*Séminaire Lotharingien de Combinatoire* **XX** (2026)
Article #73, 12 pp.

*Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)*

# Classical double Grothendieck transitions

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**Abstract.** Kirillov and Naruse have constructed double Grothendieck polynomials to represent the equivariant $K$-theory classes of Schubert varieties in the complete flag manifolds of types $B$, $C$, and $D$. We derive a recursive formula for these polynomials, extending certain transition equations known in type $A$. As an application, we obtain an identity that expands the $K$-Stanley symmetric functions in types $B$, $C$, and $D$ into positive linear combinations of $K$-theoretic Schur $P$- and $Q$-functions. We use this expansion to resolve some conjectures related to $K$-theoretic Schur $P$- and $Q$-functions.

**Keywords:** Grothendieck polynomials, $K$-theoretic symmetric functions, Coxeter groups

## 1 Introduction

This work studies certain power series introduced by Kirillov and Naruse [10] in connection with the equivariant $K$-theory rings of flag varieties in classical type. We show that the *K-Stanley symmetric functions* for types B, C, and D defined in [10] have finite, positive expansions into the *K-theoretic Schur P- and Q-functions* constructed by Ikeda and Naruse in [9]. As motivation, we start by reviewing the analogous story in type A.

### 1.1 Symmetric Grothendieck functions

Write $\mathbb{Z}$ for the set of integers, and let $\beta$ and $z_1, z_2, z_3, \ldots$ be commuting variables. We define $\mathbf{Sym}_\beta$ to be the subring of formal power series in $\mathbb{Z}[\beta][\![\mathbf{z}]\!] = \mathbb{Z}[\beta][\![z_1, z_2, z_3, \ldots]\!]$ that are invariant under all permutations of the $z$-variables. Let $S_\infty$ be the group of permutations of the positive integers with finite support and write $\ell : S_\infty \to \mathbb{Z}_{\geq 0}$ for its Coxeter length function, which assigns to each permutation its number of inversions.

Each element $w \in S_\infty$ has an associated *symmetric Grothendieck function* $G_w(\mathbf{z}) \in \mathbf{Sym}_\beta$, which is a certain symmetric formal power series of unbounded degree [5]. These functions, which were first defined by Fomin and Kirillov in [7], arise as the stable limits of the *Grothendieck polynomials* of Lascoux and Schützenberger [12]. They also generalize the *Stanley symmetric functions* from [18], which are recovered by setting $\beta = 0$.

There is a natural bialgebra structure on the module $\mathbf{G} := \mathbb{Z}[\beta]\text{-span}\{G_w : w \in S_\infty\}$ of finite $\mathbb{Z}[\beta]$-linear combinations of symmetric Grothendieck functions. To describe

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this, let $\circ$ denote the *Demazure product* for $S_\infty$, which is the unique associative binary operation with $u \circ v = uv$ if $\ell(uv) = \ell(u) + \ell(v)$ and $s \circ s = s$ if $\ell(s) = 1$.

**Proposition 1.1** ([5]). **G** is a bialgebra for the $\mathbb{Z}[\beta]$-linear (co)product operations

$$G_u(\mathbf{z})G_v(\mathbf{z}) = G_{u\times v}(\mathbf{z}) \quad \text{and} \quad \Delta(G_w(\mathbf{z})) = \sum_{\substack{u,v\in S_\infty \\ u\circ v = w}} \beta^{\ell(u)+\ell(v)-\ell(w)} G_u(\mathbf{z}) \otimes G_v(\mathbf{z}).$$

Here $u \times v$ is the image of $(u, v) \in S_n \times S_n$ under the usual inclusion $S_n \times S_n \hookrightarrow S_{2n}$ where $n \in \mathbb{Z}_{\geq 0}$ is any integer such that $u, v \in S_n := \{w \in S_\infty : w(i) = i \text{ for all } i > n\}$.

The coproduct $\Delta$ in this proposition extends the usual one on symmetric functions.

A permutation $w \in S_\infty$ is *Grassmannian* if it has at most one *descent* $i \in \mathbb{Z}_{>0}$ with $w(i) > w(i+1)$. In this case the *shape* of $w$ is the partition $\lambda = (w(i) - i, \dots, w(2) - 2, w(1) - 1)$. All Grassmannian permutations with a given shape $\lambda$ have the same symmetric Grothendieck function, which we denote by $G_\lambda(\mathbf{z})$. The subset of symmetric Grothendieck functions obtained in this way is significant for the following reasons.

Buch [5] showed that $G_\lambda(\mathbf{z})$ has a tableau generating function formula. Specifically, $G_\lambda(\mathbf{z})$ is the weight generating function for all *semistandard set-valued tableaux* of shape $\lambda$. This formula implies that the $G_\lambda(\mathbf{z})$'s are linearly independent over $\mathbb{Z}[\beta]$. Lascoux [11] proved that for each $w \in S_\infty$, there is a finite list of (not necessarily distinct) Grassmannian permutations $u_1, u_2, \dots, u_k \in S_\infty$ such that $G_w(\mathbf{z}) = \sum_{i=1}^k \beta^{\ell(u_i)-\ell(w)} G_{u_i}(\mathbf{z})$. It follows that the set $\{G_\lambda(\mathbf{z}) : \lambda \text{ any partition}\}$ is a $\mathbb{Z}[\beta]$-basis for the bialgebra **G** whose structure constants for both the product and coproduct are all elements of $\mathbb{Z}_{\geq 0}[\beta]$. Buch's work [5] provides combinatorial formulas for these coefficients.

Thus, in type A, the set of symmetric Grothendieck functions indexed by Grassmannian permutations provides a basis for a bialgebra with certain strong positivity properties. Our goal now is to explain how one can extend these results to all classical types.

## 1.2 K-Stanley symmetric functions

A *signed permutation* is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ that has $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$ and $w(i) = i$ for all but finitely many $i \in \mathbb{Z}$. Define $W_\infty^{\mathsf{B}} = W_\infty^{\mathsf{C}}$ to be the group of all signed permutations and let $W_\infty^{\mathsf{D}}$ be the subgroup of signed permutations $w$ with an even number of sign changes, that is, for which the set $\{i \in \mathbb{Z}_{>0} : w(i) < 0\}$ has even size.

