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Skew shapes, Ehrhart positivity and beyond

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Abstract. A classical result by Kreweras (1965) allows one to compute the number of plane partitions of a given skew shape and bounded parts as certain determinants. We prove that these determinants expand as polynomials with nonnegative coefficients. Our main result has numerous ramifications impacting on several areas within algebraic combinatorics. It can be reformulated in terms of order polynomials of cell posets of skew shapes, and explains important positivity phenomena about the Ehrhart polynomials of matroids, and of order polytopes. Among other applications, we show that all fence posets, including the zig-zag poset, have order polynomials with nonnegative coefficients. We discuss a general method for proving positivity which reduces to showing positivity of the linear terms of the order polynomials.

Keywords: Order polynomials, Ehrhart polynomials, plane partitions, Fence posets.

1 Introduction

Let P be a finite partially ordered set. In this article we shall be concerned with one of the most prominent polynomial invariants of P, called the *order polynomial*. Throughout, this polynomial will be denoted by $\Omega(P;t) \in \mathbb{Q}[t]$. The order polynomial encodes useful information about the structure of P, including the number of linear extensions, the number of antichains, among many other features of relevance in combinatorics.

One geometric way of viewing the order polynomial of a poset P is under the lens of the Ehrhart theory of polytopes. The *Ehrhart polynomial* of a d-dimensional lattice polytope $\mathscr{P} \subseteq \mathbb{R}^n$ is the unique polynomial $\operatorname{ehr}(\mathscr{P},t) \in \mathbb{Q}[t]$ such that $\operatorname{ehr}(\mathscr{P},t) = \#(t\mathscr{P} \cap \mathbb{Z}^n)$ for each positive integer t. The polynomiality of the map $t \mapsto \operatorname{ehr}(\mathscr{P},t)$ is a foundational result by Ehrhart [9].

The relationship between order polynomials and Ehrhart polynomials can be made explicit as follows: starting from a poset P on n elements, one can construct the so-called

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order polytope $\mathscr{O}(P) \subseteq \mathbb{R}^n$ (for the precise definition see equation (2.2) below). Then, the following relation holds:

$$\operatorname{ehr}(\mathscr{O}(P), t) = \Omega(P; t+1). \tag{1.1}$$

For general lattice polytopes $\mathscr{P}$, it can be proved that the degree of the Ehrhart polynomial is $d = \dim \mathscr{P}$ and that it has constant term equal to 1. Moreover, writing $\operatorname{ehr}(\mathscr{P}, t) = a_d\, t^d + a_{d-1}\, t^{d-1} + \cdots + a_1 t + 1$, then one has that $a_d = \operatorname{vol}(\mathscr{P})$, and $a_{d-1} = \frac{1}{2}\operatorname{vol}(\partial\mathscr{P})$, where $\operatorname{vol}(\mathscr{P})$ stands for the relative volume of the polytope $\mathscr{P}$ with respect to intersection of the lattice $\mathbb{Z}^n$ with the affine span of $\mathscr{P}$ and, similarly, $\operatorname{vol}(\partial\mathscr{P})$ stands for the sum of the relative volumes of the facets of $\mathscr{P}$. In light of this, the two highest degree coefficients and the constant one are nonnegative for all lattice polytopes $\mathscr{P}$. Unfortunately the middle coefficients can be negative—and more so, all of them can be negative at the same time. If the polytope $\mathscr{P}$ satisfies that all of the coefficients of $\operatorname{ehr}(\mathscr{P}, t)$ are nonnegative, then we say that $\mathscr{P}$ is *Ehrhart positive*. For a detailed reading about Ehrhart positivity, we refer to surveys by Liu [25] and Ferroni–Higashitani [14].

One of the most challenging endeavors in Ehrhart theory is that of finding classes of polytopes that are Ehrhart positive. As is suggested by equation (1.1), one may ask similar questions about the sign patterns of order polynomials of posets, noting that if $\Omega(P; t)$ has nonnegative coefficients, then $\mathscr{O}(P)$ is Ehrhart positive, but not conversely.

