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Kirillov–Reshetikhin dual equivalence graphs

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Abstract. We introduce Kirillov–Reshetikhin dual equivalence graphs, a generalization of Assaf's dual equivalence graphs to affine type A. Given a sequence of rectangular partitions, we use commutators of adjacent crystal operators to construct a graph on the 0-weight space of the corresponding Kirillov–Reshetikhin crystal. We give a complete characterization of the set of edges that appear in these graphs, which includes standard dual equivalence moves as a subset. Furthermore, we characterize the number of connected components in terms of the energy function of the underlying crystal. We conjecture that characters of these connected components can be computed via the plethysm of cyclic characters with a product of rectangular Schur functions.

Keywords: tableaux, dual equivalence graphs, Kirillov–Reshetikhin crystals, plethysm, promotion

1 Background

1.1 Dual equivalence graphs and crystals

Dual equivalence graphs were introduced by Assaf [3] as a method for proving that certain sums of fundamental quasisymmetric functions are symmetric and Schur positive. For each partition λ, the corresponding dual equivalence graph G_λ is a graph whose vertices are standard Young tableaux of shape λ, where there is an edge between T and T' if we can obtain T' by swapping entries i and $i+1$ in T, with the added condition that the descent sets of T and T' are incomparable. Edges in this graph correspond to dual Knuth moves on the row-reading words of the tableaux, which preserve the recording tableau under RS insertion. One can define a similar graph structure on other combinatorial objects, and if the graph satisfies certain local axioms (see [3, Definition 3.2]) then there is a descent-preserving isomorphism from each connected component to some G_λ.

Example 1.1. Let $\lambda = (3,2)$. The corresponding dual equivalence graph is given below.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \overset{t_3}{\text{———}} \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \overset{t_2}{\text{———}} \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} \overset{t_4}{\text{———}} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \overset{t_2}{\text{———}} \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$$

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The descents of each tableau are indicated in red, and the edge label t_i corresponds to swapping i and $i+1$. Note that, while we could obtain the fifth tableau from the second by swapping 4 and 5, this isn't an edge in the graph because the descent sets are not incomparable.

Crystal graphs are another family of combinatorially defined graphs with deep connections to representation theory. We briefly review the type A construction, and refer the reader to [4] for a more thorough treatment. For partition λ and fixed n, let $\mathrm{SSYT}(\lambda)$ denote the collection of semistandard Young tableaux with entries at most n. For $T \in \mathrm{SSYT}(\lambda)$, let $\mathsf{rw}(T)$ denote the word obtained by reading the entries of each row of T from left to right, taking rows from bottom to top. For each $i \in [n-1]$, we define crystal raising and lowering operators $e_i, f_i : \mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda)$ as follows.

1. Consider the subword of $\mathsf{rw}(T)$ consisting of entries $i, i+1$. Replace each $i+1$ with "(", each i with ")", and remove matched parentheses.
2. If applying e_i, replace the $i+1$ corresponding to the left-most unmatched "(" with i. If applying f_i, replace the i corresponding to the right-most unmatched ")" with $i+1$. In either case, we leave T unchanged if no such entry exists.

It is well known that the output of this process will always be a semistandard tableau, and that e_i and f_i are inverses of each other when they act nontrivially. The crystal graph B_λ is defined with vertices $\mathrm{SSYT}(\lambda)$, where we draw an i-colored edge $T \xrightarrow{i} T'$ if $f_i(T) = T'$.

Example 1.2. Suppose $\lambda = (3,3)$ and $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline\end{array}$. Then $\mathsf{rw}(T) = 234123$, and we have

$$e_2 T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & 4 \\ \hline\end{array} \quad \text{and} \quad f_2 T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline\end{array}.$$

Assaf showed that the graphs G_λ and B_λ are related. In particular, one can obtain G_λ by taking the weight 0 (i.e., standard) elements of B_λ, and connecting them with a commutator of consecutive crystal operators.

