

# Deligne–Lusztig varieties, toric orbifolds, and the $q$ -Klyachko algebra

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**Abstract.** We investigate the geometry of the  $q$ -Klyachko algebra in the sense of Nadeau–Tewari. When  $q$  is a prime power, we show that the  $q$ -Klyachko algebra is the image of the pullback map on Chow rings  $\mathrm{CH}(\mathrm{Fl}_{n+1}) \rightarrow \mathrm{CH}(\mathrm{DL}_n)$ , where  $\mathrm{DL}_n$  is a compactified Deligne–Lusztig variety inside the complete flag variety  $\mathrm{Fl}_{n+1}$ . When  $q$  is a positive rational number, we establish a Kähler package for the  $q$ -Klyachko algebra through inputs from toric geometry.

**Keywords:** matroids, Chow rings, Deligne–Lusztig varieties, Kähler packages

## 1 Introduction

Fix  $q \in \mathbf{Q}_{>0}$  and a natural number  $n \in \mathbf{Z}_{>0}$ . In [14], Nadeau and Tewari considered:

**Definition 1.1.** The  $q$ -Klyachko algebra  $\mathrm{Kly}_{n,q}$  is the finite  $\mathbf{Q}$ -algebra

$$\mathrm{Kly}_{n,q} := \mathbf{Q}[u_1, \dots, u_n] / \langle (q+1)u_i^2 = u_i u_{i+1} + q u_i u_{i-1}, \text{ for all } i = 1, \dots, n \rangle,$$

with the convention  $u_0 = u_{n+1} = 0$ . It is equipped with a degree map  $\mathrm{deg}_{n,q} : \mathrm{Kly}_{n,q}^n \rightarrow \mathbf{Q}$  by sending  $u_1 \cdots u_n$  to  $(n)_q! = \prod_{i=1}^n \frac{q^i - 1}{q - 1}$ .

The set of square-free monomials in  $u_i$  form a basis for  $\mathrm{Kly}_{n,q}$ . Thus the degree map is well-defined. The purpose of the present note is to study the geometry behind the  $q$ -Klyachko algebra.

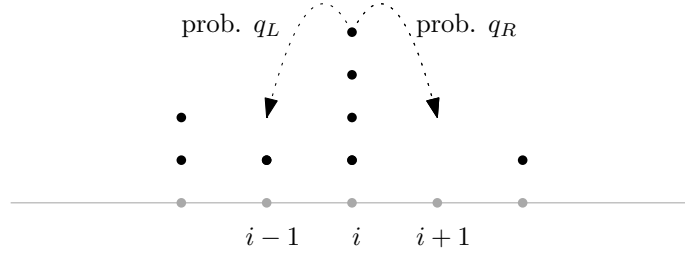
### 1.1 Motivation from probability theory

Consider a compactly supported point measure on the set of integers of the form  $\eta = \sum_{i \in \mathbf{Z}} c_i \delta_i$ , where  $c_i \in \mathbf{Z}_{\geq 0}$  vanishes for all but finitely many  $i$ , and  $\delta_i$  is the Dirac measure at  $i$ . Fix  $q_L, q_R \in \mathbf{Q}_{>0}$  with  $q_L + q_R = 1$ . We consider the following *random displacement* rule. Whenever  $\eta(i) \geq 2$  for some  $i$ , we set

$$\eta \mapsto \eta - \delta_i + \delta_{i-1} \text{ with probability } q_L, \text{ or } \eta - \delta_i + \delta_{i+1} \text{ with probability } q_R.$$

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**Figure 1:** An application of the random displacement rule at  $i$ .

That is, we move 1 unit of the mass at  $i$  leftward with probability  $q_L$  and rightward with probability  $q_R$ . See Figure 1 for an illustration. A point measure with  $\eta(i) \in \{0, 1\}$  for all  $i \in \mathbf{Z}$  is invariant under this displacement rule.

Let  $I \subset \mathbf{Z}$  be a finite subset. Denote by  $p(I; \eta)$  the probability that the point measure  $\eta$  transforms to the point measure  $\sum_{i \in I} \delta_i$  after running the random displacement process above for sufficiently many steps. Here, at every step, we select one  $i$  with  $\eta(i) \geq 2$  according to some rule.