The Ehrhart positivity of order polytopes has attracted increasing attention during the past decade. As was noted by Stanley in [34, Exercise 3.164], there exist posets whose order polynomials have negative coefficients. More strongly, there are order polytopes that fail to be Ehrhart positive. In fact, the results by Liu and Tsuchiya [26] and Liu, Xin, Zhang [24] imply that if $|P| < 14$ then $\mathscr{O}(P)$ is Ehrhart positive, but for every $n \geq 14$ there exists a poset P on n elements such that $\mathscr{O}(P)$ is not Ehrhart positive. The positivity of the coefficients of $\Omega(P; t)$ is even more rare, in the sense that one can find posets on 5 elements (for example, the 5-element poset in Figure 2c) attaining a negative coefficient.

The central result of this article establishes the nonnegativity of the order polynomials of the cell posets of arbitrary skew shapes.

Theorem 1.1 (Main Theorem 4.2)**.** *Let λ/μ be a skew shape of size n. The coefficients of the order polynomial $\Omega(P_{\lambda/\mu}; t)$ are nonnegative.*

We now explain the notation in the preceding theorem. Consider any skew shape λ/μ represented as a skew Young diagram. The poset $P_{\lambda/\mu}$ is the poset whose elements are the boxes $X(\lambda/\mu) = \{(i,j) : j \in [\mu_i + 1, \lambda_i]\}$ and covering relations given by $(i,j) \succeq (i+1, j)$, $(i,j) \succeq (i, j+1)$, see Figure 1.

The order polynomial of a skew shape λ/μ enumerates plane partitions of shape λ/μ with bounded parts. There is a famous determinantal formula due to Kreweras [22] for skew shapes and to MacMahon for straight shapes [27] for $\Omega(P_{\lambda/\mu}; t)$ (see Prop. 2.2). The content of Theorem 1.1 can be recast to say that Kreweras determinants expand as polynomials with nonnegative coefficients. In the case of straight shapes, the nonnegativity

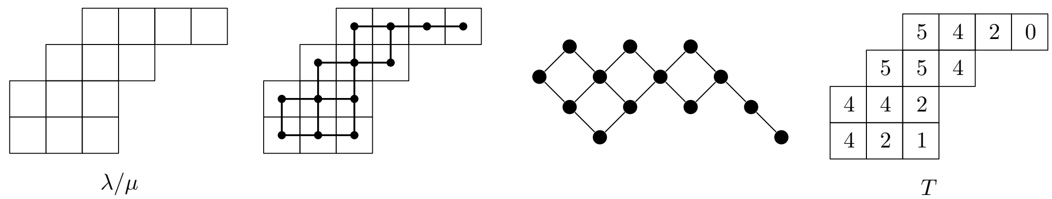

Figure 1: The skew shape $\lambda/\mu = 6533/21$, its cell poset $P_{\lambda/\mu}$, and a plane partition with entries in $\{0, \dots, 5\}$.

of this expansion was proved by Fomin and Kirillov [18, Theorem 2.1] in the context of Schubert calculus.

When the skew shape λ/μ is a ribbon, i.e., when its Young diagram does not contain a 2×2 square, the cell poset $P_{\lambda/\mu}$ is commonly referred to as a *fence poset*; see Figure 2a for an example.

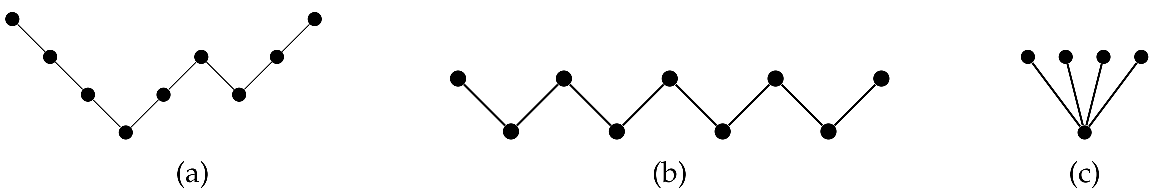

Figure 2: The Hasse diagrams of (a) a fence poset, (b) the zig-zag poset Z_9, and (c) one minimum element covered by 4 elements.