Theorem 1.3 ([2]). *The graph G_λ can be constructed by taking the 0-weight space of the type A crystal graph B_λ, and connecting vertices by commutators of consecutive crystal operators. Explicitly, T and T' are connected by an edge in G_λ if and only if $T' = e_i e_{i+1} f_i f_{i+1}(T)$ or $T' = e_{i+1} e_i f_{i+1} f_i(T)$ for some $i \in [n-1]$.*

Example 1.4. Consider the left-most edge in the graph $G_{(3,2)}$ shown in Example 1.1. It can be equivalently computed using the following composition of crystal operators (acting on row words).

$$45123 \xrightarrow{f_4} 55123 \xrightarrow{f_3} 55124 \xrightarrow{e_4} 45124 \xrightarrow{e_3} 35124$$

1.2 Kirillov–Reshetikhin crystals

Kirillov–Reshetikhin (KR) crystals are an important family of affine crystals. For a rectangular partition $\lambda = (s^r)$, the KR crystal $B^{r,s}$ consists of the classical type A crystal B_λ, with extra crystal operators e_n, f_n corresponding to the extra node in the type $\widetilde{A}$ Dynkin diagram. The action of these operators on $\mathrm{SSYT}(\lambda)$ was worked out by Shimozono [14] in terms of the *promotion* operator, pr. In particular,

$$e_n = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad \text{and} \quad f_n = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}\,.$$

Given $T \in \mathrm{SSYT}(\lambda)$, promotion consists of the following sequence of operations.

1. Remove all boxes containing n from T.
2. Perform *jeu de taquin* slides [8] at each outer corner created.
3. Put the entry 1 into each empty cell created on the inner boundary, and increase all other entries by 1.

Promotion encodes extra symmetry in $B^{r,s}$ coming from the cyclic symmetry of the type $\widetilde{A}$ Dynkin diagram. In particular, we can compute e_i (resp. f_i) by conjugating e_{i-1} (resp. f_{i-1}) by pr for any $i \in [n]$.

Example 1.5. Let λ and T be as in Example 1.2. If $n = 4$, then $\mathrm{pr}(T)$ is computed by the following steps

$$T = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3&4\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3&\bullet\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&\bullet\\\hline 2&3&3\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&\bullet&2\\\hline 2&3&3\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline \bullet&1&2\\\hline 2&3&3\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&4&4\\\hline\end{array} = \mathrm{pr}(T).$$

Let $R = (R_1, \ldots, R_k)$ be a tuple of rectangular partitions, that is, for each $i \in [k]$, $R_i = (s_i^{r_i})$ for some $s_i, r_i > 0$. We consider the tensor product of KR crystals $B^R = B^{r_1,s_1} \otimes \cdots \otimes B^{r_k,s_k}$. The vertex set of this graph consists of tuples of semistandard tableaux, i.e., $T = T_1 \otimes \cdots \otimes T_k$ where $T_i \in SSYT(R_i)$. It is sometimes useful to think of T as a disconnected skew tableau, where the northeast corner of T_i touches the southwest corner of T_{i+1}. Unlike their type A analogs (which split into connected components according to the Littlewood–Richardson rule), tensor products of KR crystals are connected [1]. In fact, it is conjectured that every connected affine crystal is isomorphic to a tensor product of KR crystals.

Tensor products of KR crystals are naturally graded by the energy function. Following [10, 14], given rectangles R_1, R_2, there is a unique affine crystal graph isomorphism $\sigma : B^{R_1} \otimes B^{R_2} \to B^{R_2} \otimes B^{R_1}$ called the *combinatorial R-matrix*. It can be easily described in terms of tableaux. Given $T = T_1 \otimes T_2 \in B^{R_1} \otimes B^{R_2}$, $\sigma(T)$ is the unique skew-shape $T_2' \otimes T_1'$ of shape $R_2 \otimes R_1$ obtainable by applying jeu de taquin slides to T (see Example 1.7 for an example of this computation). For general tensor products of KR crystals, we let σ_i denote the combinatorial R-matrix which swaps the i and $i+1$ tensor factors. The σ_i satisfy the Yang-Baxter equation, so

$$B^{R_1} \otimes \cdots \otimes B^{R_k} \cong B^{R'_1} \otimes \cdots \otimes B^{R'_k},$$

where $(R'_1, \ldots, R'_k)$ is any permutation of $(R_1, \ldots, R_k)$. The energy function for B^R is a statistic on the vertices which is invariant under the application of combinatorial R matrices. Shimozono showed that this function can be computed using the following generalization of the *charge* statistic originally defined by Lascoux and Schützenberger.