In [14, §5], Nadeau and Tewari showed that the structure constants for the square monomial basis of  $\text{Kly}_{n,q}$ , known as *remixed Eulerian numbers*, compute the probability  $p(I; \eta)$ . More precisely, setting  $q = q_L/q_R$ , if  $\eta$  is supported on  $[n]$  and is such that  $\sum_{1 \leq i \leq n} \eta(i) = n$ , then we have

$$p([n]; \eta) = \frac{1}{(n)_q!} \deg_{n,q} \prod_i u_i^{\eta(i)}.$$

Moreover, this holds for any rule of selecting  $i$  to update.

## 1.2 Statements of the main results

The first geometric object related to  $\text{Kly}_{n,q}$  is a Deligne–Lusztig variety. Fix a finite field  $k = \mathbf{F}_q$  of order  $q$ , in addition to an algebraic closure  $\bar{k}$ . Denote by  $\text{Fl}_{n+1}$  the complete flag variety over  $\bar{k}$ . Let  $\text{DL}_n$  be the compactified Deligne–Lusztig variety (§2) associated to a Coxeter element and the  $q$ -Frobenius. The space  $\text{DL}_n$  is a smooth projective subvariety of  $\text{Fl}_{n+1}$ ; let  $\iota: \text{DL}_n \rightarrow \text{Fl}_{n+1}$  be the evident embedding.

**Theorem 1.2.** *For  $1 \leq i \leq n$ , we set  $L_i$  to be the pullback of line bundle on  $\text{Fl}_{n+1}$  associated to the  $i$ th fundamental weight for the type  $A_n$  root system. Then, we have the following isomorphism of graded  $\mathbf{Q}$ -algebras,*

$$\epsilon: \text{Kly}_{n,q} \xrightarrow{\sim} \text{im}(\iota^*: \text{CH}(\text{Fl}_{n+1}) \rightarrow \text{CH}(\text{DL}_n)), \quad u_i \mapsto \iota^* c_1(L_i).$$

Our argument relies on a result of Langer [11], in which he identified the  $\text{DL}_n$  as the de Concini–Procesi wonderful model for the arrangement  $\text{PG}(n, q)$  of all  $\mathbf{F}_q$ -hyperplanes,

so  $\text{CH}(\text{DL}_n)$  is isomorphic to the matroidal Chow ring  $\text{CH}(\text{PG}(n, q))$  in the sense of Feichtner–Yuzvinsky [7]. If no confusion arises, we also denote by  $\text{PG}(n, q)$  the matroid associated to the arrangement.

**Remark 1.3.** (i) In [14, §7], Nadeau and Tewari related the  $q$ -Klyachko algebra  $\text{Kly}_{n,q}$  to the Schubert cycle expansion of the variety  $\text{DL}_n$ . This is an illustration of Theorem 1.2 at degree  $n$ .

(ii) In [9, Remark 7.13], Katz and Kutler exhibited a ring map  $\text{Kly}_{n,q} \rightarrow \text{CH}(\text{PG}(n, q))$  and asked about the geometrisation of this map. This map is, up to change of variables, given by  $\epsilon$ . Theorem 1.2, therefore, provides a positive answer to the question of Katz–Kutler.

In a different direction, we establish a second geometric interpretation for the  $q$ -Klyachko algebra, valid even when  $q$  is not a prime power.

**Theorem 1.4.** *For any positive rational number  $q$ , there exists a projective simplicial fan  $\Sigma_{n,q}$  (§3) such that the cohomology ring  $\text{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q})$  of the associated toric variety  $X_{\Sigma_{n,q}}$  is isomorphic to  $\text{Kly}_{n,q}$  as graded  $\mathbf{Q}$ -algebras. Under this isomorphism, the ample cone of  $X_{\Sigma_{n,q}}$  contains the cone*

$$\mathcal{K}_{n,q} := \mathbf{R}_{>0} u_1 + \cdots + \mathbf{R}_{>0} u_n \subseteq \text{Kly}_{n,q}^1 \otimes \mathbf{R}.$$

This result is inspired by the work of Abe and Zeng [1], who established the case when  $q = 1$ . Granting the theorem, we immediately deduce:

**Corollary 1.5.** (i) (Kähler package) *The  $q$ -Klyachko algebra  $\text{Kly}_{n,q}$  is a Poincaré duality algebra. Moreover, it satisfies the Hard Lefschetz theorem and Hodge-Riemann relations with respect to any class  $\ell \in \mathcal{K}_{n,q}$ .*

(ii) (Volume polynomial realisation) *Keeping the notation and assumptions of Section 1.1. We have*

$$\binom{n}{q}_q! \int_{X_{\Sigma_{n,q}}} (x_1 u_1 + \cdots + x_n u_n)^n = \sum_{\eta} p([n]; \eta) \prod_{i=1}^n x_i^{\eta(i)},$$

where on the left-hand side, the  $u_i$  are construed as  $\mathbf{Q}$ -divisors on  $X_{\Sigma_{n,q}}$ , and the right-hand sum runs over all point measures  $\eta$  supported on  $[n]$  with  $\sum_{i=1}^n \eta(i) = n$ .