Fence posets have received considerable attention in the last few years, due to their relevance in the study of quiver representations, cluster algebras, and valuations in matroid theory. A relevant problem that garnered much interest was the unimodality conjecture due to Morier-Genoud and Ovsienko [29] for the rank polynomials of the lattices of ideals of fence posets, which was settled by Kantarcı Oğuz and Ravichandran [20]. For order polynomials, we obtain the following corollary of Theorem 1.1.

Corollary 1.2 (Corollary 5.1)**.** *Let P be a fence poset. Then $\Omega(P; t)$ has nonnegative coefficients.*

The above result was one of the main motivations to prove Theorem 1.1. The first author raised this question[1] in a workshop at the American Institute of Mathematics in May of 2022 and, since then, various independent attempts of proving this statement have been made by several groups of researchers. Two special cases deserve an explicit mention. First, the case of fences with only two 'arms', i.e., when $\mu = \varnothing$ and λ is a hook: despite appearances, this simple case is already a fairly difficult problem. The

[1] See Problem 1.2 here `http://aimpl.org/ehrhartineq/1/`

nonnegativity of the order polynomial in that special case is one of the main results by Ferroni in [13], and reappeared as problem B5 in the 2024 Putnam competition, being solved by only one participant. Second, the case of fence posets whose Hasse diagram is a zig-zag (see Figure 2b below): the order polynomials of these posets have attracted considerable attention, e.g., in [6, 30], even though the nonnegativity of the coefficients remained open, which now follows as a special case.

An influential conjecture due to De Loera, Haws, and Köppe [7], disproved by the first author in [12], postulated that matroid polytopes were Ehrhart positive. Our results have the following implication for certain matroid polytopes.

Corollary 1.3. *Let* M *be a matroid. If* M *is a direct sum of loops, coloops, and snake matroids, then its base polytope is Ehrhart positive.*

In order to prove Theorem 1.1 we demonstrate a general strategy to establish the nonnegativity of coefficients of order polynomials. This strategy can be summarized via the following statement that can be applicable to other posets.

Theorem 1.4 (Theorem 3.4)**.** *Let* $\mathcal{F}$ *be a family of posets closed under taking lower and upper ideals, such that for every* $P \in \mathcal{F}$ *we have* $c_1(P) := [t^1]\,\Omega(P;t) \geq 0$*. Then* $|P|! \cdot \Omega(P;t)$ *has nonnegative integer coefficients for every* $P \in \mathcal{F}$*.*

Many classes of posets appearing in combinatorics display the stability property required by the preceding statement, i.e., they are closed under taking upper and lower ideals. Hence, the nonnegativity of the coefficients of the order polynomial can be reduced to a single coefficient: the linear one. This coefficient has a positive product formula for the posets $P_{\lambda/\mu}$ (see Proposition 4.1).

It is tempting to apply a similar strategy on some classes like planar posets, trees, or even binary trees; but via small examples they can be ruled out very quickly. Nonetheless, based on evidence collected by computer experiments, we conjecture that this general procedure may lead to the positivity of the order polynomial of other classes of posets: (i) posets of width two (the size of a maximal antichain is equal to 2), (ii) cell posets of cylindric skew shapes, and (iii) cell posets of shifted skew shapes. These classes of posets are easily seen to be closed under the operations of taking upper and lower order ideals, so it would suffice to establish the nonnegativity of the linear term.

This is the extended abstract of [16], which contains all the proofs and references.

2 Posets, order polynomials, and plane partitions

We will assume that the reader is acquainted with the fundamental notions about partially ordered sets. However, below we recapitulate some of the essential concepts in this topic, especially those relevant for the content of this paper. We refer to [34] for the undefined terminology and further reading.

