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When is a Schubert variety spherical?

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Abstract. Let L be a Levi subgroup of a general linear group acting on a Schubert variety. If there is a dense open orbit of a Borel subgroup, the Schubert variety is L-spherical, generalizing the concept of toric varieties. Hodges and Lakshmibai classified L-spherical Schubert varieties in Grassmannians. L-spherical Schubert varieties in complete flag varieties were treated independently by Gao–Hodges–Yong and Can–Saha. In this paper, we generalize these classifications to Schubert varieties in partial flag varieties of type A. This is achieved by applying Lakshmibai-Seshadri paths forming crystal graphs of Demazure modules, which has a root-system uniform framework.

Keywords: Schubert variety, spherical variety, Demazure module, LS-paths

1 Introduction

Let $G := GL_n$ over $\mathbb{C}$. Fix a choice of Borel subgroup $B := B_n$ and a maximal torus $T := T_n$ to be the sets of upper triangular matrices and diagonal matrices, respectively. The Weyl group is $W = S_n \cong N_G(T)/T$. Then **Schubert varieties** are defined from the Bruhat decomposition

$$G/B = \bigsqcup_{w \in W} BwB/B, \quad X(w) := \overline{BwB/B}. \tag{1.1}$$

Let $I \subseteq [n-1] := \{1, 2, \cdots, n-1\}$. Write $[n-1] - I = \{y_1, \cdots, y_{g-1}\}$ so that $y_1 < \cdots < y_{g-1}$, $y_0 := 0$ and $y_g := n$. Define a Levi subgroup of GL_n and its Borel subgroup

$$L_I := GL_{y_1-y_0} \times \cdots \times GL_{y_g-y_{g-1}}, \quad B_I := B_{y_1-y_0} \times \cdots \times B_{y_g-y_{g-1}}. \tag{1.2}$$

Write $s_i \in W$ as simple reflections for $1 \leq i \leq n-1$ and define $W_I := \langle s_i \mid i \in I \rangle$.

From the left action of G on G/B, L_I induces an action on $X(w)$ under restriction. $X(w)$ is stable under L_I-action if and only if w is the maximal representative of $W_I w$; see *e.g.* [9, §1.2]. $X(w)$ is L_I**-spherical** if there exists a dense open orbit of B_I. There are

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recent papers about classifications of L_I-spherical $X(w)$; see, *e.g.*, [6], [7] and [2]. Our first theorem gives another classification. Write

$$\mathrm{Par}_n := \{\lambda = (\lambda_1, \cdots, \lambda_n) \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0,\ \forall \lambda_i \in \mathbb{Z}\}. \tag{1.3}$$

For each $\lambda \in \mathrm{Par}_n$, it can be regarded as a character $\lambda : B \to \mathbb{C}^*$. Define an action of B on $G \times \mathbb{C}$ by $b(g,t) := (gb^{-1}, \lambda(b)t)$. Construct a G-linearized line bundle

$$L_\lambda := (G \times \mathbb{C})/B \tag{1.4}$$

over G/B. If $X(w)$ has L_I-action, so does the set of global sections $H^0(X(w), L_\lambda)$.

Theorem 1 ([9, Conjecture 4.19])**.** *Let $G = GL_n$ and $W = S_n$. Let $I \subseteq [n-1]$ and $w \in W$ be the maximal representative of $W_I w$. Fix a partition λ which is strictly decreasing. Then $X(w)$ is L_I-spherical if and only if $H^0(X(w), L_\lambda)$ is L_I-multiplicity free.*

In particular, the T-character of $H^0(X(w), L_\lambda)^*$ is given by the **key polynomial** $\kappa_{w\lambda}$; see *e.g.* [9, §4.4]. To demonstrate, let $n = 4$, $w = 2413$ and $I = \{1, 3\}$. Choose a strictly decreasing $\lambda = (3, 2, 1, 0)$ and express $\kappa_{w\lambda}$ as a sum of products of Schur polynomials

$$\kappa_{w\lambda}(x_1, x_2, x_3, x_4) = s_{3,1}(x_1, x_2)s_{2,0}(x_3, x_4) + s_{3,2}(x_1, x_2)s_{1,0}(x_3, x_4). \tag{1.5}$$

Since each product of Schur polynomials appears at most once, $X(w)$ is L_I-spherical due to Theorem 1.