Motivated by the problem of constructing equivariant $K$-theory representatives for Schubert varieties in flag varieties of all classical types, Kirillov and Naruse have introduced analogues of $G_w(\mathbf{z})$ indexed by each of the classical Weyl groups $W_\infty^{\mathsf{B}}$, $W_\infty^{\mathsf{C}}$, and $W_\infty^{\mathsf{D}}$. These power series, which we write as $G_w^{\mathsf{B}}(\mathbf{z})$, $G_w^{\mathsf{C}}(\mathbf{z})$, and $G_w^{\mathsf{D}}(\mathbf{z})$, are again elements of $\mathbf{Sym}_\beta$ and will be referred to as *K-Stanley symmetric functions*. This terminology is motivated by the fact that setting $\beta = 0$ reduces $G^X_w(\mathbf{z})$ to the *Stanley symmetric functions* for classical types introduced in [3]. For the precise definitions, see Section 3.

Fix a type $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$. Then we may consider the submodule of symmetric functions $\mathbf{G}^X := \mathbb{Z}[\beta]\text{-span}\{G^X_w(\mathbf{z}) : w \in W^X_\infty\}$. One consequence of our results (specifically, Theorem 1.3 and Corollary 1.6) is the following analogue of Proposition 1.1 for $\mathbf{G}^X$.

**Theorem 1.2.** $\mathbf{G}^\mathsf{B}$, $\mathbf{G}^\mathsf{C}$, and $\mathbf{G}^\mathsf{D}$ are sub-bialgebras of $\mathbf{G}$ with $\mathbf{G}^\mathsf{C} \subset \mathbf{G}^\mathsf{B} = \mathbf{G}^\mathsf{D}$.

These bialgebras have a more precise characterization. Let $\mathbf{SSym}_\beta$ be the subring of power series $f(\mathbf{z}) \in \mathbf{Sym}_\beta$ that have the following *K-supersymmetry property* from [9]:

$$f(t, \tfrac{-t}{1+\beta t}, z_3, z_4, z_5, \dots) = f(0, 0, z_3, z_4, z_5, \dots). \tag{1.1}$$

Ikeda and Naruse [9] have identified two distinguished pseudobases for $\mathbf{SSym}_\beta$, given by the *K-theoretic Schur P- and Q-functions* $GP_\lambda(\mathbf{z})$ and $GQ_\lambda(\mathbf{z})$. These are the generating functions for certain *semistandard set-valued shifted tableaux* of strict partition shape $\lambda$; see Section 4 for a detailed review. The sets of finite linear combinations

$$\begin{aligned} \mathbf{GP} &:= \mathbb{Z}[\beta]\text{-span}\,\{GP_\lambda(\mathbf{z}) : \lambda \text{ any strict partition}\} \quad \text{and} \\ \mathbf{GQ} &:= \mathbb{Z}[\beta]\text{-span}\,\{GQ_\lambda(\mathbf{z}) : \lambda \text{ any strict partition}\} \end{aligned} \tag{1.2}$$

are both proper submodules of $\mathbf{SSym}_\beta$.

The modules $\mathbf{GP}$ and $\mathbf{GQ}$ have some remarkable properties. It is known that both are closed under multiplication and thus form rings [14], with strict containment $\mathbf{GQ} \subsetneq \mathbf{GP} \subsetneq \mathbf{G}$ [16, §3]. Our results here will show that these subrings are sub-bialgebras.

A signed permutation $w$ is *Grassmannian* if $w(i) < w(i+1)$ for all $i \in \mathbb{Z}_{>0}$. If $w$ has this property and $n \in \mathbb{Z}_{\geq 0}$ is maximal with $w(n) \leq 0$ then we can define two strict partitions by $\lambda_\mathsf{B}(w) = \lambda_\mathsf{C}(w) := (-w(1), \dots, -w(n))$ and $\lambda_\mathsf{D}(w) := \lambda_\mathsf{B}(w) - 1^n$. If $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $w \in W^X_\infty$ is Grassmannian with $\lambda = \lambda_X(w)$, then by [10, §4.2] we have

$$G^X_w(\mathbf{z}) = \begin{cases} GP_\lambda(\mathbf{z}) & \text{when } X \in \{\mathsf{B}, \mathsf{D}\} \\ GQ_\lambda(\mathbf{z}) & \text{when } X = \mathsf{C}. \end{cases} \tag{1.3}$$

The following theorem confirms predictions of Kirillov–Naruse [10, Rem. 3] and Arroyo–Hamaker–Hawkes–Pan [1, Conj. 7.2]. Here, we write $\ell^X$ for the Coxeter length function $W^X_\infty \to \mathbb{Z}_{\geq 0}$ (see §2 for an explicit formula).

**Theorem 1.3.** Fix a type $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $w \in W^X_\infty$. Then there is a finite list of (not necessarily distinct) Grassmannian elements $u_1, u_2, \dots, u_k \in W^X_\infty$ such that

$$G^X_w(\mathbf{z}) = \textstyle\sum_{i=1}^k \beta^{\ell^X(u_i) - \ell^X(w)} G^X_{u_i}(\mathbf{z})$$

Consequently, it holds that $\mathbf{G}^\mathsf{B} = \mathbf{G}^\mathsf{D} = \mathbf{GP}$ and $\mathbf{G}^\mathsf{C} = \mathbf{GQ}$.

Our methods provide an explicit algorithm for determining the $GP$- and $GQ$-expansions of $G^X_w(\mathbf{z})$ (see Theorem 6.2), but it is an open problem to find a positive combinatorial formula. The recent article [1] provides a conjectural formula of this kind for type $\mathsf{C}$.

More of Theorem 1.2 follows as a corollary to Theorem 1.3 and known properties of $GP$ and $GQ$. It only remains to explain why each subalgebra $\mathbf{G}^X \subset \mathbf{G}$ is also a sub-coalgebra. We establish this property in the following way.

