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Asymptotics of generalized Tesler matrices *

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Abstract. This paper studies the asymptotic enumeration of generalized Tesler matrices. To suit this goal, we introduce integral Tesler matrices, linking Tesler matrices to integer partitions and obtaining sharp logarithmic asymptotics for flat hook sum vectors. We then introduce height diagrams to decompose more complicated hook sum vectors into flat pieces. Thereby, we establish the asymptotics of Tesler matrices with staircase hook sum vectors, significantly improving the best known bounds. In particular, the logarithmic asymptotic for the number of regular Tesler matrices is obtained.

Keywords: Tesler matrices, Kostant partition function, asymptotic enumeration

Introduction

Generalized Tesler matrices are upper–triangular matrices with non-negative integer entries that arise in the study of algebraic combinatorics, polyhedral geometry, and representation theory. The number of those is also known in Lie theory as the values of the type A_n Kostant partition function (KPF). Still, even for the number of regular Tesler matrices $T(\mathbf{1}^n)$, the asymptotic estimates were far from sharp. In this paper we drastically improve the previous benchmark from [2] by combining our results for Tesler matrices of flat and staircase hook sums.

In Section 1, we introduce ***integral Tesler matrices*** and obtain asymptotics for flat hook sum vectors [Theorem 14, Theorem 15, Proposition 16]. Then, in Section 2, we utilize the Kostant picture representation and introduce ***height diagrams***, which we use to ***decompose*** general hook sum vectors (weights for KPF) into flat pieces. In particular, we obtain asymptotics for Tesler matrices with 'staircase' hook sums $\mathbf{h} = (h_1, \ldots, h_n)$, $h_k > 0$ (Theorem 22). This, notably, leads to the obtaining (Corollary 23) of

$$\ln T(\mathbf{1}^n) \sim \sqrt{\frac{8}{27}}\, \pi n \sqrt{n}.$$

*This paper is an extended abstract of "Tesler matrices and Lusztig data" DOI.

1 Generalized Tesler matrices and their asymptotics

1.1 Generalized Tesler matrices and integral Tesler matrices

First of all, we shall define Tesler matrices. Let U_n be the set of $n \times n$ upper-triangular matrices with non-negative integer entries.

Definition 1. For a matrix $A = \left(a_{ij}\right) \in U_n$ we define its k^{th} ***hook sum*** h_k (Figure 1) as

$$h_k = \sum_{i=k}^{n} a_{ki} - \sum_{i=1}^{k-1} a_{ik},\ 1 \leq k \leq n,$$

and its ***hook sum vector*** as $\mathbf{h} = (h_1, h_2, \ldots, h_n)$.

$$\begin{bmatrix} \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & \color{blue}0 & 1 & 0 & 0 \\ & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & \color{blue}1 & 0 & 0 \\ & 1 & \color{blue}0 & 0 & 0 \\ & & \color{red}0 & \color{red}1 & \color{red}1 \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & \color{blue}0 & 0 \\ & 1 & 0 & \color{blue}0 & 0 \\ & & 0 & \color{blue}1 & 1 \\ & & & \color{red}0 & \color{red}2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & \color{blue}0 \\ & 1 & 0 & 0 & \color{blue}0 \\ & & 0 & 1 & \color{blue}1 \\ & & & 0 & \color{blue}2 \\ & & & & \color{red}4 \end{bmatrix}$$

Figure 1: We add the red entries and subtract the blue ones to get $\mathbf{h} = (1, 1, 1, 1, 1)$

Then the set of all U_n matrices with a given hook sum vector $\mathbf{h}$ is called the set of ***(generalized) Tesler matrices with hook sum*** $\mathbf{h}$ and is denoted as $\mathcal{T}(\mathbf{h})$, while $T(\mathbf{h}) = |\mathcal{T}(\mathbf{h})|$.

Example 2. When the hook sum is $(1, 1, \ldots, 1) = \mathbf{1}^n$, then $\mathcal{T}(\mathbf{1}^n)$ is called the set of ***regular Tesler matrices***. For $n = 2$ we have only two regular Tesler matrices:

$$\begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \text{ AND } \begin{bmatrix} 0 & 1 \\ & 2 \end{bmatrix}.$$

We are concerned with counting the number of (regular) Tesler Matrices. However, the definition above uses subtraction, which makes it awkward in combinatorial proceedings. It is natural that we address this flaw before we proceed with further analysis.

Consider the map $I : U_n \to U_n$ that sends $\left[a_{ij}\right]$ into $\left[b_{ij}\right]$ such that $b_{ij} = a_{i(j+1)}$, $j \neq n$ and $b_{in} = a_{ii}$. Then $I(\mathcal{T}(\mathbf{h}))$ has an alternative definition as follows.