Our next goal is to give a classification of L_I-spherical Schubert varieties in partial flag varieties generalizing [13], [8], [6] and [2]; see §2. Let $J \subseteq [n-1]$. Write $[n-1] - J = \{z_1, \cdots, z_{h-1}\}$, $z_1 < \cdots < z_{h-1}$, $z_0 := 0$ and $z_h := n$. Choose a parabolic subgroup P_J given by the set of block upper triangular matrices of size $z_1 - z_0, \cdots, z_h - z_{h-1}$. Then

$$G/P_J = \bigsqcup_{wW_J \in W/W_J} BwP_J/P_J, \quad X(wW_J) := \overline{BwP_J/P_J}. \tag{1.6}$$

Let $w \in W$. For each $1 \leq j \leq n$, record $1 \leq x_j \leq h$ where $z_{x_j - 1} < w^{-1}(j) \leq z_{x_j}$. Denote the word $[w, J] := x_1 x_2 \cdots x_n$. Next, insert i between x_y and x_{y+1} whenever $y \in [n-1] - I$. $x_1 x_2 \cdots x_n$ with i insertion, which is a word of numbers $1, 2, \cdots, h$ and an alphabet i, is denoted by $[w, I, J]$.

In short, equalize each part of one-line notation w^{-1} based on J and then insert i from I. For instance, if $w^{-1} = 1472356$, $I = \{1, 4, 5\}$ and $J = \{2, 4, 6\}$, then $[w, J] = 1342234$ and $[w, I, J] = 13i4i223i4$.

We say $[w, I, J]$ has $[w', I', J']$ as a **pattern** if there is a subsequence of $[w, I, J]$ in which i is inserted in the same place as in $[w', I', J']$ and the remaining numerical entries have the same relative order. If $I = [n-1]$ and $J = \emptyset$, this reduces to the *permutation pattern* of w'^{-1} in w^{-1}. For instance, in [4, Corollary 1.5], C. Gaetz showed $X(w) \subseteq G/B$ is *maximally spherical* if and only if w avoids twenty-one permutation patterns.

Returning to the previous example, $[w, I, J] = 13i4i223i4$ has $12i32i3$ as a pattern due to the subsequence $13i43i4$. It avoids a pattern $231i1$. Our second theorem, which classifies spherical Schubert varieties in partial flag varieties is stated below.

Theorem 2. *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I, J \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Then $X(wW_J)$ is L_I-spherical if and only if $[u, I, J]$ avoids the patterns as follows:*

$$\begin{array}{cccccc} 321, & 3412, & 3i312, & 231i1, & 2i21i1, & 244113, \\ 233112, & 3331122, & 2233111, & 22211i11, & 22i22111, & 22ii1i1, \\ 33ii12, & 44123, & 3312i2, & 2i2ii11, & 23ii11, & 23411, \\ 2i2311, & 221211, & 331322, & 221311, & 331422. & \end{array} \tag{1.7}$$

2 Known results

We recall some known results about spherical Schubert varieties. A Schubert variety is **toric** if there exists a dense open orbit of T. When $I = \emptyset$, $L_I = B_I = T$. In this case, the classification of spherical varieties coincides with that of toric varieties.

Theorem 3 ([13], [10])**.** *Let $G = GL_n$ and $W = S_n$. Let $w \in W$, $I = \emptyset$ and $J \subseteq [n-1]$. Then $X(wW_J)$ is toric if and only if $[w, J]$ avoids* 321, 3412, 3312, 2311 *and* 2211.

Proof. In [13, Theorem 2, 4], a classification of toric $X(w) \subseteq G/B$ was given. Using birational equivalence, this extends to $X(wW_J) \subseteq G/P_J$. Another proof can be given from the classification of multiplicity-free key polynomials in [10, Theorem 1.1]. For generalization to other Lie types, see [5, Corollary 4.12]. □

R. Hodges and V. Lakshmibai provided a classification of spherical Schubert varieties in Grassmannians.

Theorem 4 ([8, Theorem 1.1.1])**.** *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I, J \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Assume that $|J| = n-2$ so that G/P_J is a Grassmannian. Then $X(wW_J)$ is L_I-spherical if and only if $[u, I, J]$ avoids the following patterns:*

$$222i11i11, \quad 22i22i111, \quad 22i12i11, \quad 2i2i1i1, \quad 22ii1i1, \quad 2i2ii11. \tag{2.1}$$

Proof. The first three patterns are due to the case $b_L = 3$ in [8, Theorem 1.1.1], while the remaining patterns are due to the case $b_L = 4$. □

The classification of spherical Schubert varieties in complete flag varieties G/B was also done recently. However, birational equivalence fails to induce sphericality of $X(wW_J)$ from $X(w)$; see [5, §4.3].

Theorem 5 ([6, Theorem 1.3]). *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Then $X(w)$ is L_I-spherical if and only if u avoids 321 and 3412 patterns.*

Proof. In [6, §1.1], $X(w)$ is L_I-spherical if and only if u is a *standard Coxeter element*. This is equivalent to avoiding 321 and 3412 patterns; see [18, Theorem 4.3]. For generalization to other Lie types, see [7, Theorem 1.3] or [2, Theorem 1.1]. □

3 Partial flag varieties

Let $J \subseteq [n-1]$. Write

$$\mathrm{Par}_J := \{\lambda \in \mathrm{Par}_n \mid \forall j \in J,\ \ \lambda_j = \lambda_{j+1}\}. \tag{3.1}$$

As in (1.4), let L_λ be a G-linearized line bundle over G/P_J for each $\lambda \in \mathrm{Par}_J$; see *e.g.* [1].