Definition 1.6. Let $R = (R_1, \ldots, R_k)$ be a sequence of rectangles, with $R_i = (s_i^{r_i})$. The charge function is defined on B^R as follows:

1. If $k = 1$, $\text{charge}(T) = 0$ for every $T \in B^R$.
2. If $k = 2$, so $T = T_1 \otimes T_2$, then $\text{charge}(T)$ is equal to the number of cells with row index greater than $\max(r_1, r_2)$ in the tableau $P(\mathsf{rw}(T_1)\mathsf{rw}(T_2))$ obtained by row insertion.
3. If $k > 2$, then
$$\text{charge}(T) = \sum_{i<j} \text{charge}(T_{j-1}^{(i)} \otimes T_j)$$
where $T_{j-1}^{(i)}$ the $(j-1)$'st component of the image of T under the composition of R-matrices $\sigma_{j-2} \cdots \sigma_i$.

This definition differs slightly from that of [14], due to our usage of row insertion instead of column insertion, but it is equivalent.

Example 1.7. Let $R = ((3^3), (2^2), (2^2))$, and consider

$$T = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 9 & 13 & 14 \\ \hline 10 & 16 & 17 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 11 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 12 & 15 \\ \hline \end{array}$$

To compute $\text{charge}(T)$, we must compute $\text{charge}(T_1 \otimes T_2)$, $\text{charge}(T_2 \otimes T_3)$, and $\text{charge}(T_2^{(1)} \otimes T_3)$, each of which can be computed by row insertion. The first two are straightforward:

$$T_1 \leftarrow \mathsf{rw}(T_2) = \begin{array}{cccc} 2 & 5 & 6 & 8 \\ 3 & 11 & 14 & \\ 4 & 13 & 17 & \\ 9 & 16 & & \\ 10 & & & \end{array} \qquad T_2 \leftarrow \mathsf{rw}(T_3) = \begin{array}{cccc} 1 & 7 & 12 & 15 \\ 2 & 8 & & \\ 3 & 11 & & \end{array}$$

To compute the last term, we first compute $T_2^{(1)}$ via the combinatorial R-matrix σ_1:

$$T_1 \otimes T_2 = \begin{array}{ccccc} & & & 2 & 8 \\ & & & 3 & 11 \\ 4 & 5 & 6 & & \\ 9 & 13 & 14 & & \\ 10 & 16 & 17 & & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} & & 2 & 6 & 8 \\ & & 3 & 11 & 14 \\ & & 5 & 13 & 17 \\ 4 & 9 & & & \\ 10 & 16 & & & \end{array} = T_1^{(1)} \otimes T_2^{(1)}$$

Finally, we see that

$$T_2^{(1)} \leftarrow \mathsf{rw}(T_3) = \begin{array}{ccccc} 1 & 6 & 7 & 12 & 15 \\ 2 & 8 & 14 & & \\ 3 & 11 & 17 & & \\ \mathbf{5} & \mathbf{13} & & & \end{array}$$

Thus, $\operatorname{charge}(T) = 3 + 2 + 2 = 7$. Cells that contribute a $+1$ to charge are highlighted in blue.

2 KR dual equivalence graphs

Inspired by Theorem 1.3, we make the following definition.

Definition 2.1. Let $R = (R_1, \ldots, R_k)$ be a sequence of rectangles. The *Kirillov–Reshetikhin dual equivalence graph* (KR DEG), $\mathcal{T}(R)$, is a graph whose vertices consist of standard fillings of R, where T and T' are connected by an edge if $T' = e_i e_{i+1} f_i f_{i+1}(T)$ or $T' = e_{i+1} e_i f_{i+1} f_i(T)$ for some $i \in [n]$. Addition in subscripts is taken modulo n.