Definition 3. Define the k^{th} ***integral hook sum*** s_k (Figure 2) of a matrix $A = \left(a_{ij}\right) \in U_n$ as

$$s_k = \sum_{i \leq k \leq j} a_{ij},\ 1 \leq k \leq n,$$

and its ***integral hook sum vector*** as $\mathbf{s} = (s_1, s_2, \ldots, s_n)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix}$$

Figure 2: We add the red entries to get $\mathbf{s} = (1, 2, 3, 4, 5)$

Definition 4. The set of all U_n matrices with a given integral hook sum vector $\mathbf{s}$ is called the set of ***(generalized) integral Tesler matrices*** and denoted as $\mathcal{T}[\mathbf{s}]$ with $T[\mathbf{s}] = |\mathcal{T}[\mathbf{s}]|$.

Now, notice that the matrix on Figure 2 is precisely the one from Figure 1 after applying I. The map sent a matrix from $\mathcal{T}(1, 1, 1, 1, 1)$ to one from $\mathcal{T}[1, 2, 3, 4, 5]$. A general statement can be made.

Proposition 5. $I(\mathcal{T}(h_1, h_2, \ldots, h_n)) = \mathcal{T}[h_1, h_1 + h_2, \ldots, \sum_{k=1}^{n} h_k]$.

Proof. Let $A \in \mathcal{T}[s_1, s_2, \ldots, s_n]$, then observe $s_k - s_{k-1} = \sum_{i \geq k} A_{ki} - \sum_{i<k} A_{i(k-1)}$. This, in turn, is precisely $\sum_{i \geq k} I^{-1}(A)_{ki} - \sum_{i<k} I^{-1}(A)_{ik}$, which is h_k for $I^{-1}(A)$. □

Example 6. The integral version of $\mathbf{h} = \mathbf{1}^n$ is $\mathbf{s} = (1, 2, \ldots, n)$ and so the regular Tesler matrices from Example 2 correspond to the integral Tesler matrices $\mathcal{T}[1, 2]$:

$$\begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \quad \text{AND} \quad \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix}.$$

1.2 Asymptotics analysis via integral Tesler matrices and partitions

We will analyze the asymptotic behavior of the functions defined as $T[\mathbf{s}(n)] : \mathbb{Z}^+ \to \mathbb{Z}^+$, where $\mathbf{s}(n)$ is a vector-valued function. Due to exponential growth, it is only practical to study $\ln T[\mathbf{s}(n)]$, and so from now on we shall use the notation $L[\mathbf{s}] = \ln T[\mathbf{s}]$.

It is natural that the asymptotics differ considerably as one changes $\mathbf{s}(n)$. For example, it is easy to verify that $L[\mathbf{1}^n] = (n-1)\ln 2$. More importantly for us, it was shown (by a clever appeal to the Morris identity) in [6] that $L(1, 2, \ldots, n) = \sum_{k=1}^{n} \ln C_k \sim n^2 \ln 2$, where C_k denotes k^{th} Catalan number $\frac{1}{k+1}\binom{2k}{k}$. Based on this, Igor Pak raised the question whether $L(\mathbf{1}^n) \asymp n^2$ as well (for asymptotics symbols $\prec, \preccurlyeq, \precsim, \asymp$ see Appendix).

In this paper we present the answer to the question above and also the general result for a wide range of integral hook sum sequences. That is achieved via breaking complicated $\mathbf{s}(n)$ into integral hook sums of much simpler structure. Those are the basic building blocks (atoms) of our analysis, and therefore, sufficient time should be devoted to studying them. To denote those, we shall introduce a binary operation $\times$ that maps integer pairs from $\mathbb{Z}^+ \times \mathbb{Z}^+$ into vectors $\underbrace{(a,\, a,\, \ldots,\, a)}_{b\text{ times}} = a \times b \;\in\; \left(\mathbb{Z}^+\right)^b$.

The fundamental integral hook sums are then $\mathbf{s}\,(n) = H\,(n) \times W\,(n)$, where we have functions $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$. Now the aim is to build a small theory behind the asymptotics of $L\,[H\,(n) \times W\,(n)]$, or, as it is convenient to omit indices (the implicit index n), $L\,[H \times W]$. For the best previous analysis, see [2].

Note that having a fixed integral hook sum is too strict a constraint to do any counting. Then comes the idea of ***representation***. We construct a superset of $\mathcal{T}\,[H \times W]$ that is easier to work with, but which, under logarithm, has the same asymptotics. In this case, we declare that $\mathcal{T}\,[H \times W]$ represents this superset.