Let $P = (X, \preceq)$ be a partially ordered set and let $|X| = n$. Let $\mathcal{O}(P; S)$ be the set of order preserving maps $f : X \to S$. The *order polynomial* of P is the unique polynomial $\Omega(P; t) \in \mathbb{Q}[t]$ such that

$$\Omega(P; t) := \#\{f : X \to [1, \ldots, t] : f(x) \leq f(y) \text{ if } x \preceq y\} = \#\mathcal{O}(P; [1, \ldots, t]),$$

for each positive integer t. The fact that this counting function is indeed a polynomial is a classical result [34, Section 3.12]; one way to see this directly is through the formula

$$\Omega(P; t) = \sum_{k=1}^{n} \sum_{\varnothing = I_0 \subsetneq \cdots \subsetneq I_k = P} \binom{t}{k}, \tag{2.1}$$

where the sum ranges over all chains of ideals of P. If $P = \varnothing$ then we set $\Omega(P; t) = 1$. Note that when P is not the empty poset, we have that $\Omega(P; 0) = 0$.

The *order polytope* of P, introduced by Stanley in [33], is defined by

$$\mathscr{O}(P) := \{f \in [0, 1]^P : f(u) \leq f(v) \text{ for all } u \preceq v \text{ in } P\}. \tag{2.2}$$

This is a subpolytope of the unit cube $[0, 1]^P$ whose vertices are in bijection with the order ideals of the poset. In particular, it is a lattice polytope so we may consider its Ehrhart polynomial $\mathrm{ehr}(\mathscr{O}(P), t)$, as defined in the introduction. The identity in equation (1.1) implies that both the order polynomial and the Ehrhart polynomial have the same leading coefficient, which in turn is equal to the relative volume of the polytope $\mathscr{O}(P)$. In turn, a result of Stanley in [33] implies that this equals (up to the normalization by $|P|!$) the number of linear extensions of P.

Example 2.1. *Consider the poset P_n arising from an antichain of size n covering a single minimum element. For example, Figure 2c depicts this poset for $n = 4$. The order polytope $\mathscr{O}(P)$ is a pyramid over an n-dimensional cube. The order polynomial can be computed for a positive integer t as follows: $\Omega(P_n; t) = \sum_{j=1}^{t} j^n$, see, e.g., [14, Example 2.8]. The polynomial $\Omega(P_n; t)$ is known in the literature as the n-th* Faulhaber's polynomial. *When $n = 4$, this is $\Omega(P_4; t) = \frac{1}{5}t^5 + \frac{1}{2}t^4 + \frac{1}{3}t^3 - \frac{1}{30}t$ which has a negative linear term. The smallest n such that* $\mathrm{ehr}(\mathscr{O}(P_n), t) = \Omega(P_n; t + 1)$ *has negative coefficients is $n = 20$, see [26].*

A *plane partition T* is a filling of the shape λ/μ with nonnegative integers that is weakly decreasing rows and weakly decreasing columns. See Figure 1. It is immediate to verify that every plane partition corresponds to an order preserving map $T : X \to \mathbb{Z}_{\geq 0}$. We set $\mathrm{PP}_{\lambda/\mu}(t) := \#\mathcal{O}(P_{\lambda/\mu}; [0, \ldots, t])$, so $\Omega(P_{\lambda/\mu}; t) = \mathrm{PP}_{\lambda/\mu}(t - 1)$. Note that this is also the *zeta polynomial* of the interval $[\mu, \lambda]$ in Young's lattice, see [34, Ex. 3.149].

In [22], Kreweras gave the following determinantal identity for $\mathrm{PP}_{\lambda/\mu}(t)$.

Proposition 2.2 (Kreweras [22])**.** *Let $\mathrm{PP}_{\lambda/\mu}(t)$ be the number of plane partitions of shape λ/μ with entries less than or equal to t. We have that*

$$\mathrm{PP}_{\lambda/\mu}(t) = \det\left[\binom{\lambda_i - \mu_j + t}{\lambda_i - \mu_j - i + j}\right]_{i,j=1}^{\ell(\lambda)}. \tag{2.3}$$

3 Order polynomials

The goal in this section is to establish a general framework to prove the nonnegativity of order polynomials. As we will see, for certain classes of posets, this task may be reduced to just proving the positivity of *one* coefficient.

The first ingredient is the following elementary "coproduct formula" for order polynomials. This is well known to the experts and can be found, e.g., in [23, Ex. 1.2.4(d)] or [32, Theorem 2.6].