The shifted tableau generating functions for $GP_\lambda(\mathbf{z})$ and $GQ_\lambda(\mathbf{z})$ reviewed in Section 4 have natural skew analogues $GP_{\lambda/\mu}(\mathbf{z})$ and $GQ_{\lambda/\mu}(\mathbf{z})$ indexed by pairs of strict partitions with $\mu \subset \lambda$ in the sense that $\mu_i \leq \lambda_i$ for all $i$. These functions naturally arise when considering the coproduct of $\mathbf{G}$, as we have [4, Prop. 5.6]

$$\begin{aligned}\Delta(GP_\lambda(\mathbf{z})) &\in \mathbb{Z}_{\geq 0}[\beta]\text{-span}\{GP_{\lambda/\kappa}(\mathbf{z}) \otimes GP_\mu(\mathbf{z}) : \kappa \subseteq \mu \subseteq \lambda\} \quad \text{and} \\ \Delta(GQ_\lambda(\mathbf{z})) &\in \mathbb{Z}_{\geq 0}[\beta]\text{-span}\{GQ_{\lambda/\kappa}(\mathbf{z}) \otimes GQ_\mu(\mathbf{z}) : \kappa \subseteq \mu \subseteq \lambda\}.\end{aligned} \tag{1.4}$$

The following result extends [1, Thm. 1.5] from type $\mathsf{C}$ to the remaining classical types $\mathsf{B}$ and $\mathsf{D}$. This statement implies that $\mathbf{G}^X \subset \mathbf{G}$ is a sub-coalgebra.

**Theorem 1.4.** For any strict partitions $\mu \subseteq \lambda$ there are signed permutations $w_{\mathsf{B}}(\lambda/\mu) = w_{\mathsf{C}}(\lambda/\mu) \in W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty$ and $w_{\mathsf{D}}(\lambda/\mu) \in W^{\mathsf{D}}_\infty$ (which are explicitly described in Section 3) such that $G^{\mathsf{B}}_{w_{\mathsf{B}}(\lambda/\mu)}(\mathbf{z}) = G^{\mathsf{D}}_{w_{\mathsf{D}}(\lambda/\mu)}(\mathbf{z}) = GP_{\lambda/\mu}(\mathbf{z})$ and $G^{\mathsf{C}}_{w_{\mathsf{C}}(\lambda/\mu)}(\mathbf{z}) = GQ_{\lambda/\mu}(\mathbf{z})$.

We highlight some other consequences of these results. Combining the preceding theorems with (1.3) gives the following corollary, which was predicted as [4, Conj. 5.14]:

**Corollary 1.5.** There are numbers $f^\nu_{\lambda\mu}, g^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ indexed by strict partitions with

$$GP_{\lambda/\mu} = \sum_\nu f^\nu_{\lambda\mu} \cdot \beta^{|\nu|+|\mu|-|\lambda|} \cdot GP_\nu \in \mathbf{GP} \text{ and } GQ_{\lambda/\mu} = \sum_\nu g^\nu_{\lambda\mu} \cdot \beta^{|\nu|+|\mu|-|\lambda|} \cdot GQ_\nu \in \mathbf{GQ}.$$

When $\mu \subseteq \lambda$ are fixed, $f^\nu_{\lambda\mu}$ and $g^\nu_{\lambda\mu}$ are nonzero for only finitely many choices of $\nu$.

A conjectural combinatorial formula for the coefficients $g^\nu_{\lambda\mu}$ is given in [1, Conj. 6.1]. We can summarize this discussion with the following stronger statement:

**Corollary 1.6.** The modules $\mathbf{GP}$ and $\mathbf{GQ}$ are sub-bialgebras of $\mathbf{G}$, for which the respective sets $\{GP_\lambda : \lambda \text{ strict}\}$ and $\{GQ_\lambda : \lambda \text{ strict}\}$ are positive bases in the sense that

$$\begin{aligned}GP_\lambda GP_\mu &= \sum_\nu A^\nu_{\lambda\mu} \cdot \beta^{|\nu|-|\lambda|-|\mu|} \cdot GP_\nu, \quad \Delta(GP_\nu) = \sum_\lambda \sum_\mu a^\nu_{\lambda\mu} \cdot \beta^{|\lambda|+|\mu|-|\nu|} \cdot GP_\lambda \otimes GP_\mu, \\ GQ_\lambda GQ_\mu &= \sum_\nu B^\nu_{\lambda\mu} \cdot \beta^{|\nu|-|\lambda|-|\mu|} \cdot GQ_\nu, \quad \Delta(GQ_\nu) = \sum_\lambda \sum_\mu b^\nu_{\lambda\mu} \cdot \beta^{|\lambda|+|\mu|-|\nu|} \cdot GQ_\lambda \otimes GQ_\mu.\end{aligned}$$

for nonnegative integers $A^\nu_{\lambda\mu}, B^\nu_{\lambda\mu}, a^\nu_{\lambda\mu}, b^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ indexed by triples of strict partitions.

*Proof.* The positivity of the multiplicative structure constants $A^{\nu}_{\lambda\mu}$ and $B^{\nu}_{\lambda\mu}$ is established in [14, Thm. 1.6]. The claim that $\mathbf{GQ} \subsetneq \mathbf{GP} \subsetneq \mathbf{G}$ is shown in [16, §3]. The positivity of the comultiplicative structure constants is new, and follows from (1.4) and Corollary 1.5. □

Combinatorial formulas are known for $A^{\nu}_{\lambda\mu}$ from [6, 17], but it is an open problem to find similar expressions for the other coefficients in Corollary 1.6; see the conjectures in [1, §6]. The nonnegativity of $a^{\nu}_{\lambda\mu}$ and $b^{\nu}_{\lambda\mu}$ confirms [15, Conjs. 4.35 and 4.36].

### 1.3 Double Grothendieck transitions

Kirillov and Naruse's definition of the $K$-Stanley symmetric functions $G^X_w(\mathbf{z})$ is a preliminary step in their construction of certain *double Grothendieck polynomials* $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ for each classical type $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$. The objects, which belong to the ring

$$\mathbf{SSym}_{\beta}[\mathbf{x},\mathbf{y}] := \mathbf{SSym}_{\beta}[x_1,x_2,\dots,y_1,y_2,\dots],$$

generalize the *(double) Schubert polynomials* for classical groups introduced in [3, 8]. We review the definition of $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ in Section 5. One always has $\mathfrak{G}^X_w(0;0;\mathbf{z}) = G^X_w(\mathbf{z})$.