Definition 7. The disjoint union of all $\mathcal{T}\,[s_1,\, s_2,\, \ldots,\, s_n]$ such that $\sum s_k = m$ is called ***integral Tesler matrices universe***, and denoted as $\mathcal{T}\,\{m,\, n\}$. Also, $L\,\{m,\, n\} = \ln T\,\{m,\, n\}$.

We will prove later that $L\,[H \times W] \sim L\,\{HW,\, W\}$. For now, we shall think of ways to enumerate $\mathcal{T}\,\{m,\, n\}$. Here, we face the opposite problem to the one before: the defining restriction of the new set is too loose. The immediate idea is to choose a good representative of the universe, which is, in fact, pivotal. This motivates the next definition.

Definition 8. Define the k^{th} ***(weighted) row sum*** r_k of a matrix $A = \left(a_{ij}\right) \in U_n$ as

$$r_k = a_{kk} + 2a_{k(k+1)} + \cdots + (n-k+1)a_{kn} = \sum_{j=k}^{n} (j-k+1)\, a_{kj}$$

and its ***(weighted) row sum vector*** as $\mathbf{r} = (r_1,\, r_2,\, \ldots,\, r_n)$.

Definition 9. The set of all U_n matrices with fixed row sum vector $\mathbf{r}$ shall be called the set of ***row Tesler matrices*** and denoted as $\mathcal{T}_r\,(\mathbf{r})$. As before, $T_r\,(\mathbf{r}) = |\mathcal{T}_r\,(\mathbf{r})|$, and $L_r\,(\mathbf{r}) = \ln T_r\,(\mathbf{r})$.

Example 10. Here are the only two matrices in $\mathcal{T}_r\,(2,\, 1)$:

$$\begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \quad \text{AND} \quad \begin{bmatrix} 2 & 0 \\ & 1 \end{bmatrix}.$$

We shall elaborate on the choice of definition. Consider the sum of all integral hook sums $\sum s_k$ of some $A = (a_{ij}) \in U_n$ and count the coefficients before each a_{ij} in it (see Figure 3). As a result, those coefficients in each row perfectly match the ones from the definition of r_i. Therefore, $\sum r_k = \sum s_k$, and so the following proposition holds.

Proposition 11. *Integral Tesler matrices universe is a universal set for $\mathcal{T}_r$:*

$$\mathcal{T}\,\{m,\, n\} = \bigcup_{r_1 + r_2 + \cdots + r_n = m} \mathcal{T}_r\,(r_1,\, r_2,\, \ldots,\, r_n)\,.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ & 1 & 2 & 2 & 2 \\ & & 1 & 1 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ & 1 & 2 & 3 & 3 \\ & & 1 & 2 & 2 \\ & & & 1 & 1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{bmatrix}$$

Figure 3: Counting coefficients before every a_{ij} in partial sums $s_1 + s_2 + \cdots + s_k$

The proposition above importantly enables the use of the following trick: if we have a decomposition of a set $A = \cup_{k=1}^{m} A_k$, then provided $\ln m \prec \ln |A|$, we have $\ln \max |A_k| \leq \ln |A| \sim \ln \max |A_k| + \ln m$, and so $\ln |A| \sim \ln \max |A_k|$. Due to extensive use, we shortcut by saying that the decomposition ***induces the representative*** $\ln \max |A_k|$.

So far, we have united all the integral Tesler matrices into a superset, and then split it into pieces of the same number as before unification. However, amazingly, those new pieces are extremely easy to count.

Proposition 12. *Let $p\,(n;\ l,\ r)$ denote the number of unordered partitions of n in which the size of each part s satisfies $l \leq s \leq r$, then*

$$T_r\,(r_1, r_2, \ldots, r_n) = \prod_{k=1}^{n} p\,(r_k;\ 1;\ n-k+1)\,.$$

Here we first find an unforeseen connection between Tesler matrices and integer partitions. The following fact will prove crucial.