Proposition 3.1. *For variables x, y we have the following polynomial identity*

$$\Omega(P; x+y) = \sum_{I \subset P} \Omega(I; x)\Omega(P \setminus I; y),$$

where the sum goes over all ideals I of P (including $\varnothing$ and P).

Remark 3.2. *The zeta polynomial of a poset P, denoted $Z(P; t) \in \mathbb{Q}[t]$, enumerates multichains of elements in P of size t. Defining $Z([u,v]; t)$ as the zeta polynomial of the interval $[u,v] \subseteq P$, it is easy to see that*

$$Z([u,w]; x+y) = \sum_{u \preceq v \preceq w} Z([u,v]; x)Z([v,w]; y).$$

The order polynomial of a poset P equals the zeta polynomial of its lattice of order ideals $J(P)$ (e.g., see [34, Section 3.12]). One can use this to derive another proof of the last proposition.

Proposition 3.3. *For any poset P, let $\Omega(P; t) = \sum_{k=1}^{n} c_k(P) t^k$ expand with coefficients $c_k(P)$. Then for every $k \geq 2$ we have that $c_k(P) = \dfrac{1}{2^k - 2}\left(\sum_{I \subsetneq P, I \neq \varnothing} \sum_{i=1}^{k-1} c_i(I) c_{k-i}(P \setminus I)\right)$, where the sum goes over all nonempty ideals $I \neq P$.*

Proof. Apply Proposition 3.1 with $x = y = t$, we have that

$$\Omega(P; 2t) = \sum_{I \subset P} \Omega(I; t)\Omega(P \setminus I; t).$$

Subtracting the terms with $I = \varnothing$ and $I = P$ we have

$$\Omega(P; 2t) - 2\Omega(P; t) = \sum_{I \subsetneq P; I \neq \varnothing} \Omega(I; t)\Omega(P \setminus I; t).$$

Finally, expand both sides as polynomials in t and compare coefficients at t^k, noting that

$$\Omega(P; 2t) - 2\Omega(P; t) = \sum_{k=1}^{n} c_k(P) 2^k t^k - 2c_k(P) t^k = \sum_{k=2}^{n} (2^k - 2) c_k(P) t^k. \qquad \square$$

Now let $\mathcal{F}$ be a family of posets closed downwards, i.e., if $P \in \mathcal{F}$ and $I \subset P$ is an ideal then $I, P \setminus I \in \mathcal{F}$. Assume $\varnothing \in \mathcal{F}$ and necessarily $\{1\} \in \mathcal{F}$ (poset of 1 element).

Theorem 3.4. *Let $\mathcal{F}$ be a family of posets closed under taking lower and upper ideals, such that for every $P \in \mathcal{F}$ we have $c_1(P) := [t^1]\,\Omega(P;t) \geq 0$. Then $c_k(P) \geq 0$ for every $k \geq 1$ and $P \in \mathcal{F}$. If $n = |P|$, then $n! \cdot \Omega(P;t)$ has nonnegative integer coefficients.*

Proof. The proof follows by strong induction on n, the size of the posets in the family. We have $\Omega(\varnothing;t) = 1$ and $\Omega(\{1\};t) = t$, both in $\mathbb{Z}_{\geq 0}[t]$. Suppose now that the coefficients of the order polynomial of every $P \in \mathcal{F}$ with $|P| \leq n-1$ are nonnegative. Let $P \in \mathcal{F}$ be a poset with n elements. By assumption we have that $c_0(P) = 0$ and $c_1(P) \geq 0$. For $k \geq 2$ we invoke the formula from Proposition 3.3, noting that $c_i(I) \geq 0$ and $c_{k-i}(P \setminus I) \geq 0$ since $I, P \setminus I \in \mathcal{F}$ and have at most $n-1$ elements each. Thus $c_k(P) \geq 0$ and this finishes the induction.