The results in [10, §6] show that certain specializations of $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ represent the (opposite) Schubert classes in the equivariant $K$-theory ring of the complete flag variety $G/B$ in type $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$. These representatives are distinguished by several desirable stability properties. In particular, the structure constants that expand the product of $\mathfrak{G}^X_v(\mathbf{x};\mathbf{y};\mathbf{z})$ and $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ are the same as the ones for the Schubert basis of the equivariant $K$-theory ring $K_T(G/B)$.

These rings have been extensively studied in the literature. Most notably, Lenart and Postnikov [13] have proved an equivariant Chevalley formula giving a multiplication rule in $K_T(G/B)$ for all types. By translating this rule to a polynomial identity, we are able to prove a *transition formula* (Theorem 6.1) that gives a recurrence relation for the power series $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$. Setting $\mathbf{x} = \mathbf{y} = 0$ in this result provides an $\mathbb{Z}_{\geq 0}[\beta]$-linear recurrence for $G^X_w(\mathbf{z})$ which can be used to show the main results in the previous section. This approach is similar to the methods used for the cohomological case in [2, 8].

Complete proofs and further connections to equivariant $K$-theory will be discussed in the full length version of this extended abstract. Here, we just focus on applications to power series identities involving $\mathfrak{G}^X(\mathbf{x};\mathbf{y};\mathbf{z})$ and $G^X(\mathbf{z})$.

## 2 Classical Weyl groups

The functions of interest in this work are indexed by the elements of the following Coxeter groups. A bijection $w : \mathbb{Z} \to \mathbb{Z}$ is a *signed permutation* if $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$ and $w(i) = i$ for all but finitely many $i \in \mathbb{Z}$. Let $W^{\mathsf{B}}_{\infty} = W^{\mathsf{C}}_{\infty}$ be the group of all signed permutations, and let $W^{\mathsf{D}}_\infty$ be the subgroup of signed permutations $w$ such that $|\{i > 0 : w(i) < 0\}|$ is even. These are Coxeter groups with simple generating sets

$$W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty = \langle t_0, t_1, t_2, t_3, \dots \rangle \quad \text{and} \quad W^{\mathsf{D}}_\infty = \langle t_{-1}, t_1, t_2, t_3, \dots \rangle$$

where $t_{-1} = (1,-2)(-1,2)$, $t_0 = (-1,1)$, and $t_i = (i, i+1)(-i-1, -i)$ for $i > 0$. We implicitly identify $S_\infty$ with the subgroup $W^{\mathsf{A}}_\infty := \langle t_1, t_2, t_3, \dots \rangle \subset W^{\mathsf{D}}_\infty \subset W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty$.

The Coxeter length functions of $S_\infty \cong W^{\mathsf{A}}_\infty$, $W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty$, and $W^{\mathsf{D}}_\infty$ are given by [2, §3]

$$\ell^{\mathsf{A}}(w) := \mathsf{inv}(w), \quad \ell^{\mathsf{B}}(w) = \ell^{\mathsf{C}}(w) := \tfrac{\mathsf{inv}(w) + \ell_0(w)}{2}, \quad \text{and} \quad \ell^{\mathsf{D}}(w) := \tfrac{\mathsf{inv}(w) - \ell_0(w)}{2} \tag{2.1}$$

where $\mathsf{inv}(w) = |\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i < j,\ w(i) > w(j)\}|$ and $\ell_0(w) = |\{i > 0 : w(i) < 0\}|$.

The *reflections* in a Coxeter group are the elements conjugate to the simple generators. In $S_\infty$ these elements are the transpositions $(i,j)$ for $1 \leq i < j$. Define

$$t_{ij} := (i,j)(-j,-i) \in W^{\mathsf{D}}_\infty \quad \text{and} \quad t_{0k} := (-k,k) \in W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty \tag{2.2}$$

for all $i, j, k \in \mathbb{Z}$ with $k \neq 0$. Note that $t_{jj} = t_{-j,j} = 1$ and $t_{ij} = t_{ji} = t_{-i,-j} = t_{-j,-i}$. The distinct reflections in $W^{\mathsf{B}}_\infty = W^{\mathsf{C}}_\infty$ consist of the elements $t_{ij}$ with $|i| < j$. The reflections in $W^{\mathsf{D}}_\infty$ are given by the same set but with $t_{0j}$ excluded for all $j \in \mathbb{Z}_{>0}$.

# 3 K-Stanley symmetric functions

We now review Kirillov and Naruse's definition of *K-Stanley symmetric functions* for signed permutations.

The *Demazure product* on a Coxeter group $W$ with length function $\ell$ is the unique associative operation $\circ : W \times W \to W$ that satisfies $u \circ v = uv$ if $\ell(uv) = \ell(u) + \ell(v)$ and $s \circ s = s$ if $\ell(s) = 1$. Fix a type $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and an element $w \in W^X_\infty$. Let $\mathcal{H}^X(w)$ be the set of integer sequences $a = (a_1, a_2, \dots, a_k)$ with $a_i \in \{-1, 0, 1, 2, \dots\}$ such that

$$w = t_{a_1} \circ t_{a_2} \circ \cdots \circ t_{a_k}$$

when the Demazure product $\circ$ is computed relative to $W = W^X_\infty$. All entries of a word $a \in \mathcal{H}^X(w)$ are in the set $\{0, 1, 2, 3, \dots\}$ when $X \in \{\mathsf{B}, \mathsf{C}\}$, and in $\{-1, 1, 2, 3, \dots\}$ when $X = \mathsf{D}$. Note that $\mathcal{H}^{\mathsf{B}}(w) = \mathcal{H}^{\mathsf{C}}(w)$. We refer to elements of $\mathcal{H}^X(w)$ as *Hecke words* for $w$.