Example 2.2. Let $R = ((2,2),(1))$. The corresponding KR DEG is given in Figure 1. The orange edges correspond to the crystal commutators which realize each edge ($C_i = e_i e_{i+1} f_i f_{i+1}$), and the black edges correspond to the characterization described in Theorem 2.5. While some edges can be realized in multiple ways, we still draw at most one edge between any two tableaux. Note that if we remove the edges that can only be realized using affine crystal operators (the vertical ones), the graph splits up into a top and bottom component, which correspond to $G_{(3,2)}$ and $G_{(2,2,1)}$ respectively. This agrees with the Schur expansion of $s_{2,2} \cdot s_1$.

Definition 2.3. Let $T \in \mathcal{T}(R)$. We say $i \in [n-1]$ is a *descent* of T if $i+1$ appears before i in $\mathsf{rw}(T)$. Additionally, we say that n is a descent of T if 1 is a descent of $\operatorname{pr}(T)$. Denote by $D(T)$ the set of all descents of T, and by $\overline{D}(T) := D(T) \setminus \{n\}$ the set of standard descents.

The following proposition shows that $\mathcal{T}(R)$ inherits the cyclic symmetry of B^R. Of particular interest to us is that it plays very nicely with the descent set $D(T)$.

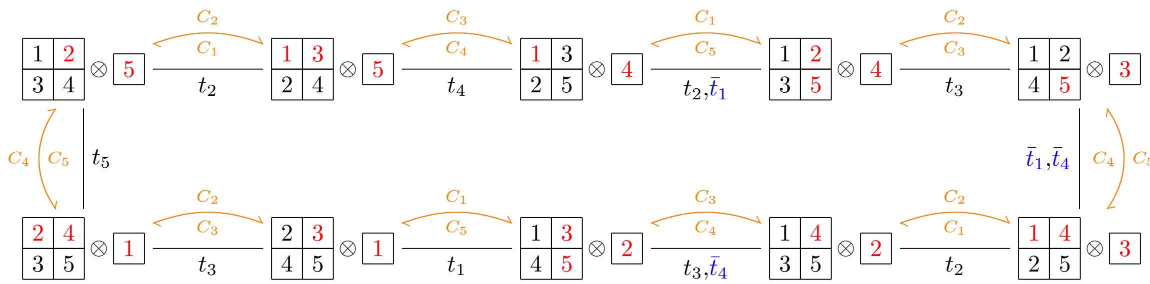

Figure 1: The KR DEG $\mathcal{T}((2,2),(1))$

Proposition 2.4. *The map* $\mathrm{pr} : \mathcal{T}(R) \to \mathcal{T}(R)$ *is a graph automorphism with the property that* $i \in D(T)$ *if and only if* $i+1 \in D(\mathrm{pr}(T))$ *for every* $T \in \mathcal{T}(R)$.

Next, we give a description of the edges of $\mathcal{T}(R)$, that is easier to compute than a composition of crystal operators.

Theorem 2.5. *For any* $T, T' \in \mathcal{T}(R)$ *and* $i \in \{1, 2, \cdots, n\}$, *we connect* T *and* T' *with an edge if* $D(T)$ *and* $D(T')$ *are incomparable, and* T' *can be obtained from* T *by doing one of the following sequences of moves.*

1. *For* $1 \leq i \leq n-1$, *we swap* i *and* $i+1$. *In this case we write* $t_i(T) = T'$. *These edges correspond to those using only "classical" crystal operators.*

2. *For* $i = n$, *if* $n \in T_\ell$ *and* $i+1 = 1 \in T_j$ *are not in the same tensor factor, we compute* $\mathrm{jdt}_n(T_\ell)$ *with replacing the created inner box with* 1, *and compute* $\mathrm{jdt}_1(T_j)$ *with replacing the created outer box with* n. *We obtain* $T' = t_n(T)$ *by changing these two tensor factors, and leaving any others unchanged.*

3. *If* $1, 2$ *and* n *are in the same tensor factor* T_j *of* T *such that* $\mathsf{rw}(T_j) = (n \cdots 1\, 2 \cdots)$ *and* $n \in D(T)$, *then we can find* T' *by the sequence of moves: (i) Apply* $\mathrm{jdt}_n(T_j)$ *to get an inner box* $*$; *(ii) Switch* 1 *and* 2; *(iii) Apply* jdt_* *to the tableau obtained from the last move, and replace the resulting outer box with* n. *We denote this edge by* $\bar{t}_1(T) = T'$.