Finally, observe that $n! \cdot \Omega(P;t) \in \mathbb{Z}[t]$ for every poset P. This can be easily seen from formula (2.1) since $n! \cdot \binom{t}{k} = n(n-1)\cdots(k+1)t(t-1)\cdots(t-k+1) \in \mathbb{Z}[t]$. $\square$

4 Positivity of skew plane partitions

Let $\mathcal{F}$ be the family of all posets $P_{\lambda/\mu}$ for all skew shapes λ/μ. This family is closed under taking lower or upper ideals, since if $I \subset P_{\lambda/\mu}$ is an ideal then $I = P_{\nu/\mu}$ for some ν and $P \setminus I = P_{\lambda/\nu}$. More specifically, if $(i, \nu_i) \in I$ is the last element in I from row i of $[\lambda/\mu]$ then $(i,j) \in I$ for all $j \leq \nu_i$, and $\nu_{i+1} \leq \nu_i$, or else $(i+1, \nu_{i+1}) \succeq (i, \nu_{i+1}) \notin I$.

We can then apply Theorem 3.4 if we show that $c_1(P_{\lambda/\mu}) \geq 0$ for every skew shape.

Proposition 4.1. *Let λ/μ be a skew shape of size n and length ℓ with order polynomial $\Omega(P_{\lambda/\mu};t) = \sum_{k=1}^{n} c_k(P)\, t^k$. Then if λ/μ is not connected, i.e., $\lambda_i = \mu_i$ or $\lambda_i \leq \mu_{i-1}$ for some i we have $c_1(P_{\lambda/\mu}) = 0$. If λ/μ is connected then $c_1(P_{\lambda/\mu}) = \frac{(\lambda_1-\mu_\ell-1)!(\ell-1)!}{(\lambda_1-\mu_\ell-1+\ell)!}$.*

Proof. If $\ell(\lambda) = 1$ then

$$\Omega(P_{\lambda/\mu};t) = \binom{t+\lambda_1-\mu_1-1}{\lambda_1-\mu_1} = \frac{t(t+1)\cdots(t+\lambda_1-\mu_1-1)}{(\lambda_1-\mu_1)!},$$

and so $c_1(P_{\lambda/\mu}) = \frac{1}{\lambda_1-\mu_1}$.

If λ/μ is not connected then it consists of two or more skew shapes $\theta^1, \theta^2, \ldots$ and $P_{\lambda/\mu} = P_{\theta^1} + P_{\theta^2} + \cdots$, where $P + Q$ denotes the direct sum of posets P and Q. Then $\Omega(P_{\lambda/\mu};t) = \prod_i \Omega(P_{\theta^i};t)$ is divisible by t^2 since each of the terms is divisible by t and so $c_1 = 0$.

For $\ell := \ell(\lambda) > 1$ we use Kreweras's formula from Proposition 2.2.

Denote the entries in this matrix by $A_{i,j}(t)$. In that formula, if $\lambda_i - \mu_j - i + j < 0$ then the term is considered 0. For every $j \geq i$ we then have

$$A_{i,j}(t) = \binom{\lambda_i - \mu_j + t - 1}{\lambda_i - \mu_j - i + j} = \frac{(t - 1 + \lambda_i - \mu_j) \cdots (t + i - j)}{(\lambda_i - \mu_j - i + j)!}.$$

Since $\lambda_i > \mu_i \geq \mu_j$ then there is a term t in the above product and the binomial is divisible by t, so $A_{i,j}(t) = tB_{i,j}(t)$ for $i \leq j$.

We now expand $\det \left[A_{i,j}(t)\right]_{i,j=1}^{\ell} = \sum_{w \in \mathfrak{S}_\ell} \operatorname{sign}(w) \prod_i A_{i,w(i)}(t)$. The products are all divisible by t since $w(i) \geq i$ for some i always. In order to get a linear term t then exactly one of $A_{i,w(i)}(t)$ should be divisible by t, i.e., there is exactly one i for which $i \leq w(i)$. We must then have that $w(i) = \ell$ since $\ell \geq w^{-1}(i)$ for all i, and then $w(j) < j$ for all $j \neq i$. Since $w(1) \not< 1$ we must then have $i = 1$, so $w(1) = \ell$. Since the remaining entries in the permutation matrix of w should be below $i = j$, we must have $w(i) = i - 1$ for all $i = 2, \ldots, \ell$. Then the only term in the determinant expansion that gives a linear term is

$$A_{1,\ell}(t) \prod_{i=2}^{\ell} A_{i,i-1}(t) = \binom{t - 1 + \lambda_1 - \mu_\ell}{\lambda_1 - \mu_\ell - 1 + \ell} \prod_{i=2}^{\ell} \binom{t - 1 + \lambda_i - \mu_{i-1}}{\lambda_i - \mu_{i-1} - 1}.$$