Theorem 6. *Let $G = GL_n$ and $W = S_n$. Let $I, J \subseteq [n-1]$. Let $w \in W$ be the maximal representative of $W_I w$. Then $X(wW_J)$ is L_I-spherical if and only if for all $\lambda \in \mathrm{Par}_J$, $H^0(X(wW_J), L_\lambda)$ is L_I-multiplicity free.*

Proof. Analogous to [6, Theorem 4.13]. In [3], $X(wW_J)$ is a normal variety. $X(wW_J)$ is L_I-spherical if and only if for any L_I-linearized line bundle L over $X(wW_J)$, $H^0(X(wW_J), L)$ is L_I-multiplicity free; see [17, Theorem 2.1.2]. □

In particular, the B-module $H^0(X(w), L_\lambda)^* \cong H^0(X(wW_J), L_\lambda)^*$ is called *Demazure module*. In §4, we define it on a semisimple Lie algebra. Define **Bruhat order** on W/W_I

$$uW_I \leq vW_I \quad \Leftrightarrow \quad X(uW_I) \subseteq X(vW_I). \tag{3.2}$$

Lemma 1. *Let $G = GL_n$ and $W = S_n$. Let $I, J \subseteq [n-1]$, $\lambda \in \mathrm{Par}_J$ and assume that $u, v \in W$ are maximal representatives of $W_I u$ and $W_I v$, respectively. Then if $uW_J \leq vW_J$, there is a natural L_I-module monomorphism*

$$H^0(X(uW_J), L_\lambda)^* \hookrightarrow H^0(X(vW_J), L_\lambda)^*. \tag{3.3}$$

Proof. The natural map is defined from the inclusion $X(uW_J) \subseteq X(vW_J)$. Injectivity follows from the fact that $H^0(X(uW_J), L_\lambda)^* \to H^0(G/P_J, L_\lambda)^*$ is injective; see *e.g.* [11, Proposition 14.15]. □

For each $a, b \in \mathbb{Z}$ with $a \leq b$, write $[a, b] := \{a, a+1, \cdots, b\}$.

Lemma 2 (I-sorting). *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I, J \subseteq [n-1]$. Let u and w be the minimal and maximal representative of vW_I. Denote $[n-1] - I = \{y_1 < \cdots < y_{g-1}\}$ and $y_0 := 0$, $y_g := n$. Choose $1 \leq p < q \leq g$ and*

$$K := [y_{p-1} + 1, y_q - 1] \subseteq [n-1]. \tag{3.4}$$

Let u' be the minimal representative of $W_K u$ and w' be the maximal representative of $W_I u'$. Write $[u, J] = x_1 \cdots x_n$ and $[u', J] = x'_1 \cdots x'_n$. Then we have the following:

- *For $\lambda \in \mathrm{Par}_J$, if $H^0(X(w'W_J), L_\lambda)^*$ has L_I-multiplicities, so does $H^0(X(wW_J), L_\lambda)^*$.*
- *If $i, j \in [y_{p-1} + 1, y_q]$, $x'_i < x'_j$ implies $i < j$. If $j \notin [y_{p-1} + 1, y_q]$, then $x_j = x'_j$.*

Proof. Due to the relation between K and I, u' is still the minimal representative of $W_I u'$. From $u' \leq u$, we have $w' \leq w$. Due to Lemma 1, we have the first statement. From the construction of K, we have the second statement. □

Lemma 3 (J-sorting). *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I, J \subseteq [n-1]$. Let u and w be the minimal and maximal representative of vW_I. Denote $[n-1] - J = \{z_1 < \cdots < z_{h-1}\}$ and $z_0 := 0$, $z_h := n$. Choose $1 \leq p < q \leq h$ and*

$$K := [z_{p-1} + 1, z_q - 1] \subseteq [n-1]. \tag{3.5}$$

Let u' be the minimal representative of uW_K and w' be the maximal representative of $W_I u'$. Write $[u, J] = x_1 \cdots x_n$ and $[u', J] = x'_1 \cdots x'_n$. Then we have the followings:

- *For $\lambda \in \mathrm{Par}_J$, if $H^0(X(w'W_J), L_\lambda)^*$ has L_I-multiplicities, so does $H^0(X(wW_J), L_\lambda)^*$.*
- *For any $i, j \in [n]$, if $p \leq x'_i < x'_j \leq q$, then $i < j$. If $x'_j \notin [p, q]$, then $x_j = x'_j$.*

Proof. Analogous to the proof of I-sorting Lemma 2. □

4 Demazure modules

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ of rank $n-1$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subseteq \mathfrak{h}^*$ be the subset of non-zero roots and Δ be the subset of simple roots. Let $\Lambda \subseteq \mathfrak{h}^*$ be the lattice of integral weights and Λ^+ be the subset of dominant integral weights. In particular, write the simple root as α_i and the fundamental weight as ϖ_i satisfying $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$. Write the Chevalley basis of $\mathfrak{g}$ as $\{H_i\}_{1 \leq i \leq n-1} \cup \{X_\alpha\}_{\alpha \in \Phi}$.