Lemma 1. *Let $l\,(n) \prec \sqrt{n} \prec r\,(n)$, then*

$$\ln p\,(n;\ l,\ r) \sim \ln p\,(n;\ 1,\ n) \sim \sqrt{\frac{2}{3}}\pi\sqrt{n}.$$

Proof. The second equivalence is the famous Hardy–Ramanujan formula, so we derive the first one. Any partition is first partitioning n into three numbers n_1, n_2, n_3 and then partitioning them into parts of size from 1 to $l-1$, l to r, and $r+1$ to n respectively. This leads to the following bounds:

$$\ln p\,(n;l,r) \leq \ln p\,(n;1,n) \leq \ln \binom{n+2}{2} + \ln p\,(n;1,l-1) + \ln p\,(n;l,r) + \ln p(n;r+1,n).$$

It follows from the asymptotic formula of [5] that all the summands to the right are asymptotically negligible compared to $\sqrt{\frac{2}{3}}\pi\sqrt{n}$, except for $\ln p\,(n;\ l,\ r)$. $\square$

It is now high time to move back to $L\,[H\,(n) \times W\,(n)]$. We shall first do so for $W^2 \succ H$.

Theorem 13. *Let $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $W^2 \succ H \succ 1$. Then*

$$L\left\{HW,\, W\right\} \sim L_r\left(H \times W\right) = L_r\,\underbrace{\left(H(n),\, H(n),\, \ldots,\, H(n)\right)}_{W(n)\ \textit{times}} \sim \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

Proof. We combine Proposition 12 with Lemma 1 (we can use it as $1 \prec \sqrt{H} \prec W$) to obtain $L_r\left(H \times W\right) = \sum_{k=1}^{W} \ln p\left(H;\ 1,\, W-k+1\right) \sim W \cdot \sqrt{\frac{2}{3}}\pi\sqrt{H}$.

What is left is to obtain a strict upper-bound for $L\left\{HW,\, W\right\}$. Proposition 11 induces the representative $L\left\{HW,\, W\right\} \sim \max L_r\left(\mathbf{r}\right)$, which we can compute as

$$\max \sum_{k=1}^{W} \ln p\left(r_k;\ 1,\, W-k+1\right) \precsim \sqrt{\frac{2}{3}}\pi \cdot \max \sum_{k=1}^{W} \sqrt{r_k} = \sqrt{\frac{2}{3}}\pi W\sqrt{H},$$

since for fixed $\sum_{k=1}^{W} r_k$, the sum $\sum_{k=1}^{W} \sqrt{r_k}$ is maximized under $r_1 = r_2 = \cdots = r_W$. □

We shall now show that $\mathcal{T}\left[H \times W\right]$ is indeed a representative of $\mathcal{T}\left\{HW \times W\right\}$.

Theorem 14. *Let $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $W^2 \succ H \succ 1$. Then*

$$L\left[H \times W\right] \sim \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

Proof. We shall "carve" (almost) a subset of $\mathcal{T}\left[H \times W\right]$ from $\mathcal{T}\left\{HW,\, W\right\}$ by imposing several restrictions loose enough though to preserve the asymptotics.

We want all s_k equal. However, the corresponding summation regions differ significantly, so we need to define a restriction first. Let $l \sim \alpha^{-1}\sqrt{H}$, $r \sim \alpha\sqrt{H}$ be some functions, where $1 \prec \alpha \prec \min\left(W/\sqrt{H},\, \ln H\right)$. Then let $\mathcal{A} \subseteq \mathcal{T}\left\{HW,\, W\right\}$ be the subset of matrices with non-zero entries a_{ij} having either $j > W - r + 1$ or $l \leq j - i + 1 \leq r$. Now we have non-zero elements only on portions of consecutive entries in each row (except for the last $r-1$ ones), which we shall call ***segments*** (see Figure 4).

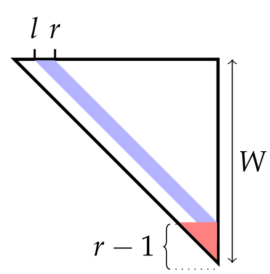

Figure 4: Consecutive non-zero entries in each row (segments) are colored blue, the last rows without segments are colored red, zero entries are colored white.

We would like all segments to have equal entries, equalizing all s_k. However, this drastically reduces cardinality, so we need to adjust the restrictions. We divide every segment into consecutive ***blocks*** of equal length $\gamma \sim \alpha^2 \ln H$ so that there are $\beta = (r-l+1)/\gamma \sim \alpha^{-1}\sqrt{H}/\ln H$ blocks in any segment. Let m_{ik} denote the weighted sum $\sum (j-i+1)\, a_{ij}$ over the k^{th} block of the i^{th} segment. Those can be put in block sum matrix $M = [m_{ik}]$. Our idea now is to equate sequences $(m_{ik})_{1\le k\le\beta}$ over all $1 \le i \le W-r+1$.