**Definition 3.1.** Suppose $a = (a_1, a_2, \dots, a_k)$ is any integer sequence. Define $\ell(a) = k$ and

$$o^{\mathsf{B}}(a) = |\{i \in [k] : a_i = 0\}|, \quad o^{\mathsf{C}}(a) = 0, \quad \text{and} \quad o^{\mathsf{D}}(a) = |\{i \in [k] : a_i = \pm 1\}|.$$

Let $\mathcal{C}^X(a)$ be the set of positive integer sequences $b = (b_1 \leq b_2 \leq \cdots \leq b_k)$ such that:

(a) if $|a_{i-1}| \leq |a_i| \geq |a_{i+1}|$ for some $i \in \{2, 3, \dots, k-1\}$ then $b_{i-1} < b_{i+1}$; and

(b) if $X = \mathsf{B}$ (respectively, $\mathsf{D}$) and $a_i = a_{i+1} = 0$ (respectively, $\pm 1$) then $b_i < b_{i+1}$.

Given $b \in \mathcal{C}^X(a)$, let $\mathbf{z}_b = z_{b_1} z_{b_2} \cdots z_{b_k}$ and $|b| = |\{b_1, b_2, \dots, b_k\}|$, and define

$$\gamma(a,b) = |\{i \in [k-1] : a_i = a_{i+1} \text{ and } b_i = b_{i+1}\}|. \tag{3.1}$$

The following is not the first definition of $K$-Stanley symmetric functions given in [10], but we find it to be more self-contained and explicit than [10, Def. 7]. Our statement corrects a minor error in the type $\mathsf{B}$ formula in [10, Prop. 17].

**Definition 3.2** ([10, Prop. 17])**.** Fix $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$. Write $\ell = \ell^X$ and suppose $w \in W^X_\infty$. The *K-Stanley symmetric function* of $w$ for type $X$ is the formal power series

$$G^X_w(\mathbf{z}) = \sum_{a \in \mathcal{H}^X(w)} \sum_{b \in \mathcal{C}^X(a)} 2^{|b| - \gamma(a,b) - o^X(a)} \cdot \beta^{\ell(a) - \ell(w)} \cdot \mathbf{z}_b \in \mathbb{Z}[\beta][\![\mathbf{z}]\!].$$

# 4 Supersymmetric functions

There are more explicit tableau formulas that give $G^X_w(\mathbf{z})$ for certain classes of signed permutations. Recall from Section 1.2 that $\mathbf{SSym}_\beta$ is the subring of power series $f(\mathbf{z}) \in \mathbf{Sym}_\beta$ with the extra symmetry property (1.1). While not evident from Definition 3.2, it turns out [10, Lem. 6] that $G^X_w(\mathbf{z}) \in \mathbf{SSym}_\beta$ for all $X \in \{\mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $w \in W^X_\infty$. Following [9, 10], we refer to elements of $\mathbf{SSym}_\beta$ as *K-supersymmetric functions*.

The ring $\mathbf{SSym}_\beta$ was first considered by Ikeda and Naruse in [9] since it consists of exactly the formal power series that can be expressed as infinite $\mathbb{Z}[\beta]$-linear combinations of their *K-theoretic Schur P- and Q-functions*, whose definition we now recall.

A *partition* is a decreasing sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ with finite sum $|\lambda| = \sum_i \lambda_i$. A partition is *strict* if $\lambda_i = \lambda_{i+1}$ implies that $\lambda_i = \lambda_{i+1} = 0$. The *shifted diagram* of a strict partition $\lambda$ is the set $\mathsf{SD}_\lambda = \{(i, i+j-1) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : 1 \leq j \leq \lambda_i\}$ which we treat as positions in a matrix. We write $\mu \subseteq \lambda$ if $\mathsf{SD}_\mu \subseteq \mathsf{SD}_\lambda$ and define $\mathsf{SD}_{\lambda/\mu} = \mathsf{SD}_\lambda \setminus \mathsf{SD}_\mu$ and $|\lambda/\mu| = |\lambda| - |\mu|$.

A *set-valued shifted tableau* of shape $\lambda/\mu$ is a map $T$ from $\mathsf{SD}_{\lambda/\mu}$ to the set of finite, nonempty subsets of the totally ordered alphabet $\{1' < 1 < 2' < 2 < \dots\}$. If $T$ is such a tableau, then we write $(i,j) \in T$ to indicate that $(i,j)$ is in its domain and let $T_{ij}$ denote the set assigned by $T$ to this position. We also define $|T| = \sum_{(i,j) \in T} |T_{ij}|$ and

$$\mathbf{z}^T = \prod_{k=1}^{\infty} z_k^{a_k + b_k} \quad \text{where} \begin{cases} a_k \text{ is the number of pairs } (i,j) \in T \text{ with } k \in T_{ij}, \\ b_k \text{ is the number of pairs } (i,j) \in T \text{ with } k' \in T_{ij}. \end{cases} \tag{4.1}$$

A set-valued shifted tableau is *semistandard* if

(a) $\max(T_{ij}) \leq \min(T_{i,j+1})$ and $T_{ij} \cap T_{i,j+1} \subset \{1, 2, 3, \dots\}$ when $(i,j), (i,j+1) \in T$; and

(b) $\max(T_{ij}) \leq \min(T_{i+1,j})$ and $T_{ij} \cap T_{i+1,j} \subset \{1', 2', 3', \dots\}$ when $(i,j), (i+1,j) \in T$.

Write $\mathsf{SetShTab}^+(\lambda/\mu)$ for the set of semistandard set-valued shifted tableaux of shape $\lambda/\mu$. Let $\mathsf{SetShTab}(\lambda/\mu)$ be the subset of $T \in \mathsf{SetShTab}^+(\lambda/\mu)$ with no primed numbers in any positions on the main diagonal. We abbreviate by writing $\mathsf{SetShTab}(\lambda) = \mathsf{SetShTab}(\lambda/\varnothing)$ and $\mathsf{SetShTab}^+(\lambda) = \mathsf{SetShTab}^+(\lambda/\varnothing)$.