We have $\binom{t-1+\lambda_i-\mu_{i-1}}{\lambda_i-\mu_{i-1}-1} = \frac{(t+1)\cdots(t+\lambda_i-\mu_{i-1}-1)}{(\lambda_i-\mu_{i-1}-1)!} = 1 + O(t)$ and

$$\binom{t - 1 + \lambda_1 - \mu_\ell}{\lambda_1 - \mu_\ell - 1 + \ell} = t\frac{(\lambda_1 - \mu_\ell - 1)!(-1)^{\ell-1}(\ell - 1)!}{(\lambda_1 - \mu_\ell - 1 + \ell)!} + O(t^2).$$

Since $\operatorname{sign}(w) = (-1)^{\ell-1}$ we obtain the formula. □

We can now derive Theorem 1.1 by combining Proposition 4.1 and the Theorem 3.4.

Theorem 4.2. *Let λ/μ be a skew shape of size n. Then the coefficients of its order polynomial $\Omega(P_{\lambda/\mu}; t)$ are all nonnegative and $|\lambda/\mu|! \cdot \Omega(P_{\lambda/\mu}; t) \in \mathbb{Z}_{\geq 0}[t]$.*

5 Fence posets and snake matroids

5.1 Fence posets

When the skew shape λ/μ has a Young diagram that does not contain a 2×2 square, we say it is a *ribbon* or a *border strip*. The posets that arise as cell posets of border strips are called *fence posets*. They appear naturally in the study of quiver representations, cluster algebras and, as we will explain below, matroid theory. See [28] for a more detailed discussion on some of these connections.

Specializing Theorem 1.1 to the case in which λ/μ is a ribbon shape leads to the following result.

Corollary 5.1. *Fence posets have order polynomials with nonnegative coefficients.*

Among all fence posets, the most pervasive is the zig-zag poset, which corresponds to a fence poset with covering relations of the form $x_1 \succ x_2 \prec x_3 \succ x_4 \prec \cdots \succ x_{2n}$ or $x_1 \succ x_2 \prec x_3 \succ x_4 \prec \cdots \prec x_{2n+1}$. In this paper we will denote by Z_n the zig-zag poset on n elements. For a depiction of the Hasse diagram of Z_9, see Figure 2b.

The order polynomial and the order polytope of Z_n have been studied in several articles, including [33, 6, 30]. Despite its relevance, not many results regarding $\Omega(Z_n;t)$ have been obtained. Moreover, it has been a long-standing folklore open problem to prove that $\mathscr{O}(Z_n;t)$ is Ehrhart positive. Evidently, now we can obtain a proof as an easy special case of Theorem 4.2 since $Z_n = P_{\zeta_n}$, where ζ_n is a zig-zag ribbon of size n:

$\zeta_n = (k+1,k,\dots,2)/(k-1,k-2,\dots,1)$ if $n = 2k$ and $\zeta_n = (k+1,k,\dots,1)/(k-1,k-2,\dots,1)$, if $n = 2k+1$.

Corollary 5.2. *For every $n \geq 2$, the zig-zag poset Z_n has an order polynomial with nonnegative coefficients. In particular, $\mathscr{O}(Z_n)$ is Ehrhart positive.*

Even though we have confirmed the nonnegativity of the order polynomials of zig-zag posets, we do not[2] have a combinatorial interpretation for the coefficients of $n! \cdot \Omega(Z_n;t)$. By Proposition 2.2 we have that $\Omega(Z_n;t) = \det\left[\binom{t+1-i+j}{2j-2i+2}\right]_{i,j=1}^{k}$, where $k = \lfloor n/2 \rfloor$. Using this determinant formula or by a direct computation one can see that the values that $n!\Omega(Z_n;t)$ takes for $n \geq 1$ are, respectively,

$$t, \qquad t^2+t, \qquad 2t^3+3t^2+1, \qquad 5t^4+10t^3+7t^2+2t, \qquad \dots \tag{5.1}$$