This would equate all the row sums, so we shall require $\mathcal{A} \subseteq \mathcal{T}_r(H\times W)$. From Proposition 12 and Lemma 1 we have $\ln|\mathcal{A}| \sim \sqrt{2/3}\pi W\sqrt{H}$. Now, denote by $\mathcal{A}_M \subseteq \mathcal{A}$ the set of matrices from $\mathcal{A}$ with block sum matrix $M = [m_{ik}]$. Then, since each partition in $p(H;\, l,\, r)$ corresponds to a sequence of weighted block sums, we choose the sequence $m'_1, m'_2, \dots, m'_\beta$ associated with the largest number of partitions. From it we construct a matrix $M' = [m_{ik}]\,,\, m_{ik} = m'_k$. Clearly, $|\mathcal{A}_{M'}| = \max|\mathcal{A}_M|$, and so it is the induced representative of $\mathcal{A}$.

Some basic computations show $s_k \sim H,\, r \le k \le W-r+1$ for the matrices of $\mathcal{A}_{M'}$. Now, as we want to connect this to $\mathcal{T}[H\times W]$, we shall restrict $\mathcal{A}_{M'}$ to just one integral hook sum vector (as for now it contains matrices with different vectors). This is easily done, as we can decompose $\mathcal{A}_{M'}$ into subsets with fixed $\mathbf{s}$, and induce some representative $\mathcal{B} \subseteq \mathcal{A}_{M'}$.

Finally, our argument runs as follows: the above holds for every H as long as $W^2 \succ H \succ 1$, so we can replace every H by some qH, $q<1$ to get $\mathcal{B}$ with $s_k \sim qH$, $r \le k \le W-r+1$. For sufficiently large n, we, in fact, have $s_k(n) \le H(n)$, and so it is easily obtained $L[H\times W] \gtrsim \ln|\mathcal{B}|$, culminating in $L[H\times W] \gtrsim \sqrt{\frac{2}{3}}\pi W\sqrt{qH} \xrightarrow{q\to 1} \sqrt{\frac{2}{3}}\pi W\sqrt{H}$. $\square$

As it turns out, all of this can be much easier done for $H \succ W^2$.

Theorem 15. *Let $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $H \succ W^2 \succ 1$. Then*

$$L[H\times W] \sim L\{HW,\, W\} \sim L_r(H\times W) \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

Proof. The proof is analogous to Theorem 13. Thus we proceed with

$$L_r[H\times W] = \sum_{k=1}^{W} \ln p(H;\, 1,\, k) \sim \sum_{k=1}^{W} k\ln\left(\frac{H}{k^2}\right) \sim \int_1^W k\ln\left(\frac{H}{k^2}\right) dk \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right),$$

where we use the $p(H;\, 1,\, k)$ asymptotics from [5]. Now, we induce the representative $L\{HW,\, W\} \sim \max L_r(\mathbf{r})$, which we shall expand utilizing the same partition asymptotic formula as

$$\max L_r(\mathbf{r}) = \max \sum_{k=1}^{W} \ln p(r_k;\, 1,\, W-k+1) \sim \max \sum_{k=1}^{W} k\ln\left(\frac{r_k}{k^2}\right).$$

As this is maximized under $r_k = 2kH/W$, we can write

$$L\{HW, W\} \precsim \sum_{k=1}^{W} k \ln\left(\frac{2H}{kW}\right) \sim \int_1^W k \ln\left(\frac{2H}{kW}\right) dk \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

Finally, we shall provide a sharp lower-bound for $L[H \times W]$. To do this, consider all the integral Tesler matrices with sum over all entries being H to derive

$$L[H \times W] \geq \ln \binom{H + W(W-1)/2 - 1}{H} \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right). \qquad \square$$

While the asymptotics of $T[H \times W]$ has been investigated, with an improved lower bound presented in [2, Theorem 1.1], it was done so for a constant $H(n) = H_0$ and $W(n) \to \infty$. Here, in contrast, both $H, W \to \infty$, allowing us to use asymptotics for the number of partitions to produce such sharp estimates.

Proposition 16. *Let $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $H \asymp W^2 \succ 1$. Then*

$$L[H \times W] \asymp L\{HW, W\} \asymp L_r(H \times W) \asymp H.$$

Proof. Let $W_1 \prec \sqrt{H} \prec W_2$, then we can bound all these functions by the previous theorems as $W_1^2 \ln\left(\frac{\sqrt{H}}{W_1}\right) \precsim L[H \times W], L\{HW, W\}, L_r(H \times W) \precsim \sqrt{\frac{2}{3}}\pi W_2 \sqrt{H}$. If we now let $W_1, W_2 \to \sqrt{H}$, we would get the desired result. $\square$

2 Kostant pictures and their asymptotics

2.1 Kostant pictures and height diagrams

As claimed in the introduction to the question of asymptotics analysis, we shall connect a large class of $\mathbf{s}(n)$, including $\mathbf{s}(n) = (1, 2, \ldots, n)$, to the atomic $\mathbf{s}(n) = H(n) \times W(n)$ in this section. Our chief tool in doing so will be ***Kostant pictures***.