Let W be the Weyl group associated to $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ generated by $\{H_i\}_{1 \leq i \leq n-1} \cup \{X_\alpha\}_{\alpha \in \Phi^+}$. Finite dimensional irreducible $\mathfrak{g}$-modules are characterized by highest weights $\lambda \in \Lambda^+$. We write an irreducible module as V_λ of highest weight λ with highest weight vector v^+.

For each $w \in W$, there exists a unique weight vector of weight $w\lambda$ in V_λ up to scalar multiplication. We call it **extremal weight vector** and denote it by v_w^+. Define a $\mathfrak{b}$-module

$$V_\lambda(w) := \mathfrak{b} \cdot v_w^+. \tag{4.1}$$

$V_\lambda(w)$ is called a **Demazure module**.

Theorem 7 ([12, Theorem 3.4]). *Let $\lambda \in \Lambda^+$ and $w \in W$. For each $\alpha \in \Phi^+$, define*

$$k_\alpha := \begin{cases} -(w\lambda, \alpha^\vee), & (w\lambda, \alpha^\vee) \leq 0 \\ 0, & (w\lambda, \alpha^\vee) > 0 \end{cases} \tag{4.2}$$

Then

$$\operatorname{Ann}_{\mathcal{U}(\mathfrak{b})} v_w^+ = \sum_{\alpha \in \Phi^+} \mathcal{U}(\mathfrak{b}) X_\alpha^{k_\alpha+1} + \sum_{1 \leq i \leq n-1} \mathcal{U}(\mathfrak{b}) \left(H_i - (w\lambda, \alpha_i^\vee) \cdot 1\right). \tag{4.3}$$

Let $I \subseteq [n-1]$. Write

$$\Delta_I := \{\alpha_i \mid i \in I\}, \quad \Phi_I := \{\alpha \in \Phi \mid \alpha = \sum_{i \in I} c_i \alpha_i\}. \tag{4.4}$$

Define Lie subalgebras of $\mathfrak{g}$

$$\mathfrak{g}_I := \langle H_i \mid i \in I \rangle \oplus \langle X_\alpha \mid \alpha \in \Phi_I \rangle, \quad \mathfrak{b}_I := \langle H_i \mid i \in I \rangle \oplus \langle X_\alpha \mid \alpha \in \Phi_I^+ \rangle. \tag{4.5}$$

Write $[n-1] - I = \{y_1 < y_2 < \cdots < y_{g-1}\}$, $y_0 := 0$ and $y_g := n$. For $1 \leq i \leq g$, define

$$\Lambda_i := \bigoplus_{j=y_{i-1}+1}^{y_i - 1} \mathbb{Z}\varpi_j, \tag{4.6}$$

and the projection $p_i : \Lambda \to \Lambda_i$.

Let $\lambda \in \Lambda^+$ and $w \in W$. Let N be a large enough positive integer. Define

$$\xi := \sum_{i \notin I} \varpi_i. \tag{4.7}$$

Due to [16, Theorem 1], $V_\lambda(w) \otimes V_{N\xi}(e)$ has a filtration of which subquotients are Demazure modules. In other words,

$$V_\lambda(w) \otimes V_{N\xi}(e) = \mathcal{F}_m \supseteq \mathcal{F}_{m-1} \supseteq \cdots \supseteq \mathcal{F}_0 = 0, \quad \mathcal{F}_j/\mathcal{F}_{j-1} \cong V_{\lambda_j}(w_j) \quad 1 \leq j \leq m, \tag{4.8}$$

for some $w_j \in W$ and $\lambda_j \in \Lambda^+$. On the other hand,

$$\operatorname{res}^{\mathfrak{b}}_{\mathfrak{b}_I} V_\lambda(w) \cong \operatorname{res}^{\mathfrak{b}}_{\mathfrak{b}_I} \left(V_\lambda(w) \otimes V_{N\xi}(e)\right), \tag{4.9}$$

since $V_{N\xi}(e)$ is a trivial module after restriction.