4. *If* $1, n-1$ *and* n *are in the same tensor factor* T_j *of* T *such that* $\mathsf{rw}(T_j) = (\cdots n \cdots n - 1 \cdots 1 \cdots)$ *and* $n \notin D(T)$, *then we can find* T' *by the sequence of moves: (i) Apply* $\mathrm{jdt}_n(T_j)$ *and fill the inner box with* 1; *(ii) Apply* jdt_{n-1} *to the tableau obtained from the last move and get an inner box* $*$; *(iii) Apply* jdt_1 *for the* 1 *in the first row of the tableau followed by applying* jdt_*, *and then fill the two outer boxes with* $n-1$ *and* n. *We denote this edge by* $\bar{t}_{n-1}(T) = T'$.

Example 2.6. We have the following examples to illustrate the edges of $\mathcal{T}(R)$ described in Theorem 2.5.

1. Let $T = T_1 \otimes T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$ with $D(T) = \{2,4,7,8\}$. For $i = 3$, $T' = t_3(T) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$ with $D(T') = \{3,7,8\}$. For $i = 8$, we find $\mathrm{jdt}_8(T_2)$ and replace the created inner box with 1 to obtain $T_2' = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 6 & 7 \\ \hline \end{array}$. Also, find $\mathrm{jdt}_1(T_1)$ and replace the created outer box with 8 gives $T_1' = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 8 \\ \hline \end{array}$. Thus $T' = t_8(T) = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 8 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 6 & 7 \\ \hline \end{array}$ with $D(T') = \{1,2,4,7\}$.

2. Let $T = T_1 \otimes T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 5 & 10 & 12 \\ \hline 7 & 11 & 13 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 & 9 \\ \hline \end{array}$ with $D(T) = \{3,4,6,9,10,12,13\}$. T falls into Case 3 of Theorem 2.5. We take the sequence of moves as
$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 5 & 10 & 12 \\ \hline 7 & 11 & 13 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_{13}} \begin{array}{|c|c|c|}\hline * & 2 & 8 \\ \hline 1 & 5 & 10 \\ \hline 7 & 11 & 12 \\ \hline \end{array} \xrightarrow[1,2]{\text{switch}} \begin{array}{|c|c|c|}\hline * & 1 & 8 \\ \hline 2 & 5 & 10 \\ \hline 7 & 11 & 12 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_*} \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 10 & 12 \\ \hline 7 & 11 & 13 \\ \hline \end{array},$$
and
$$\bar{t}_1(T) = T' = T_1' \otimes T_2' = \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 10 & 12 \\ \hline 7 & 11 & 13 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 & 9 \\ \hline \end{array}$$
with $D(T') = \{1,3,4,6,9,10,12\}$.
Now T' falls into Case 4 of Theorem 2.5. We take the sequence of moves on T_1' as
$$T_1' = \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 10 & 12 \\ \hline 7 & 11 & 13 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_{13}} \begin{array}{|c|c|c|}\hline 1 & 1 & 8 \\ \hline 2 & 5 & 10 \\ \hline 7 & 11 & 12 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_{12}} \begin{array}{|c|c|c|}\hline * & 1 & 8 \\ \hline 1 & 5 & 10 \\ \hline 2 & 7 & 11 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_1} \begin{array}{|c|c|c|}\hline * & 5 & 8 \\ \hline 1 & 7 & 10 \\ \hline 2 & 11 & 12 \\ \hline \end{array} \xrightarrow{\mathrm{jdt}_*} \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 7 & 10 \\ \hline 11 & 12 & 13 \\ \hline \end{array},$$
and
$$\bar{t}_{12}(T') = T'' = T_1'' \otimes T_2'' = \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 7 & 10 \\ \hline 11 & 12 & 13 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 6 \\ \hline 4 & 9 \\ \hline \end{array}$$
with $D(T'') = \{1,3,4,6,9,10,13\}$.