It has been shown [3] that the number of generalized Tesler matrices is connected to the type A_n ***Kostant partition function*** which arises in representation theory. The combinatorial model below is due to [1].

Definition 17. A ***Kostant picture*** is a diagram consisting of $n \in \mathbb{Z}^+$ ordered vertices and loops around consecutive vertices. Denote the number of loops enclosing vertices $i, i+1, \ldots j+1$ as a_{ij}. Additionally, let $(e_1, e_2, \ldots, e_{n+1})$ be the standard orthonormal vector basis of $\mathbb{R}^{n+1}$, and define $\alpha_{ij} = e_i - e_{j+1}$. Then, the vector $\nu = \sum a_{ij}\alpha_{ij}$ is called the ***weight*** of this Kostant picture. The set of all possible Kostant pictures for a given ν is denoted by $\mathcal{K}(\nu)$, while their number $\mathrm{KPF}(\nu)$ defines the ***Kostant partition function***.

Lemma (Mészáros, Morales, Rhoades [3])**.** $T(\nu) = \mathrm{KPF}(\nu)$.

As we have already seen, it is beneficial to "integrate" the hook sum vector, and so we shall do the same with weight. To each weight $\nu = (\nu_1, \nu_2, \ldots, \nu_{n+1})$ we associate the height $\psi = (\nu_1, \nu_1 + \nu_2, \ldots, \sum_{k=1}^{n} \nu_k)$. This way, ψ_k denotes the number of loops containing vertex k. We denote the set of all Kostant pictures with height ψ by $\mathcal{K}[\psi]$ (analogously to how we defined $\mathcal{T}[\mathbf{s}]$), and we already know that $|\mathcal{K}[\psi]| = T[\psi]$. We shall stick to this T notation as we have already used it extensively.

Height ψ can be visually represented by its ***height diagram***, which is simply the graph of $y = \psi(x)$ in the xy-plane. It is a discrete function that is zero on its entire domain except for a finite number of arguments (a subset of $\{1, 2, \ldots, n\}$). That being said, we will draw it continuously in the next sections for the sake of clarity. The utility of height diagrams lies in the following easy observation.

Proposition 18. *Let ψ_1, ψ_2 be two heights such that $\psi_1(x) \geq \psi_2(x + a)$, $x \in \mathbb{Z}$ holds for some integer a. Then* $\mathrm{KPF}[\psi_1] \geq \mathrm{KPF}[\psi_2]$.

Proof. Consider the subset of $\mathcal{K}[\psi_1]$, which contains all the Kostant pictures with the number of loops α_{ii} at least $\psi_1(i) - \psi_2(i + a)$ for all i. Then there is an obvious bijection between this set and $\mathcal{K}[\psi_2]$: given a picture, for all i delete $\psi_1(i) - \psi_2(i + a)$ loops of the type α_{ii} and change each α_{ij} by $\alpha_{i-a,\, j-a}$. □

With no work at all we can already use height diagrams to connect a known and an unknown asymptotics to the atomic ones.

Proposition 19. $L(\mathbf{1}^n) \asymp L[n \times n]$ AND $L(1, 2, \ldots, n) \asymp L\left[n^2 \times n\right]$.

Proof. Note that weight $\mathbf{1}^n$ corresponds to the height $\psi(x) = x$, and so by looking at Figure 5 we can deduce (recall Proposition 18) $L\left[\frac{n}{2} \times \frac{n}{2}\right] \leq L(\mathbf{1}^n) \leq L[n \times n]$. As $L\left[\frac{n}{2} \times \frac{n}{2}\right] \asymp L[n \times n]$, we get the desired result. The same can be stated for the second proposition (see Figure 6). □

2.2 Asymptotics analysis via colorings and height decomposition

In the previous section, asymptotics for the type $\psi = H \times W$ was derived. Now, we shall analyze the type $\psi(n) = (a_1, a_2, \ldots, a_n)$ for some fixed integer sequence $(a_n)_{n \geq 1}$. In other words, one obtains $\psi(n+1)$ by appending a_{n+1} to $\psi(n)$. This will give us the asymptotic formula for the number of regular Tesler matrices on substituting $a_n = n$ (answering questions in [2, 4]).