5.2 Matroids

In this section we will address Ehrhart polynomials of matroids. Let M be a matroid on $E=[n]$. The *matroid base polytope* of M is the polytope $\mathscr{P}(\mathsf{M}) \subseteq \mathbb{R}^n$ defined by

$$\mathscr{P}(\mathsf{M}) := \text{convex hull}(e_B : B \in \mathscr{B}),$$

where $e_B := \sum_{i\in B} e_i$ stands for the indicator vector of the basis of M.

Ehrhart polynomials of matroid base polytopes have been intensively studied during the last two decades. The main motivation was the Ehrhart positivity conjecture posed by De Loera, Haws, and Köppe in [7, Conjecture 2], and a stronger version by Castillo and Liu in [3] for all generalized permutohedra. Even though these conjectures were disproved in [12], in many interesting special cases Ehrhart positivity indeed holds true. Some instances appearing in the literature are uniform matroids [11], matroids of rank

[2] Yakob Kahane (private communication) announced an interpretation for such coefficients of zigzags and fence posets.

2 [15], panhandle matroids [8], Catalan matroids [5], generalized permutohedra that are Minkowski sums of standard simplices [31], and various special matroids of small size. More so, the linear term of the Ehrhart polynomials of matroids and generalized permutohedra has been proved nonnegative by Jochemko and Ravichandran [19] and, using different tools, by Castillo and Liu [4]. Further related articles include [1, 21]

The main focus in this section is to explain which matroids can be proved to be Ehrhart positive by applying Theorem 1.1. To this end we recapitulate the essentials on *snake matroids*, assuming that the reader is familiar with lattice path matroids [2].

Definition 5.3. *A matroid* M *is said to be a* snake matroid *if* M *satisfies the following three properties:*

1. M *has at least two elements and is connected.*
2. M *is isomorphic to a lattice path matroid.*
3. M *does not have a* $\mathsf{U}_{2,4}$ *minor, i.e.,* M *is binary.*

We refer to [1], and [17, Appendix A] for a detailed discussion on snakes. When presented as lattice path matroids, snakes have diagrams that look like border strips. The following result is essentially a reformulation of a result by Benedetti, Knauer, and Valencia-Porras appearing in [1, Theorem 3.3].

Theorem 5.4. *Let* P *be a finite poset. The following conditions are equivalent.*

(i) $\mathscr{O}(P)$ *is integrally equivalent to a matroid polytope.*
(ii) $\mathscr{O}(P)$ *is integrally equivalent to a matroid polytope coming from a direct sum of loops, coloops, and snakes.*
(iii) P *is a disjoint union of fences and single element posets.*

Notice that if two polytopes are integrally equivalent, then their Ehrhart polynomials must coincide. In particular, the preceding theorem says that the only Ehrhart polynomials that Theorem 1.1 captures within the realm of matroids are precisely those of the direct sums of snakes, loops and coloops. In particular, we obtain the following result.

Corollary 5.5. *Let* M *be a matroid that is a direct sum of snakes, loops, and coloops. Then,* $\mathscr{P}(\mathsf{M})$ *is Ehrhart positive.*

In [17, Appendix A] Ferroni and Schröter proved that direct sums of snake matroids, loops and coloops, span the valuative group of matroids introduced by Eur, Huh, and Larson [10]. In other words, the Ehrhart polynomial of every matroid can be written as an integer combination of Ehrhart polynomials of matroids that are direct sums of snakes, loops and coloops. As a consequence of Corollary 1.3, we observe the following property of Ehrhart polynomials of matroid base polytopes: they are nonnegative on a spanning set of the valuative group, yet they fail to be nonnegative in general due to the counterexamples found in [12]. To the best of the authors' knowledge, this is the first valuative invariant of matroids arising 'in nature' for which this phenomenon shows.

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