The Kähler package in Item (i) is a consequence of Hodge theory for simplicial polytopes in the sense of Stanley and McMullen [15, 13]. Item (ii), in addition to the Khovanskii–Teissier inequality, implies the probability  $p([n]; \eta)$  satisfies log-concavity properties of the form

$$p([n]; \eta)^2 \geq p([n]; \eta - \delta_i + \delta_{i-1}) \cdot p([n]; \eta - \delta_i + \delta_{i+1}), \quad \text{for all } i \text{ with } \eta(i) \geq 2.$$

More broadly speaking, volume polynomials are special cases of Lorentzian polynomials in the sense of Brändén and Huh [4, Definition 2.1, Theorem 4.6].

Finally, we say a few words about the dichotomy between the geometric situations when  $q = 1$  and when  $q$  is a prime power. The arrangement  $\text{PG}(n, q)$  can be seen as the  $q$ -analogue of the arrangement of coordinate hyperplanes in  $\mathbf{P}^n$ . This comes from viewing  $\text{PG}(n, q)$  as the vanishing locus of the *Moore determinant*  $\det(x_i^{q^j})_{0 \leq i, j \leq n}$ , which is a  $q$ -analogue of the usual Vandermonde determinant  $\det(x_i^j)_{0 \leq i, j \leq n}$ . Moreover, the wonderful compactification of the coordinate arrangement [6] is the  $n$ -dimensional permutohedral variety  $X_{A_n}$ . From this perspective, one can view  $\text{DL}_n$  as the  $q$ -analogue of the permutohedral variety  $X_{A_n}$ , and Theorem 1.2 as the  $q$ -analogue of the well-known isomorphism, usually attributed to Klyachko [10], between  $\text{Kly}_{1,n}$  and the image of the restriction map  $\text{CH}(\text{Fl}_{n+1}) \rightarrow \text{CH}(X_{A_n})$ . We summarise the dichotomy in the following table.

$q = 1$	$q$ prime power
Permutohedral toric variety $X_{A_n}$	Deligne–Lusztig variety $\text{DL}_n$
Coordinate arrangement	The arrangement $\text{PG}(n, q)$ of all $\mathbf{F}_q$ -hyperplanes
Klyachko algebra $\text{Kly}_{n,1}$	$q$ -Klyachko algebra $\text{Kly}_{n,q}$
$\text{Kly}_{n,1} \simeq \text{im}(\text{CH}(\text{Fl}_{n+1}) \rightarrow \text{CH}(X_{A_n}))$ (Klyachko [10])	$\text{Kly}_{n,q} \simeq \text{im}(\text{CH}(\text{Fl}_{n+1}) \rightarrow \text{CH}(\text{DL}_n))$ (Theorem 1.2)
The fan $Y(A_n) = \Sigma_{1,n}$ in [1, 3, 8]	The fan $\Sigma_{n,q}$ constructed in §3.
$\text{Kly}_{n,1} \simeq \text{H}^\bullet(X_{\Sigma_{1,n}}, \mathbf{Q})$ (Abe–Zeng [1])	$\text{Kly}_{n,q} \simeq \text{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q})$ (Theorem 1.4)
Peterson variety $Y_n$	???

For the last row, we note that over the complex numbers, the Peterson variety  $Y_n$  has the property that its cohomology ring is the Klyachko algebra,  $\text{Kly}_{n,1} \simeq \text{H}^\bullet(Y_n, \mathbf{Q})$ . In [1], Abe and Zeng established a map  $Y_n \rightarrow X_{\Sigma_{1,n}}$  that realises this isomorphism in cohomology. It is natural to ask if there is a similar subvariety of  $\text{Fl}_{n+1}$  whose cohomology ring equals  $\text{Kly}_{n,q}$ . The variety  $Y_n$  is a regular nilpotent Hessenberg variety with Hessenberg function  $h = (2, 3, \dots, n, n)$ . The regular semisimple Hessenberg variety associated to the same Hessenberg function  $h$  is exactly the permutohedral variety  $X_{A_n}$ , whose role is taken by  $\text{DL}_n$  in our  $q$ -analogue situation.