**Example 4.1.** If $\lambda = (5,3,2)$ and $\mu = (2)$ then one element $T \in \mathsf{SetShTab}(\lambda/\mu)$ is

$$T = \begin{array}{|c|c|c|c|c|} \hline \blacksquare & \blacksquare & 2'3' & 4 & 457 \\ \hline \multicolumn{1}{c|}{} & 12 & 3'3 & 5' & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & 5'57 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} \qquad \text{for which } T_{23} = \{3', 3\},\ |T| = 15, \text{ and } \mathbf{z}^T = z_1 z_2^2 z_3^3 z_4^3 z_5^4 z_7^2.$$

When $\mu \not\subset \lambda$ we let $\mathsf{SetShTab}(\lambda/\mu) = \mathsf{SetShTab}^+(\lambda/\mu) = \varnothing$ be the empty set.

**Definition 4.2** (See [9, 4]). If $\mu$ and $\lambda$ are strict partitions, then the *K-theoretic Schur P- and Q-functions* indexed by $\lambda/\mu$ are the formal power series in $\mathbb{Z}_{\geq 0}[\beta][\![\mathbf{z}]\!]$ given by

$$GP_{\lambda/\mu}(\mathbf{z}) = \sum_{T \in \mathsf{SetShTab}(\lambda/\mu)} \beta^{|T| - |\lambda/\mu|} \mathbf{z}^T \quad \text{and} \quad GQ_{\lambda/\mu}(\mathbf{z}) = \sum_{T \in \mathsf{SetShTab}^+(\lambda/\mu)} \beta^{|T| - |\lambda/\mu|} \mathbf{z}^T.$$

Let $GP_\lambda(\mathbf{z}) = GP_{\lambda/\varnothing}(\mathbf{z})$ and $GQ_\lambda(\mathbf{z}) = GQ_{\lambda/\varnothing}(\mathbf{z})$.

Ikeda and Naruse [9] showed that $GP_\lambda(\mathbf{z})$ and $GQ_\lambda(\mathbf{z})$ are elements of $\mathbf{SSym}_\beta$ for all strict partitions $\lambda$. The same holds for $GP_{\lambda/\mu}(\mathbf{z})$ and $GQ_{\lambda/\mu}(\mathbf{z})$ by [4, Cor. 5.13].

The formula (1.3) implies that every $GP_\lambda(\mathbf{z})$ occurs as a $K$-Stanley symmetric function of type $\mathsf{B}$ or $\mathsf{D}$, and every $GQ_\lambda(\mathbf{z})$ occurs as a $K$-Stanley symmetric function of type $\mathsf{C}$. As foretold in Theorem 1.4, this is also true for $GP$- and $GQ$-functions indexed by skew shapes, if we use the following elements to index our $K$-Stanley symmetric functions.

**Definition 4.3.** Fix strict partitions $\mu \subseteq \lambda$. Let $T^{\mathsf{B}}_{\lambda/\mu}$ be the shifted tableau of shape $\lambda/\mu$ with $j - i$ in box $(i,j) \in \mathsf{SD}_{\lambda/\mu}$. Then let $T^{\mathsf{D}}_{\lambda/\mu}$ be the shifted tableau of shape $\lambda/\mu$ with $j - i + 1$ in box $(i,j) \in \mathsf{SD}_{\lambda/\mu}$ if $i \neq j$ and with $(-1)^i$ in this box if $i = j$. Finally, let

$$w_{\mathsf{B}}(\lambda/\mu) = w_{\mathsf{C}}(\lambda/\mu) = t_{a_1} t_{a_2} \cdots t_{a_k} \quad \text{and} \quad w_{\mathsf{D}}(\lambda/\mu) = t_{b_1} t_{b_2} \cdots t_{b_k}$$

where $k = |\lambda/\mu|$ and where $a_1 a_2 \cdots a_k$ and $b_1 b_2 \cdots b_k$ are the entries of $T^{\mathsf{B}}_{\lambda/\mu}$ and $T^{\mathsf{D}}_{\lambda/\mu}$ read row-by-row from left to right, starting with the smallest row index.

Theorem 1.4 holds for this definition of $w_{\mathsf{B}}(\lambda/\mu)$, $w_{\mathsf{C}}(\lambda/\mu)$, and $w_{\mathsf{D}}(\lambda/\mu)$.

**Example 4.4.** When $\lambda = (5,3,1)$ and $\mu = (2)$ our two shifted tableaux are

$$T^{\mathsf{B}}_{(5,3,1)/(2)} = \begin{array}{ccccc} \, & \, & 2 & 3 & 4 \\ & 0 & 1 & 2 & \\ & & 0 & & \end{array} \quad \text{and} \quad T^{\mathsf{D}}_{(5,3,1)/(2)} = \begin{array}{ccccc} \, & \, & 3 & 4 & 5 \\ & -1 & 2 & 3 & \\ & & +1 & & \end{array}$$

so we have $w_{\mathsf{B}}(\lambda/\mu) = t_2t_3t_4t_0t_1t_2t_0 = \overline{3}4\overline{1}52$ and $w_{\mathsf{D}}(\lambda/\mu) = t_3t_4t_5t_{-1}t_2t_3t_1 = 4\overline{2}5\overline{1}63$. Here, the one-line notation "$\overline{3}4\overline{1}52$" means the signed permutation that sends $1 \mapsto -3$, $2 \mapsto 4$, $3 \mapsto -1$, $4 \mapsto 5$, and $5 \mapsto 2$, while fixing all integers $i > 5$.

Tamvakis derives another family of tableau formulas for certain instances of $G^X_w(\mathbf{z})$ in [19]. However, Theorem 1.4 does not seem to be a special case of these results.

# 5 Double Grothendieck polynomials

This section reviews Kirillov and Naruse's construction in [10] of the *double Grothendieck polynomials* of classical type $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ mentioned in Section 1.3. To begin, we must first recall that definition of the (single-variable) Grothendieck polynomials in type A.

The group $S_\infty$ acts on $\mathbb{Z}[\beta][\mathbf{x}] = \mathbb{Z}[\beta][x_1, x_2, \dots]$ by permuting the $x$-variables, and

$$\pi_i f(\mathbf{x}) := \tfrac{(1+\beta x_{i+1}) f(\mathbf{x}) - (1+\beta x_i) s_i f(\mathbf{x})}{x_i - x_{i+1}} \tag{5.1}$$

defines a $\mathbb{Z}[\beta]$-linear map $\mathbb{Z}[\beta][\mathbf{x}] \to \mathbb{Z}[\beta][\mathbf{x}]$ for each $i \in \mathbb{Z}_{>0}$. The operators $\pi_i$ satisfy the Coxeter braid relations for $S_\infty$ along with the identity $\pi_i\pi_i = -\beta\pi_i$.