We note that, in this case, we can apply both $\bar{t}_1$ and $\bar{t}_{n-1}$ to the same T. This justifies our notation for these edges (instead of also notating them by t_n).

In both crystal graphs and dual equivalence graphs, the connected components of the graph encode important representation-theoretic information. Thus, it is natural to want a description of the connected components of $\mathcal{T}(R)$. This question turns out to have a nice answer in terms of the energy/charge function of the underlying tensor product of KR crystals.

Theorem 2.7. *Let R be a sequence of rectangles. Suppose that rectangle R_i appears with multiplicity m_i in R, and let $d_R = \gcd(m_1, \ldots, m_k)$. Then $\mathcal{T}(R)$ has d_R connected components, each consisting of all vertices of $\mathcal{T}(R)$ with a fixed charge modulo d_R.*

Example 2.8. Let $R = ((1^3), (1^3))$, so $d_R = 2$. $\mathcal{T}(R)$ is given in Figure 2. The left and right connected components consist of vertices with even and odd charge respectively.

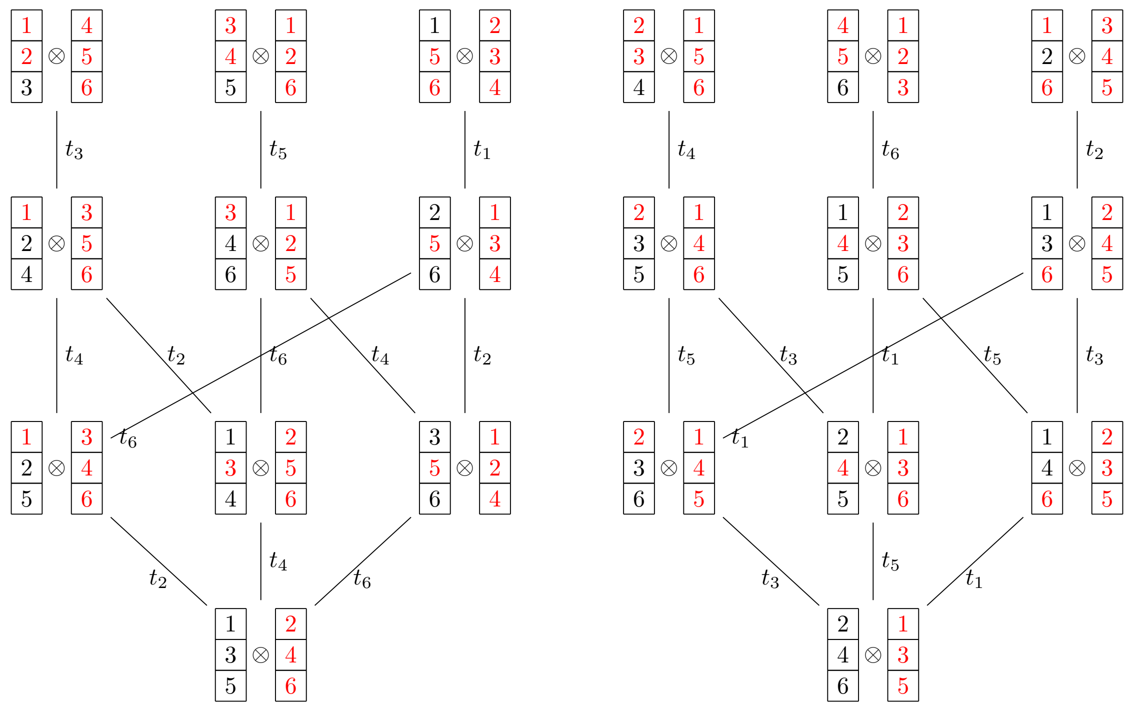

Figure 2: The KR DEG $\mathcal{T}((1^3), (1^3))$

We end this section with some comments on the proof of Theorem 2.7. It suffices to prove the special case where $R = (R_1^{\otimes m_1}, \ldots, R_\ell^{\otimes m_\ell})$, i.e., when rectangles of the same shape are grouped together. Any R where partitions of the same shape are not grouped together can be permuted to this form using combinatorial R-matrices without changing the structure of $\mathcal{T}(R)$.