Concerning organisation, we start in Section 2 explaining the connection among the  $q$ -Klyachko algebra, the Deligne–Lusztig variety, and the projective geometry matroid, proving Theorem 1.2. In §3, we give the construction of the fan  $\Sigma_{n,q}$  and prove Theo-

rem 1.4.

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## 2 The Deligne–Lusztig variety of a Coxeter element

Let us first introduce some notation. First, we define the  $q$ -integers and  $q$ -factorials

$$(m)_q := \frac{1 - q^m}{1 - q} = 1 + \cdots + q^{m-1}, \quad (m)_{q!} := \prod_{1 \leq i \leq m} (i)_q.$$

Put  $k = \mathbf{F}_q$  for  $q$  a prime power. Let  $G = \mathrm{GL}_{n+1}$  be the general linear group over  $k$  and  $B \subseteq G_{\bar{k}}$  a Borel subgroup. We define  $\mathrm{Fl}_{n+1} := G_{\bar{k}}/B$  as the (complete) flag variety over  $\bar{k}$ . Let  $\pi_d: \mathrm{Fl}_{n+1} \rightarrow \mathrm{Gr}(d, n+1)$  be the map that forgets all but the  $d$ th subspace.

**Definition 2.1.** Let  $w \in \mathfrak{S}_{n+1}$ . The *Deligne–Lusztig variety*  $\mathrm{DL}_n^\circ(w)$  associated to  $w$  is defined as the fibre product

$$\Gamma_{\mathrm{Frob}} \times_{\mathrm{Fl}_{n+1} \times \mathrm{Fl}_{n+1}} [G.(B, wB)]$$

where  $\mathrm{Frob}: \mathrm{Fl}_{n+1} \rightarrow \mathrm{Fl}_{n+1}$  is the absolute Frobenius,  $G$  acts diagonally on  $\mathrm{Fl}_{n+1} \times \mathrm{Fl}_{n+1}$ , and  $G.(B, wB)$  is the  $G$ -orbit of  $(B, wB)$ . Now, consider the permutation  $c = (1, 2)(2, 3) \cdots (n, n+1) \in \mathfrak{S}_{n+1}$ . The *Coxeter variety* is defined to be  $\mathrm{DL}_n^\circ := \mathrm{DL}_n^\circ(c)$ . The *compactified Coxeter variety*  $\mathrm{DL}_n$  is closure of  $\mathrm{DL}_n^\circ$  in  $\mathrm{Fl}_{n+1}$ .

The space  $\mathrm{DL}_n$  is a smooth projective variety of dimension  $n$  with simple normal crossing boundary  $\mathrm{DL}_n \setminus \mathrm{DL}_n^\circ$ . Analysing the fibre product above, one can describe  $\mathrm{DL}_n^\circ$  very explicitly:

**Observation 2.2.** Restricting the map  $\pi_1: \mathrm{Fl}_{n+1} \rightarrow \mathbf{P}^n$  yields an isomorphism:

$$\pi_1|_{\mathrm{DL}_n^\circ}: \mathrm{DL}_n^\circ \xrightarrow{\sim} \mathbf{P}^n \setminus \bigcup \{H : H \text{ is a } k\text{-rational hyperplane}\}, \quad (V_i) \mapsto V_1.$$

The target is the complement of the arrangement  $\mathrm{PG}(n, q)$ .

Let  $\mathrm{DL}'_n$  be the wonderful compactification of the arrangement complement  $\mathrm{DL}_n^\circ$  in the sense of de Concini and Procesi [6]. This is obtained as a series of blow-ups on  $\mathbf{P}^n$ : first blow up all  $k$ -points, then the strict transforms of all  $k$ -lines, and then  $k$ -planes, and so on. Let  $\pi': \mathrm{DL}'_n \rightarrow \mathbf{P}^n$  be the composition of these blow-ups.