Let M be a finite dimensional $\mathfrak{h}$-module. Since M has a weight space decomposition, it admits formal character denoted by $\operatorname{ch}(M)$. In particular, it is defined in the group ring $\mathbb{Z}[\Lambda]$ with basis e^λ where $\lambda \in \Lambda$. Define a ring homomorphism

$$q_I : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda], \quad e^{\varpi_j} \mapsto e^{\varpi_j} \quad (j \in I), \quad e^{\varpi_j} \mapsto 1 \quad (j \notin I). \tag{4.10}$$

Lemma 4. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank $n-1$ with Weyl group W. Let λ be a dominant integral weight, $w \in W$ and $I \subseteq [n-1]$. Then there exists a sequence of integral weights λ_j and $w_j \in W$ such that*

$$q_I\left(\mathrm{ch}\left(V_\lambda(w)\right)\right) = \sum_j \prod_{1\le i\le g} \mathrm{ch}\left(V_{p_i(\lambda_j)}(w_j|_{\Lambda_i})\right), \tag{4.11}$$

where

$$\mathrm{ch}\left(V_\lambda(w)\right)\cdot \mathrm{ch}\left(V_{N\xi}(e)\right) = \sum_j \mathrm{ch}\left(V_{\lambda_j}(w_j)\right), \tag{4.12}$$

for large enough integer N and $\xi := \sum_{i\notin I}\varpi_i$.

Proof. We need to show that if N is a large enough integer, then as $\mathfrak{b}_I$-modules,

$$\mathrm{res}^{\mathfrak{b}}_{\mathfrak{b}_I} V_{\lambda_j}(w_j) \cong V_{p_1(\lambda_j)}(w_j|_{\Lambda_1}) \boxtimes \cdots \boxtimes V_{p_g(\lambda_j)}(w_j|_{\Lambda_g}). \tag{4.13}$$

It is enough to prove that

$$\mathcal{U}(\mathfrak{b})v_w^+ = \mathcal{U}(\mathfrak{b}_I)v_w^+, \tag{4.14}$$

where v_w^+ is the extremal weight vector of weight $w\lambda$ in $V_\lambda(w)$. Let $\alpha \in \Phi^+$ and write $\alpha = \sum_{i=1}^{n-1} c_i\alpha_i$. Then

$$(w_j\lambda_j, \alpha^\vee) = \frac{2}{(\alpha,\alpha)}\sum_{i=1}^{n-1} c_i(w_j\lambda_j, \alpha_i). \tag{4.15}$$

Assume that $\alpha \notin \Phi_I^+$. Then there exists $y \notin I$ such that $c_y > 0$. Since $w_j\lambda_j$ appears as a weight in $V_\lambda(w)\otimes V_{N\xi}(e)$, $(w_j\lambda_j, \alpha_y)$ is sufficiently large enough to make $(w_j\lambda_j, \alpha^\vee)$ positive. Due to Theorem 7,

$$X_\alpha v_w^+ = 0. \tag{4.16}$$

□

We interpret Lemma 4 in terms of key polynomials. Write

$$\mathrm{Comp}_n := \{\alpha = (\alpha_1, \cdots, \alpha_n) \mid \forall \alpha_i \in \mathbb{Z}_{\ge 0}\}. \tag{4.17}$$

Since $\kappa_\beta(x_1,\cdots,x_m)\kappa_\gamma(y_1,\cdots,y_n)$ forms a linear basis of $\mathbb{C}[x_1,\cdots,x_m,y_1,\cdots,y_n]$,

$$\kappa_\alpha(x_1,\cdots,x_m,y_1,\cdots,y_n) = \sum_{\beta\in\mathrm{Comp}_m,\gamma\in\mathrm{Comp}_n} c^\alpha_{\beta,\gamma}\kappa_\beta(x_1,\cdots,x_m)\kappa_\gamma(y_1,\cdots,y_n), \tag{4.18}$$

for any $\alpha \in \mathrm{Comp}_{m+n}$ and for some $c^\alpha_{\beta,\gamma} \in \mathbb{R}$. According to Lemma 4, $c^\alpha_{\beta,\gamma} \in \mathbb{Z}_{\ge 0}$. In fact, if α, β, γ are weakly increasing *i.e.* reversed partitions λ, μ, ν, then $c^\alpha_{\beta,\gamma}$ is the Littlewood-Richardson coefficient of λ, μ, ν.

L_I-multiplicity freeness is achieved when $c^\alpha_{\beta,\gamma}$ is either zero or one. We use *Lakshmibai-Seshadri paths* to compute it in §5.

5 Lakshmibai-Seshadri paths

In this section, we introduce *LS-monomials* in [14] for semisimple Lie algebra $\mathfrak{g}$ using the same setting as in §4.

Let $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be defined. A **path** is a piecewise-linear map $\pi : [0,1] \to \Lambda_{\mathbb{R}}$ up to reparametrization. Write $\pi_1 * \pi_2$ as concatenation of two paths. Formally,

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & 0 \le t \le 1/2 \\ \pi_2(2t-1) & 1/2 \le t \le 1 \end{cases}. \tag{5.1}$$

The **weight** of a path π is $\pi(1)$ and denoted by $\mathrm{wt}(\pi)$. For each $\lambda \in \Lambda$, write the path $\pi^\lambda : [0,1] \to \Lambda_{\mathbb{R}}$ as a straight line *i.e.* $\pi^\lambda(t) := t\lambda$. Let $\lambda \in \Lambda^+$. Write the stabilizer subgroup W_λ of W for λ. An **LS-chain** of λ is a pair of lists

$$(\tau_1 > \tau_2 > \cdots > \tau_q;\ 0 = a_0 < a_1 < \cdots < a_q = 1), \tag{5.2}$$

where $\tau_j \in W/W_\lambda$ and $a_j \in \mathbb{Q}$. Here, we have an additional condition that for each j, there exists a chain of Bruhat order $\tau_j = \sigma_0 > \sigma_1 > \cdots > \sigma_p = \tau_{j+1}$ with $l(\sigma_{k+1}) = l(\sigma_k) + 1$ and $a_j(\sigma_{k+1}\lambda - \sigma_k\lambda)$ contained in the root lattice $\bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$.