**Definition 5.1** (See [7, 12]). The *(type* A*) Grothendieck polynomials* $\mathfrak{G}_w(\mathbf{x})$ for $w \in S_\infty$ are the unique elements of $\mathbb{Z}[\beta][\mathbf{x}]$ satisfying

$$\mathfrak{G}_{n\cdots 321}(\mathbf{x}) = \textstyle\prod_{i=1}^n x_i^{n-i} \text{ for all } n \in \mathbb{Z}_{>0} \quad \text{and} \quad \mathfrak{G}_{ws_i}(\mathbf{x}) = \pi_i \mathfrak{G}_w(\mathbf{x}) \text{ if } w(i) > w(i+1).$$

We may now introduce $\mathfrak{G}^X(\mathbf{x};\mathbf{y};\mathbf{z})$ by a formula which involves Grothendieck polynomials for type A, $K$-Stanley symmetric functions, and the Demazure product.

**Definition 5.2** ([10, Prop. 2]). Fix $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$ and let $\ell = \ell^X$. Then, for $w \in W^X_\infty$ define

$$\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z}) = \sum_{(\sigma,u,\tau)} \beta^{\ell(\sigma)+\ell(u)+\ell(\tau)-\ell(w)} \cdot \mathfrak{G}_\sigma(\mathbf{y}) \cdot G^X_u(\mathbf{z}) \cdot \mathfrak{G}_\tau(\mathbf{x}) \in \mathbf{SSym}_\beta[\mathbf{x};\mathbf{y}]$$

where the sum is over all triples $(\sigma, u, \tau) \in W^{\mathsf{A}}_\infty \times W^X_\infty \times W^{\mathsf{A}}_\infty$ such that $\sigma^{-1} \circ u \circ \tau = w$ (computing the Demazure product $\circ$ relative to the Coxeter group structure on $W^X_\infty$).

Setting $\beta = 0$ transforms $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ to the *double Schubert polynomials* for classical groups introduced in [8]; further setting $\mathbf{y} = 0$ recovers the *Schubert polynomials* constructed by Billey and Haiman in [3]. Notice that we always have $\mathfrak{G}^X_w(0;0;\mathbf{z}) = G^X_w(\mathbf{z})$.

**Example 5.3.** If $w = \overline{2}1 = t_1t_0$ then the triples $(\sigma, u, \tau)$ indexing the sum that defines $\mathfrak{G}^{\mathsf{B}}_w(\mathbf{x};\mathbf{y};\mathbf{z})$ are $(1, t_1t_0, 1)$, $(t_1, t_0, 1)$, and $(t_1, t_1t_0, 1)$. As $\mathfrak{G}_1(\mathbf{x}) = 1$ and $\mathfrak{G}_{t_1}(\mathbf{x}) = \mathfrak{G}_{21}(\mathbf{x}) = x_1$ and as $t_0$ and $t_1t_0$ are Grassmannian, $\mathfrak{G}^{\mathsf{B}}_{\overline{2}1}(\mathbf{x};\mathbf{y};\mathbf{z}) = y_1GP_{(1)}(\mathbf{z}) + (1+\beta y_1)GP_{(2)}(\mathbf{z})$.

As mentioned in the introduction, Kirillov and Naruse show that $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z})$ represents the Schubert classes in the equivariant $K$-theory ring of $G/B$ in types B, C, and D [10, §6]. This connection is instrumental in our proofs of the results in the next section.

# 6 Transition equations and applications

Fix $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$. For any parameter $a$ define $\ominus a = \frac{-a}{1+\beta a}$ and $y_{-i} = \ominus y_i$ for $i > 0$. Let

$$\mathbb{A}_\beta := \mathbb{Z}[\beta]\left[y_1, \tfrac{1}{1+\beta y_1}, y_2, \tfrac{1}{1+\beta y_2}, y_3, \tfrac{1}{1+\beta y_3}, \dots\right] \quad \text{and notice that} \quad \tfrac{1}{1+\beta y_{-i}} = 1+\beta y_i$$

We define certain linear operators on finite $\mathbb{A}_\beta$-linear combinations of $\mathfrak{G}^X$-functions. Fix integers $i < j$ and $k$ and $j, k \in \mathbb{Z}_{>0}$. Let $\mathfrak{t}^X_{ij}$ be the $\mathbb{A}_\beta$-linear operator sending

$$\mathfrak{t}^X_{ij} : \mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z}) \mapsto \begin{cases} \mathfrak{G}^X_{wt_{ij}}(\mathbf{x};\mathbf{y};\mathbf{z}) & \text{if } t_{ij} \in W^X_\infty \text{ and } \ell^X(w)+1 = \ell^X(wt_{ij}) \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

for each $w \in W^X_\infty$. Notice that we have $\mathfrak{t}^{\mathsf{D}}_{0j} = 0$ and $\mathfrak{t}^{\mathsf{B}}_{-j,j} = \mathfrak{t}^{\mathsf{C}}_{-j,j} = \mathfrak{t}^{\mathsf{D}}_{-j,j} = 0$. Next, define $\mathfrak{n}^{\mathsf{B}}_k$ to be the $\mathbb{A}_\beta$-linear map sending

$$\mathfrak{n}^{\mathsf{B}}_k : \mathfrak{G}^{\mathsf{B}}_w(\mathbf{x};\mathbf{y};\mathbf{z}) \mapsto \begin{cases} \frac{1}{1+\beta y_{w(k)}}\mathfrak{G}^{\mathsf{B}}_{w(-k,k)}(\mathbf{x};\mathbf{y};\mathbf{z}) & \text{if } \ell^{\mathsf{B}}(w)+1 = \ell^{\mathsf{B}}(w(-k,k)) \\ 0 & \text{otherwise,} \end{cases}$$

while setting $\mathfrak{n}^{\mathsf{C}}_k = \mathfrak{n}^{\mathsf{D}}_k = 0$. Finally, define $\mathfrak{R}^X_k = \prod^{[>]}_{k>j>-\infty}(1+\beta\mathfrak{t}^X_{jk}) \cdot (1+\beta\mathfrak{n}^X_k)$ where $\prod^{[>]}$ indicates a product taken in decreasing order. This infinite product is a well-defined linear operator on $\mathbb{A}_\beta$-linear combinations of double Grothendieck polynomials.