Langer observed the following; see [11, Proposition 7.1],

**Proposition 2.3** (Langer). *The Coxeter variety  $\mathrm{DL}_n$  is isomorphic to  $\mathrm{DL}'_n$  as subvarieties of  $\mathrm{Fl}_{n+1}$ , and  $\pi_1|_{\mathrm{DL}_n}$  coincides with  $\pi'$  under this isomorphism.*

**Corollary 2.4.** *The rational Chow ring of  $DL_n$  is given by the Chow ring of the matroid  $PG(n, q)$ :*

$$\begin{aligned} \mathrm{CH}(DL_n)_{\mathbf{Q}} &\simeq \mathrm{CH}(PG(n, q))_{\mathbf{Q}} \\ &:= \frac{\mathbf{Q}[x_F : F \subset \mathbf{P}^n \text{ a } k\text{-rational proper subspace}]}{I_{n,q} + J_{n,q}}, \end{aligned}$$

where  $I_{n,q}$  is generated by linear relations

$$\sum_{F \subset H} x_F - \sum_{F \subset H'} x_F, \quad \text{for any pair of } k\text{-rational hyperplanes } H, H' \subset \mathbf{P}^n,$$

and  $J_{n,q}$  is generated by monomials

$$x_F x_{F'} \quad \text{for every pair } F, F' \text{ of subspaces not contained in one another.}$$

*Proof.* Thanks to Proposition 2.3, we can compute the Chow ring of  $DL_n$  by viewing it as a wonderful model of hyperplane arrangements. Then the assertion follows immediately from the presentation of the Chow ring of wonderful models established in [7, Corollary 2].  $\square$

Here, the class  $x_F$  comes from the exceptional divisor of blowing up (the strict transform of) a  $k$ -linear subspace  $F \subset \mathbf{P}^n$ . Picking a  $k$ -rational hyperplane  $H \subseteq \mathbf{P}^n$ , we define  $\alpha := \sum_{F \subset H} x_F$ —the relations from the ideal  $I_{n,q}$  guarantee the linear equivalence class of  $\alpha$  is independent of the choice of the hyperplane  $H$ . Geometrically, this class of  $\alpha$  is the first Chern class  $c_1(\pi_1^* \mathcal{O}_{\mathbf{P}^n}(1))$ .

Recalling that  $N = (n+1)_q - 1$ , we define divisor classes  $\gamma_k$  on  $X_{A_N}$  as the pullback of the class of the  $N$ -dimensional hypersimplex  $\Delta(N+1, (k)_q)$  in the sense of [2] and [9]; geometrically, this comes from pulling back the Plücker line bundle on the Grassmannian  $\mathrm{Gr}((k)_q, N+1)$  via the composite

$$X_{A_N} \hookrightarrow \mathrm{Fl}_{N+1} \rightarrow \mathrm{Gr}((k)_q, N+1).$$

We will also denote by  $\gamma_k \in \mathrm{CH}^1(DL_n)$  the restriction of  $\gamma_k$  to  $DL_n$ . Also, since  $DL_n$  is a subvariety of  $\mathrm{Fl}_{n+1}$ , we can define  $L_k \in \mathrm{CH}^1(DL_n)$  as the restriction of the first Chern class of the  $k$ th Plücker line bundle on  $\mathrm{Fl}_{n+1}$ . In the case when  $q = 1$  and  $DL_n$  is replaced by the permutohedral variety  $X_{A_n}$ , the two types of divisors agree;  $L_i = \gamma_i$  for all  $1 \leq i \leq n$ . The following lemma is a  $q$ -deformed version of this coincidence.

**Lemma 2.5.** *For all  $1 \leq i \leq n$ , we have  $\gamma_i = q^i L_i$  as divisor classes on  $DL_n$ .*

*Proof.* By [11, Lemma 6.2], we have

$$L_i = (n+1-i)_q \alpha - \sum_{F: \mathrm{codim} F \geq i} (\mathrm{codim} F - i)_q x_F.$$

On the other hand, by [9, Lemma 2.4], we have

$$\gamma_i = ((n+1)_q - (i)_q)\alpha - \sum_{F: \text{codim } F \geq i} ((\text{codim } F)_q - (i)_q)x_F.$$

The claim follows immediately from the identity  $\frac{(a)_q - (b)_q}{(a-b)_q} = q^b$ .  $\square$

Finally, the matroid  $\text{PG}(n, q)$  is the special case of a more general class of matroids called *perfect matroid designs*. In [9, Lemma 7.10], the authors established quadratic relations among  $\gamma_i$  for general perfect matroid designs. Applying their formula to  $\text{PG}(n, q)$ , we obtain the following.

**Lemma 2.6.** *The divisor classes  $\gamma_i$  satisfy the  $q$ -Klyachko relation*

$$(q+1)\gamma_i^2 = \gamma_i\gamma_{i+1} + q\gamma_i\gamma_{i-1}, \quad \text{for all } 1 \leq i \leq n.$$

We are now in a position to establish the main claim.