To prove Theorem 2.7, we define a modified version of the charge statistic (called *semicharge* and denoted scharge), which agrees with charge modulo d_R in this special case, and use it to prove the following statements:

1. For any $T, T' \in \mathcal{T}(R)$, if T and T' are connected by an edge, then $\mathrm{scharge}(T) \equiv \mathrm{scharge}(T') \bmod d_R$.

2. For any $0 \le i < d_R$, there exists $T \in \mathcal{T}(R)$ such that $\mathrm{scharge}(T) \equiv i \bmod d_R$.

3. For any $T, T' \in \mathcal{T}(R)$ such that $\mathrm{scharge}(T) \equiv \mathrm{scharge}(T') \bmod d_R$, T and T' are connected in $\mathcal{T}(R)$.

Steps 1 and 2 end up being straightforward arguments, and together they show that $\mathcal{T}(R)$ has at least d_R components. Step 3 ends up being quite a bit more involved, and shows that $\mathcal{T}(R)$ has at most d_R components. The main technical challenge comes from the fact that the edges t_n, $\bar{t}_1$, and $\bar{t}_{n-1}$ are harder to work with, and so any attempt at describing a path using many of these edges becomes quite complicated. To get around this issue, we do the following two things.

1. Find a collection $\mathcal{A} \subset \mathcal{T}(R)$ such that each $A_i \in \mathcal{A}$ is connected to $\mathrm{pr}^{d_i}(A_i)$ for some d_i, with the added condition that $\gcd(\{d_i\}) = d_R$.

2. Find a path (via the *row-combing* algorithm) consisting of either edges in $\mathcal{T}(R)$ *or* powers of promotion that brings *any* $T \in \mathcal{T}(R)$ to a fixed $T' \in \mathcal{T}(R)$.

In particular, we only ever explicitly work with the simpler dual equivalence edges, and leverage Proposition 2.4 to fill in the gaps. The row-combing algorithm we describe is a generalization of Shi's combing algorithm [13] for affine permutations.

Remark 2.9. In [5], Chmutov, Lewis, and Pylyavskyy described the Knuth equivalence classes of affine permutations via the *Affine Matrix Ball Construction* (AMBC) [6], whose image gives a pair of tabloids P and Q. The Knuth moves on permutations preserve their P tabloids and determine the Knuth moves on their Q tabloids. A modified charge statistic determines the equivalence classes of these Q tabloids, and the induced graph is called the Kazhdan–Lusztig dual equivalence graph (KL DEG). If $R = (R_1, \dots, R_k)$ with $R_i = (s_i)$, i.e., R is a sequence of single rows, our KR DEG $\mathcal{T}(R)$ reduces to the KL DEG, whose vertex set consists of all Q tabloids of shape $(s_k, \cdots, s_1)$. They were able to prove Theorem 2.7 in this special case.

3 Future work

Recall that, for $A \subset [n-1]$, the corresponding fundamental quasisymmetric function is given by

$$F_A(x) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ a \in A \Rightarrow i_a < i_{a+1}}} x_{i_1} \cdots x_{i_n}.$$

For the dual equivalence graph G_λ, we have

$$\sum_{T \in G_\lambda} F_{\overline{D}(T)} = s_\lambda.$$

If we consider the same generating function over $\mathcal{T}(R)$, then the theory of type A crystals and dual equivalence graphs tells us the following.

Theorem 3.1. *Let $R = (R_1, \dots, R_k)$. Then,*

$$\sum_{T \in \mathcal{T}(R)} F_{\overline{D}(T)} = s_{R_1} s_{R_2} \cdots s_{R_k}.$$

However, in the cases where $d_R > 1$, we can consider the same sum restricted to one of the d_R connected components of $\mathcal{T}(R)$. This sum will be a Schur-positive symmetric function, but a better description of its Schur expansion remains elusive.