A *descent* of a signed permutation $w$ is an integer $i > 0$ with $w(i) > w(i+1)$. The following main theorem is a $K$-theoretic extension of [8, Prop. 6.12] and [2, Thm. 4].

**Theorem 6.1.** Fix $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$ and $w \in W^X_\infty$. Suppose $a \in [1,\infty)$ is maximal with $w(a) > w(a+1)$, define $b \in [a+1,\infty)$ to be maximal with $w(a) > w(b)$, and set $v = wt_{ab}$ and $c = v(a)$. Then $\mathfrak{G}^X_w(\mathbf{x};\mathbf{y};\mathbf{z}) = \beta^{-1}\Big((1+\beta y_c)(1+\beta x_a)\mathfrak{R}^X_a\mathfrak{G}^X_v(\mathbf{x};\mathbf{y};\mathbf{z}) - \mathfrak{G}^X_v(\mathbf{x};\mathbf{y};\mathbf{z})\Big)$.

Continue to fix integers $j < k$ with $k > 0$. Given $w \in W^X_\infty$ we now define

$$T^X_{jk}G^X_w(\mathbf{z}) = \begin{cases} G^X_{wt_{jk}}(\mathbf{z}) & \text{if } t_{ij} \in W^X_\infty \text{ and } \ell^X(w)+1 = \ell^X(wt_{jk}) \\ 0 & \text{otherwise} \end{cases}$$

and extend by $\mathbb{Z}[\beta]$-linearity. Unlike $\mathfrak{t}^X_{jk}$, our definition of $T^X_{jk}$ in this context only makes sense as a formal symbolic operator since the $K$-Stanley symmetric functions $G^X_w(\mathbf{z})$ are not linearly independent. We compose these operators to define

$$\begin{aligned} R^{\mathsf{B}}_k &= \prod^{[>]}_{k>j>-\infty}(1+\beta T^{\mathsf{B}}_{jk})\cdot(1+\beta T^{\mathsf{B}}_{0k}), \\ R^{\mathsf{C}}_k &= \prod^{[>]}_{k>j>-\infty}(1+\beta T^{\mathsf{C}}_{jk}), \end{aligned} \qquad \text{and} \qquad R^{\mathsf{D}}_k = \prod^{[>]}_{\substack{k>j>-\infty \\ j\neq 0}}(1+\beta T^{\mathsf{D}}_{ik}). \tag{6.2}$$

We interpret $R^X_k$ as a linear operator on the module $\mathbf{G}^X$ of finite $\mathbb{Z}[\beta]$-linear combinations of $K$-Stanley symmetric functions. Setting $\mathbf{x} = \mathbf{y} = 0$ in Theorem 6.1 gives this result:

**Theorem 6.2.** Fix $X \in \{\mathsf{B},\mathsf{C},\mathsf{D}\}$ and $w \in W^X_\infty$ with at least one descent. Suppose $a \in [1,\infty)$ is maximal with $w(a) > w(a+1)$, define $b \in [a+1,\infty)$ to be maximal with $w(a) > w(b)$. Then $G^X_w(\mathbf{z}) = \beta^{-1}(R^X_a - 1)G^X_{wt_{ab}}(\mathbf{z})$.

For the type A version of Theorems 6.1 and 6.2, see [20, Thm. 3.1] and [11, Thm. 4].

**Example 6.3.** Applying Theorem 6.2 to $w = \bar{3}4\bar{1}52 \in W^{\mathsf{B}}_\infty$ gives

$$G^{\mathsf{B}}_{\bar{3}4\bar{1}52}(\mathbf{z}) = G^{\mathsf{B}}_{\bar{3}42\bar{1}}(\mathbf{z}) + G^{\mathsf{B}}_{\bar{3}4\bar{2}1}(\mathbf{z}) + \beta\cdot G^{\mathsf{B}}_{\bar{3}4\bar{2}\bar{1}}(\mathbf{z}) + \beta\cdot G^{\mathsf{B}}_{\bar{3}41\bar{2}}(\mathbf{z}) + \beta^2\cdot G^{\mathsf{B}}_{\bar{3}4\bar{1}\bar{2}}(\mathbf{z})$$

We can recursively use Theorem 6.2 to expand each term on the right, until we reach a linear combination of double Grothendieck polynomials indexed by Grassmannian signed permutations. This iterative process results in the formula

$$\begin{aligned} G^{\mathsf{B}}_{\bar{3}4\bar{1}52}(\mathbf{z}) = 4\cdot G^{\mathsf{B}}_{\bar{4}\bar{2}\bar{1}3}(\mathbf{z}) + 2\cdot G^{\mathsf{B}}_{\bar{4}\bar{3}12}(\mathbf{z}) + 2\cdot G^{\mathsf{B}}_{\bar{5}\bar{2}134}(\mathbf{z}) \\ + 5\beta\cdot G^{\mathsf{B}}_{\bar{4}\bar{3}\bar{1}2}(\mathbf{z}) + 5\beta\cdot G^{\mathsf{B}}_{\bar{5}\bar{2}\bar{1}34}(\mathbf{z}) + 3\beta\cdot G^{\mathsf{B}}_{\bar{5}\bar{3}124}(\mathbf{z}) + 6\beta^2\cdot G^{\mathsf{B}}_{\bar{5}\bar{3}\bar{1}24}(\mathbf{z}). \end{aligned}$$

To prove Theorem 1.3, it suffices to show that the inductive process illustrated in the preceding example always terminates. One way to show this is by a careful analysis of the descents of the signed permutations indexing the terms in the expansion provided in Theorem 6.2. This is a more complicated version of the methods in [2].

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