Now we want a more accurate version of Proposition 19. So we can do ***height decomposition*** into pieces that can be well approximated with $\psi = H \times W$. For this, we introduce the following concepts. $\mathcal{K}^k[\psi]$ denotes the set of Kostant pictures with height ψ, in which every loop is colored with one of $\lfloor k \rfloor$ distinct colors. Also, $T^k[\psi] = \left|\mathcal{K}^k[\psi]\right|$

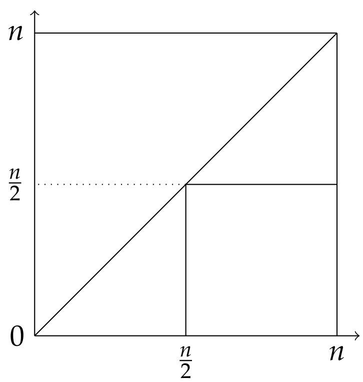

Figure 5: $L\left(\mathbf{1}^n\right) \asymp L\left[n \times n\right]$

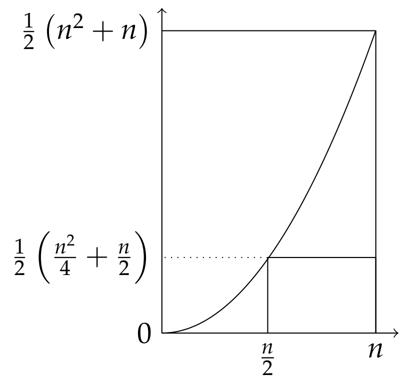

Figure 6: $L\left(1, 2, \ldots, n\right) \asymp L\left[n^2 \times n\right]$

Figure 7: Basic height diagrams usage

and $L^k\left[\psi\right] = \ln T^k\left[\psi\right]$. $C_k\left[\psi\right]$ denotes the maximal number of colorings a Kostant picture from $\mathcal{K}\left[\psi\right]$ can have into $\lfloor k \rfloor$ distinct colors.

Proposition 20. $\max\left(T\left[\psi\right], C_k\left[\psi\right]\right) \leq T^k\left[\psi\right] \leq T\left[\psi\right] \cdot C_k\left[\psi\right]$.

Theorem 21 (Decomposition Theorem)**.** *Let $\psi = \left(\psi_1, \psi_2, \ldots, \psi_{kW}\right)$ be some height. Define sequences $\phi_t = \left(\psi_{(t-1)W+i}\right)_{1 \leq i \leq W}$, $1 \leq t \leq k$ and*

$$S = \sum_{i=1}^{k} T\left[\psi_{(i-1)W+1}, \psi_{(i-1)W+2}, \ldots, \psi_{iW}\right].$$

Then we can lower-bound and upper-bound $T\left[\psi\right]$ by

$$T\left[\psi\right] \geq \max\left(S, T^{\frac{W^2}{4}}\left[\min\left(\min\phi_1, \min\phi_2\right), \ldots, \min\left(\min\phi_{k-1}, \min\phi_k\right)\right]\right),$$

$$T\left[\psi\right] \leq S \cdot T^{W^2}\left[\min\left(\max\phi_1, \max\phi_2\right), \ldots, \min\left(\max\phi_{k-1}, \max\phi_k\right)\right].$$

Proof. Decompose ψ into k blocks of length W each. If we ignore all possible loops α_{ij} that are contained within several such divisions, we have $T\left[\psi\right] \geq S$.

Now, consider all such missing α_{ij} as determined by two parameters: the set of divisions the loop covers and the relative positions of its ends in the first and the last divisions. Note that in order to get a lower-bound, this situation can be modeled by

$$\mathcal{K}^{(W/2)^2}\left[\min\left(\min\phi_1, \min\phi_2\right), \min\left(\min\phi_2, \min\phi_3\right), \ldots, \min\left(\min\phi_{k-1}, \min\phi_k\right)\right],$$

as we have $k-1$ division lines to intersect, and at least $\frac{W}{2}$ possible choices per loop end so that the loops with disjoint division sets do not cross. To get an upper bound, W possible choices per end should be allowed and all min replaced with max. $\square$

For this theorem to be utilized we shall first have some asymptotics of $T^k[\psi]$, or alternatively, $C_k[\psi]$.