Construct a path $\pi : [0,1] \to \Lambda_{\mathbb{R}}$ from the LS-chain

$$\pi(t) := \sum_{j=1}^{k-1} (a_j - a_{j-1})\tau_j\lambda + (t - a_{k-1})\tau_k\lambda, \quad a_{k-1} \le t \le a_k. \tag{5.3}$$

We call it **LS-path** of λ. Due to the construction, an LS-path is defined by unique LS-chain. We write an LS-path $\pi = (\tau_1 > \cdots > \tau_q; a_0 < \cdots < a_q)$. For instance, π^λ is also an LS-path of λ *i.e.* $\pi^\lambda = (e; 0 < 1)$.

For each $1 \le i \le n-1$, we want to define a **lowering root operator** f_i acting on a path π. Let Q be the minimum value of $(\pi(t), \alpha_i^\vee)$ for $t \in [0,1]$. Let t_1 be the largest t satisfying $Q = (\pi(t), \alpha_i^\vee)$. Let t_2 be the smallest $t > t_1$ satisfying $Q + 1 = (\pi(t), \alpha_i^\vee)$, if it does exist. Split the path π into $\pi_1 * \pi_2 * \pi_3$ in accordance to three intervals $[0, t_1]$, $[t_1, t_2]$ and $[t_2, 1]$. Then $f_i(\pi) := \pi_1 * s_i\pi_2 * \pi_3$ where $s_i\pi_2(t) := s_i(\pi_2(t))$. If such t_2 does not exist, then $f_i(\pi)$ is undefined.

Similarly, define a **raising root operator** e_i as follows. Let Q be defined the same as before. Let t_2' be the smallest t satisfying $Q = (\pi(t), \alpha_i^\vee)$. Choose t_1' the largest $t < t_2'$ with $Q + 1 = (\pi(t), \alpha_i^\vee)$, if it does exist. Split the path π into three pieces using $[0, t_1']$, $[t_1', t_2']$ and $[t_2', 1]$. Same as before, define $e_i(\pi) := \pi_1 * s_i\pi_2 * \pi_3$. If such t_1' does not exist, then $e_i(\pi)$ is undefined.

Let $w \in W$ and $\lambda \in \Lambda$. Fix a reduced word expression of $w = s_{i_1}s_{i_2}\cdots s_{i_l}$. Record this into $\mathbf{i} := (i_1, i_2, \cdots, i_l)$. Write $\lambda = \sum_i l_i\varpi_i$ and for each $1 \le j \le l$, define

$$m_j = \begin{cases} l_k & \text{if } i_j = k \text{ and } i_{j'} \ne k \text{ for all } j' > j \\ 0 & \text{otherwise} \end{cases}. \tag{5.4}$$

Write $\mathbf{m} = (m_1, m_2, \cdots, m_l)$ and $\lambda_j := m_j \varpi_{i_j}$. A **tableau** of shape $(\lambda_1, \cdots, \lambda_l)$ is a concatenation $\Pi = \pi_1 * \cdots * \pi_l$ where each π_j is an LS-path of shape λ_j.

Write each LS-path π_j as $(\tau_{j,1} > \tau_{j,2} > \cdots > \tau_{j,p_j}; a_{j,0} < a_{j,1} < \cdots < a_{j,p_j})$ where each $\tau_{j,p} \in W/W_{m_j\varpi_j}$. $\Pi = \pi_1 * \cdots * \pi_l$ is called a **standard tableau** or an **LS-monomial** if there exists a chain of index sets

$$[n] \supseteq J_{1,1} \supseteq \cdots \supseteq J_{1,p_1} \supseteq \cdots \supseteq J_{l,1} \supseteq \cdots \supseteq J_{l,p_l}, \tag{5.5}$$

such that the subword $\mathbf{i}(J_{j,p} \cap [j])$ is a reduced expression and the Weyl group element

$$w\left(\mathbf{i}(J_{j,p} \cap [j])\right) \equiv \tau_{j,p} \quad \text{modulo } W_{m_j\varpi_j}. \tag{5.6}$$

Lastly, we say a path is **dominant** if $(\pi(t), \alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $1 \leq i \leq n-1$.