We conjecture that the characters of the connected components appear in the following representation theory problem. Let $V = \mathbb{C}^n$ and let U be a polynomial representation of $GL(V)$ defined by a tensor product of representations corresponding to multiples of fundamental weights (i.e. rectangles). There is a natural action of the cyclic group C_k on $U^{\otimes k}$ by cycling tensor factors, which commutes with the diagonal action of $GL(V)$. We can decompose $U^{\otimes k}$ into C_k isotypic components, and since the two actions commute, each component is a $GL(V)$ representation.

It is not hard to show that the characters of these components will be given by the plethysm of a *cyclic character* $\ell_k^{(i)}$ with the character of U (which is a product of rectangle Schur functions). Explicitly, the cyclic character $\ell_k^{(i)}$ is the Fröbenius image of the representation $\mathrm{Ind}_{C_k}^{S_k} \chi^i$, where χ^i is the irreducible C_k character sending a generator to the i'th power of a primitive k'th root of unity. Such symmetric functions were first studied by Foulkes [7], and later work of Kraśkiewicz and Weyman [11] as well as Lascoux, Leclerc and Thibon [12] connected them to charge, which coincides with the energy function for KR crystals.

Conjecture 3.2. *Suppose that R has distinct rectangles R_j that appear with multiplicity m_j (so $d_R = \gcd(m_j)$). Then for $i \in [d_R]$, the charge i connected component of $\mathcal{T}(R)$ has character given by the plethysm*

$$\ell_{d_R}^{(i)} \left[s_{R_1}^{m_1/d_R} \cdots s_{R_k}^{m_k/d_R} \right].$$

A Schur expansion for the above plethysm is known in a few special cases. If the R_i are all rows (or all columns), then it is known due to Lascoux, Leclerc and Thibon [12]. If all R_i are the same rectangle, it is known due to Iijima [9].

Example 3.3. Consider $R = ((1^3), (1^3))$ as in Example 2.8. One can compute that $\ell_2^{(0)} = s_2$ and $\ell_2^{(1)} = s_{1,1}$. Reading off the descents from Figure 2 (where we ignore descents at 6), we see that the character of the first connected component is given by

$$2F_{1245} + F_{1234} + F_{2345} + F_{135} + F_{1235} + F_{1345} + F_{134} + F_{235} + F_{24},$$

and the second is

$$F_{1235} + F_{12345} + F_{1345} + F_{124} + F_{234} + F_{245} + F_{1245} + F_{134} + F_{235} + F_{135}.$$

One can verify that both of these quasisymmetric functions are symmetric, and their Schur expansions are $s_{2,1,1,1,1} + s_{2,2,2}$ and $s_{1,1,1,1,1,1} + s_{2,2,1,1}$ respectively, which are exactly $s_2[s_{1,1,1}]$ and $s_{1,1}[s_{1,1,1}]$.

Example 3.4. Let $R = ((1^2), (1^2), (2), (2))$, so $d_R = 2$. $\mathcal{T}(R)$ has 2520 vertices and two connected components. The characters of the two components are exactly $\ell_2^{(0)}[s_2 s_{1,1}]$ and $\ell_2^{(1)}[s_2 s_{1,1}]$:

$$\begin{aligned}\ell_2^{(0)}[s_2 s_{1,1}] &= s_{6,2} + 2s_{5,2,1} + 2s_{5,1,1,1} + s_{4,4} + 2s_{4,3,1} + 3s_{4,2,2} + 2s_{4,2,1,1} + 2s_{4,1,1,1,1} + 3s_{3,3,1,1} \\ &\quad + 2s_{3,2,2,1} + 2s_{3,2,1,1,1} + s_{2,2,2,2} + s_{2,2,1,1,1,1}, \\ \ell_2^{(1)}[s_2 s_{1,1}] &= s_{6,1,1} + s_{5,3} + 2s_{5,2,1} + s_{5,1,1,1} + 2s_{4,3,1} + s_{4,2,2} + 4s_{4,2,1,1} + s_{4,1,1,1,1} + 2s_{3,3,2} \\ &\quad + s_{3,3,1,1} + 2s_{3,2,2,1} + 2s_{3,2,1,1,1} + s_{3,1,1,1,1,1} + s_{2,2,2,1,1}.\end{aligned}$$

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