeq X_m$, and every $f_j : X_{j-1} \to X_j$ is non-zero and non-invertible, for $1 \leq j \leq m$. Then, we call A a *brick-directed* algebra if mod A contains no brick-cycles. Each representation-directed algebra is brick-directed, but the converse is far from being true. In particular, there exist brick-directed algebras which are representation-infinite, or even brick-infinite (see Theorem 9).

In the following, we show how brick-directed algebras play an important role in the study of the lattice of torsion classes and highlight some conceptual links with the lattice theoretical notions considered in the preceding sections. Before doing that, let us give alternative and more tractable characterizations of brick-directed algebras. In particular, by $Q^b(A)$ we denote the *brick-quiver* of A, constructed as follows: The vertices of $Q^b(A)$ are in bijection with elements of brick A, and for each pair of distinct (hence non-isomorphic) modules X and Y in brick A, we put an arrow from X to Y if $\mathrm{Hom}_A(X, Y) \neq 0$. For an explicit example, see [1, Example 4.11]. As shown in [1, Prop. 3.6], $Q^b(A)$ is connected and has no loops. Moreover, $Q^b(A)$ has finitely many arrows if and only if it has finitely many vertices (which is the case if and only if A is brick-finite).

The following theorem gives several characterizations of brick-directed algebras.

Theorem 7. *For an algebra A, the following are equivalent:*

1. *A is brick-directed;*
2. *$Q^b(A)$ has no cycle;*
3. *There exists a maximal chain $\{\mathcal{T}_i\}_{i\in I}$ in* tors A *such that each $\mathcal{T}_i$ is a brick-splitting torsion class.*

As an important consequence of the above theorem, we obtain our second main result of this section. More specifically, the following theorem gives a full characterization of those algebras A for which tors A is a trim lattice; compare with Theorem 1.

Theorem 8. *If A is a brick-finite algebra, the following are equivalent:*

1. *A is brick-directed;*
2. tors A *is a left modular lattice;*
3. tors A *is an extremal lattice;*
4. tors A *is a trim lattice.*

4 Construction of some left modular and extremal lattices

In this section, we briefly recall our explicit construction of many brick-directed algebras via a process known as "gluing". As an immediate consequence, we obtain an abundance of finite and infinite lattice of torsion classes which satisfy the assumptions of our main results (i.e., Theorems 1, 3, 6, 7, and 8). For more details and explicit examples of algebras produced via this construction, we refer to [1, Section 4.3].

Let $A = kQ/I$ and $B = kQ'/I'$ be connected algebras, where I and I' are assumed to be admissible ideals. Let R and R' denote two sets of admissible relations, respectively, in kQ and kQ', which generate I and I'. Assume $Q_0 = \{x_1, \ldots, x_n\}$ and $Q'_0 = \{y_1, \ldots, y_m\}$ denote, respectively, the vertex set of A and B. Provided that x_n is a sink in Q, and y_1 is a source in Q', then by $\Lambda := k\tilde{Q}/\tilde{I}$ we denote the algebra obtained from *gluing* of A and B at the vertices x_n and y_1, with the bound quiver $(\tilde{Q}, \tilde{I})$ constructed as follows:

$\tilde{\mathbf{Q}}$: The quiver obtained from Q and Q', by identifying the vertices x_n and y_1, and keeping the rest untouched. Hence, the set of vertices of $\tilde{Q}$ is $\tilde{Q}_0 = (Q_0 - \{x_n\}) \cup (Q'_0 - \{y_1\}) \cup \{v\}$, where vertex v is obtained from the identification of x_n and y_1. Thus, $\tilde{Q}_0 = \{x_1, \ldots, x_{n-1}, v, y_2, \ldots, y_m\}$, and the set of arrows is given by $\tilde{Q}$ is $\tilde{Q}_1 = Q_1 \cup Q'_1$.

$\tilde{\mathbf{I}}$: The admissible ideal in $k\tilde{Q}$ is generated by the set $\tilde{R} := R \cup R' \cup R_v$. Here, R_v denotes the set of all the paths of length 2 in $\tilde{Q}$ that pass through vertex v. More specifically, each element of R_v is of the form $\beta\alpha$, for an arrow $\alpha \in Q_1$ ending at vertex x_n, and an arrow $\beta \in Q'_1$ starting at vertex y_1.

Observe that $(\tilde{Q}, \tilde{I})$ contains a copy of (Q, I), as well as a copy of (Q', I'). Moreover, other than vertex v in $\tilde{Q}$, any other vertex in $\tilde{Q}_0$ belongs either to Q_0 or Q'_0, and not both. We note that, if $\Lambda = k\tilde{Q}/\tilde{I}$ is obtained from gluing A and B, then Λ is a connected algebra of rank $|Q_0| + |Q'_0| - 1$. Furthermore, the following hold:

1. Each $M \in \operatorname{ind} \Lambda$ is either entirely supported over Q or entirely supported over Q'.
2. Λ is brick-directed if and only if both A and B are brick-directed.

Starting from brick-directed algebras of smaller ranks, via the gluing construction we can create new brick-directed algebras of higher ranks. Before we state the following theorem, recall that, by Drozd's trichotomy theorem, each algebra A falls into exactly one of the three disjoint subfamilies, given by representation-finite, (representation-infinite) tame or wild algebras. For the proof and more details, see [1].

Theorem 9. *For any integer $n > 1$, there exists a connected brick-directed algebra Λ of rank n of any of the following types:*

(I) representation-finite algebra;

(II) brick-finite tame algebra;

(III) brick-infinite tame algebra;

(IV) brick-finite wild algebra;

(V) brick-infinite wild algebra.

In [1, Section 4.3], we give an explicit construction for each type of the algebras in the preceding theorem.

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