Theorem 8 ([14, Theorem 2])**.** *Let $\lambda, \mu \in \Lambda^+$ and $w \in W$. Fix $(\mathbf{i}, \mathbf{m})$ as before. Then the crystal graph on the set of $\{\pi^\mu * \Pi\}$ is a disjoint union of crystal graphs of Demazure modules where Π runs over LS-monomials of shape $(\lambda_1, \cdots, \lambda_l)$. In particular, there is a bijection between each copy of Demazure modules and LS-monomials which are dominant.*

Proof. See [14, Proposition 12]. Each copy of Demazure modules can be bijectively mapped to a LS-monomial Π where $e_i(\Pi)$ is undefined for all i. This is equivalent to $(\Pi(t), \alpha_i^\vee) \geq 0$ so that $Q = 0$ and $t_2' = 0$. □

6 Proof of the main theorems

Let $G = GL_n$ and $W = S_n$. In this case, LS-monomials can be introduced using partitions in [15]. W gives a left action on $\mathbb{R}^n$ in a way that for each $w \in W$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$w \cdot \alpha := (\alpha_{w^{-1}(1)}, \alpha_{w^{-1}(2)}, \cdots, \alpha_{w^{-1}(n)}). \tag{6.1}$$

Write the fundamental weights $\varpi_1, \cdots, \varpi_{n-1} \in \mathrm{Par}_n$

$$\varpi_1 = (1, 0, \cdots, 0), \quad \varpi_2 = (1, 1, 0, \cdots, 0), \cdots, \varpi_{n-1} = (1, \cdots, 1, 0). \tag{6.2}$$

Let $w \in W$ and $w = s_{i_1} \cdots s_{i_l}$ be a reduced word expression of w. Record it into $\mathbf{i} = (i_1, \cdots, i_l)$. Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathrm{Par}_n$ and for each $1 \leq j \leq l$, define

$$m_j = \begin{cases} \lambda_k - \lambda_{k+1} & \text{if } i_j = k \text{ and } i_{j'} \neq k \text{ for all } j' > j \\ 0 & \text{otherwise} \end{cases}. \tag{6.3}$$

In [15], LS-monomials are defined as $\{\alpha_{j,k}\}_{1 \leq j \leq l, 1 \leq k \leq m_j}$ associated to $(\mathbf{i}, \mathbf{m})$ where $\alpha_{j,k} \in \mathbb{Z}^n$. The condition imposed on $\alpha_{j,k}$ is that there exists a nested sequence of index sets

$$[n-1] \supseteq J_{1,1} \supseteq \cdots \supseteq J_{1,m_1} \supseteq \cdots \supseteq J_{l,1} \supseteq \cdots \supseteq J_{l,m_l}, \tag{6.4}$$

where $\mathbf{i}(J_{j,k} \cap [j])$ is a reduced word expression and

$$w(\mathbf{i}(J_{j,k} \cap [j])) \cdot \varpi_{i_j} = \alpha_{j,k}. \tag{6.5}$$

As before, the weight of an LS-monomial $\Pi = \{\alpha_{j,k}\}_{1\le j\le l, 1\le k\le m_j}$ is defined as

$$\mathsf{wt}\,(\Pi) = \sum_{1\le j\le l} \left(\sum_{1\le k\le m_j} \alpha_{j,k} \right). \tag{6.6}$$

We say an LS-monomial $\Pi = \{\alpha_{j,k}\}_{1\le j\le l, 1\le k\le m_j}$ is I**-dominant** if for any $i \in I$, $1 \le j \le l$ and $1 \le k \le m_j$, $\beta_i^{(j,k)} \ge \beta_{i+1}^{(j,k)}$ where $\beta^{(j,k)} \in \mathbb{Z}^n$ is defined as

$$\beta^{(j,k)} := \sum_{1\le j'\le j-1} \left(\sum_{1\le k'\le m_{j'}} \alpha_{j',k'} \right) + \sum_{1\le k'\le k} \alpha_{j,k'}. \tag{6.7}$$

Theorem 9. *Let $G = GL_n$ and $W = S_n$. Let $v \in W$ and $I, J \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Choose $(\mathbf{i}, \mathbf{m})$ from a reduced word expression of u and $\lambda \in \mathrm{Par}_J$. Then $H^0(X(wW_J), L_\lambda)^*$ is L_I-multiplicity free if and only if there are no two distinct I-dominant LS-monomials of $(\mathbf{i}, \mathbf{m})$ which have the same weight.*

Proof. $H^0(X(w), L_\lambda)^*$ is L_I-multiplicity free if and only if $V_\lambda(w)$ is $\mathfrak{g}_I$-multiplicity free where $\mathfrak{g} = \mathfrak{sl}_n$. Due to Lemma 4, this is equivalent to $V_\lambda(w) \otimes V_{N\xi}(e)$ being $\mathfrak{g}_I$-multiplicity free where N is a large enough positive integer and $\xi = \sum_{i\notin I} \varpi_i$. Due to Theorem 8, this is also equivalent to non-existence of a pair of I-dominant LS-monomials of $(\mathbf{i}', \mathbf{m}')$ of w and λ with the same weight. In particular, we may replace w and $(\mathbf{i}', \mathbf{m}')$ with u and $(\mathbf{i}, \mathbf{m})$ using Demazure operators. □