Lemma 2. *Let ψ, $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ be two function such that $k \succ \psi^2 \succ 1$, then*

$$\ln C_k[\psi(1), \psi(2), \dots, \psi(n)] \sim \sum_{i=1}^{n} \psi(i) \ln\left(\frac{k}{\psi(i)}\right).$$

Proof. Let $A = [a_{ij}] \in \mathcal{T}[\psi(1), \psi(2), \dots, \psi(n)]$ correspond to the Kostant picture that maximizes the number of colorings into k distinct colors, in other words,

$$C_k[\psi(1), \psi(2), \dots, \psi(n)] = \prod_{v \in [a_{ij}]} \binom{v+k-1}{k-1}.$$

Note that in every row of A we have entry values in descending order since otherwise we could permute values to get the same value of the product above, while lowering some integral hook sums. Now, let $a = a_{ii}$, $b = a_{i(i+1)}$, $c = a_{(i+1)(i+1)}$. In the sum of integral hook sums $\sum s_i$ those entries are counted as $a + 2b + c$ (see Figure 3). So, if we denote sum $= a + 2b + c$, then we shall consider maximization of

$$\binom{a+k-1}{k-1} \cdot \binom{c+k-1}{k-1} \cdot \binom{\frac{\text{sum}-a-c}{2}+k-1}{k-1}.$$

The expression above is maximized when $b \sim a^2/k \sim c^2/k \prec 1$, and so we can assume values only on the main diagonal, which leads to

$$\ln C_k[\psi(1), \psi(2), \dots, \psi(n)] \sim \sum_{i=1}^{n} \ln\binom{\psi(i)+k-1}{k-1} \sim \sum_{i=1}^{n} \psi(i) \ln\left(\frac{k}{\psi(i)}\right). \qquad \square$$

Theorem 22. *Let $\psi : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function such that $n^2 \succ \psi(n) \succ 1$ and $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ \psi(n)$. Then*

$$L[\psi(1), \psi(2), \dots, \psi(n)] \sim \sqrt{\frac{2}{3}}\pi \sum_{i=1}^{n} \sqrt{\psi(i)}.$$

Proof. Let W be a function such that $n \succ W(n) \succ \sqrt{\psi(n)}$. We shall now use Theorem 21 with $n = kW$. Consider S first. We can easily bound it with Theorem 14 to derive

$$\sqrt{\frac{2}{3}}\pi W \sum_{i=1}^{k-1} \sqrt{\psi(iW)} \precsim S \precsim \sqrt{\frac{2}{3}}\pi W \sum_{i=1}^{k} \sqrt{\psi(iW)}.$$

Since $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ \psi(n)$, we can choose W to satisfy $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ W\sqrt{\psi(n)}$, and so asymptotically equate the bounds above to get $S \sim$ RHS of the Theorem statement.

It is only left to show that the bigger coloring term from Theorem 21 is asymptotically insignificant. Use Proposition 20 and the previous Lemma to get the upper-bound for it: $L\left[\psi\left(2W\right), \psi\left(3W\right), \ldots, \psi\left(n\right)\right] + \sum_{i=1}^{k-1} \psi\left(iW\right) \ln\left(\frac{W^2}{\psi(iW)}\right)$. For the second summand, it is straightforward: $\sum_{i=1}^{n} \sqrt{\psi\left(i\right)} \succ \psi\left(n\right) = k^2 \cdot \frac{\psi(n)}{k^2} \geq k^2 \ln\left(\frac{\psi(n)}{k^2}\right)$. For the first one, first upper-bound it by $L\left[\psi\left(n\right) \times k\right]$, then $\sum_{i=1}^{n} \sqrt{\psi\left(i\right)} \succ n \succ n \cdot \frac{\sqrt{\psi(n)}}{W} = k\sqrt{\psi\left(n\right)}$. $\square$

Corollary 23. $L\left(\mathbf{1}^n\right) \sim \sqrt{\frac{8}{27}\pi n \sqrt{n}}$.

For a proof use Theorem 22 with $\sum_{i=1}^{n} \sqrt{i} \sim 2/3n\sqrt{n}$. Note that the asymptotics presented in Corollary 23 drastically improve the previous bound $L\left(\mathbf{1}^n\right) \geq \frac{n}{4} \ln^2 n - \mathcal{O}(n \ln n)$ from [2, Theorem 1.2].

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Appendix: Asymptotics

Definition 24. As $x \to \infty$, we write $f \sim g$ iff $f/g \to 1$; $f \prec g$ iff $f/g \to 0$; $f \asymp g$ iff $ag \leq f \leq bg$ for some $a, b \in \mathbb{R}^+$; $f \precsim g$ iff $f \leq ag$ for every $a > 1$ ($x \to \infty$ for every a independently); $f \preccurlyeq g$ iff $f \leq ag$ for some $a \in \mathbb{R}^+$.

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