Lemma 5 (I-expansion)**.** *Let $G = GL_n$ and $W = S_n$. Let $v \in W$, $I \subseteq I' \subseteq [n-1]$ and $J \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Similarly, write u' and w' as the minimal and maximal representative of $W_{I'} v$, respectively. Then if $X(wW_J)$ is L_I-spherical, $X(w'W_J)$ is $L_{I'}$-spherical.*

Proof. Suppose $X(w'W_J)$ is not $L_{I'}$-spherical. Then due to Theorem 6, there exists $\lambda \in \mathrm{Par}_J$ so that $H^0(X(w'W_J), L_\lambda)^*$ has $L_{I'}$-multiplicities. Choose a reduced expression of u'

$$u' = s_{i'_1} s_{i'_2} \cdots s_{i'_{l'}}, \tag{6.8}$$

and record it into $(\mathbf{i}', \mathbf{m}')$ from λ. Due to Theorem 9, there exists I'-dominant LS-monomials $\{\alpha'_{j,k}\}$ and $\{\beta'_{j,k}\}$ of same weights. Write the corresponding chains of indices $\{J'_{j,k}\}$ and $\{K'_{j,k}\}$.

Since $u \in W_{I'}u'$, we may choose a reduced expression of u

$$u = s_{i_1}s_{i_2}\cdots s_{i_l}s_{i'_1}s_{i'_2}\cdots s_{i'_{l'}}. \tag{6.9}$$

By computing $\mathbf{m}$ from $\mathbf{i}$ and λ,

$$m_{j+l} = m'_j \quad j \in [l']. \tag{6.10}$$

Define

$$J_{j,k} = \begin{cases} [l+1, l+l'] & j \in [l] \\ l + J'_{j-l,k} & j \in [l+1, l+l'] \end{cases}, \quad K_{j,k} = \begin{cases} [l+1, l+l'] & j \in [l] \\ l + K'_{j-l,k} & j \in [l+1, l+l'] \end{cases}, \tag{6.11}$$

by adding l to the indices in $J'_{j-l,k}$ and $K'_{j-l,k}$. From $\{J_{j,k}\}$ and $\{K_{j,k}\}$, we have LS-monomials $\{\alpha_{j,k}\}$ and $\{\beta_{j,k}\}$, respectively. Due to $I \subseteq I'$, they are I-dominant. Therefore, $H^0(X(wW_J), L_\lambda)^*$ has L_I-multiplicities, proving the given statement. □

Lemma 6 (J-expansion)**.** *Let $G = GL_n$ and $W = S_n$. Let $v \in W$, $I \subseteq [n-1]$ and $J \subseteq J' \subseteq [n-1]$. Write u and w as the minimal and maximal representative of $W_I v$, respectively. Then if $X(wW_J)$ is L_I-spherical, $X(wW_{J'})$ is L_I-spherical.*

Proof. Analogous to the proof of I-expansion Lemma 5. □

Proof of Theorem 1. Suppose $X(w)$ is not L_I-spherical. Let u be the minimal representative of $W_I w$. Due to Theorem 5, u has 321 or 3412 patterns. We first assume that u has 3412 patterns. Using I-sorting Lemma 2 and J-sorting Lemma 3, we may assume that there exists $2 \leq p \leq n-2$ such that $u = s_p s_{p+1} s_{p-1} s_p$. For any strictly decreasing partition λ, we construct LS-monomials and use Theorem 9 to conclude that $H^0(X(w), L_\lambda)$ has L_I-multiplicities. Apply the same argument for 321 pattern. □

Proof of Theorem 2. We first prove that if $[u, I, J]$ has 321, 3412, 3*i*312, 231*i*1, 2*i*21*i*1, 244113 or 233112 patterns, $X(wW_J)$ is not L_I-spherical. For instance, suppose $[u, I, J]$ has 3*i*312 pattern. Use I-expansion Lemma 5 and J-expansion Lemma 6 and we may assume $[u, I, J]$ only has $1, 2, 3$ as number entries and two insertions of i. Choose $\lambda \in \mathrm{Par}_J$. Using I-sorting Lemma 2 and J-sorting Lemma 3, we can reduce possible cases and apply Theorem 9 to show L_I-multiplicities of $H^0(X(wW_J), L_\lambda)$. Due to Theorem 6, this leads to $X(wW_J)$ not being L_I-spherical.

If $[u, I, J]$ does not have 321, 3412, 3*i*312, 231*i*1, 2*i*21*i*1, 244113 and 233112 patterns, then we reduce it into Grassmannian case. □

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