*Proof of Theorem 1.2.* First, we check that Plücker classes  $L_i$  also satisfy a  $q$ -Klyachko relation, we have

$$\begin{aligned} (q+1)L_i^2 &= \frac{q+1}{q^{2i}}\gamma_i^2 = \frac{1}{q^{2i}}(\gamma_i\gamma_{i+1} + q\gamma_i\gamma_{i-1}) \\ &= \frac{1}{q^{2i}}(q^{2i+1}L_iL_{i+1} + q^{2i}L_iL_{i-1}) = qL_iL_{i+1} + L_iL_{i-1}, \end{aligned}$$

by Lemma 2.5. Therefore, the map  $\iota^*: \text{CH}(\text{Fl}_{n+1}) \rightarrow \text{CH}(\text{DL}_n)$  factors through the map  $\tau: \text{Kly}_{n,q} \rightarrow \text{CH}(\text{DL}_n)$  that sends  $u_i$  to  $L_{n-i}$ . Since the Chow ring  $\text{CH}(\text{Fl}_{n+1})$  is generated by  $L_i$ , we get a surjection

$$\tau: \text{Kly}_{n,q} \twoheadrightarrow \text{im } \iota^*.$$

But [9, Theorem 7.12] combined with Lemma 2.5 shows the degree map on  $\text{CH}(\text{DL}_n)$  restricts to the degree map on  $\text{Kly}_{n,q}$  given by  $q$ -divided symmetrisation, so  $\tau$  must also inject.  $\square$

### 3 The toric orbifold $X_{\Sigma_{n,q}}$

#### 3.1 A $q$ -deformation of the type $A$ Cartan matrix

We give a different construction that realises the  $q$ -Klyachko algebra. The plan is to exhibit the  $q$ -Klyachko as the Stanley–Reisner ring of a toric orbifold, and then the Kähler package follows from Hodge theory for simple polytopes.

Assume now  $q \in \mathbf{Q}_{>0}$ . Consider the  $n$ -by- $n$  matrix,

$$A(n, q) = \begin{pmatrix} q+1 & -1 & & & \\ -q & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -q & q+1 & \end{pmatrix}$$

where the omitted entries are zero. When  $q = 1$ , this is the Cartan matrix of the root system of type  $A_n$ . This matrix also appeared in [14, Lemma 3.6]. The following list of properties is clear:

**Lemma 3.1.** (i) *The matrix  $A(n, q)$  has determinant  $(n+1)_q$ .*

(ii) *For any subset  $J \subseteq [n]$ , the submatrix  $A(n, q)_{i,j \in J}$  is nonsingular.*

(iii) *The matrix  $A(n, q)$  satisfies  $(A(n, q)^{-1})_{ij} > 0$  for all  $1 \leq i, j \leq n$ .*

(iv) *For any subset  $J \subseteq [n]$ , the submatrix  $A_J = A(n, q)_{i,j \in J}$  satisfies  $(A_J^{-1})_{ij} \geq 0$  for all  $i, j \in J$ .*

Recall that when  $q = 1$ , the matrix  $A(n, 1)_{i,j \in J}$  is a Cartan matrix of the root system of type  $A_n$  for any  $J \subseteq [n]$ . Then Item (iv) above is an illustration of the fact that the inverse  $[A(n, 1)_{i,j \in J}]^{-1}$  consists of nonnegative rational numbers [12].

### 3.2 The simplicial fan $\Sigma_{n,q}$ : construction and properties

Denote by  $e_i$  the standard basis vector in  $N = \mathbf{Z}^n$ , and let  $\alpha_1, \dots, \alpha_n \in \mathbf{Q}^n$  be the column vectors of  $A(n, q)$ .

Let  $\Sigma_{n,q}$  be the set of cones of the form

$$\sigma_{J,K} = \text{cone}\{e_i : i \in J\} - \text{cone}\{\alpha_k : k \in K\}$$

for disjoint subsets  $J, K \subseteq [n]$ . Here, our convention is that  $\text{cone } \emptyset$  is the origin  $\{0\}$ , so that  $\sigma_{\emptyset, \emptyset} = \{0\}$ . As a matter of notation, we write  $A = A(n, q)$  and  $A_J = A_{i,j \in J}$ .

Using Lemma 3.1, one can easily observe:

**Lemma 3.2.** *For  $J, K \in [n]$  so that  $J \cap K = \emptyset$ , the cone  $\sigma_{J,K}$  is a strongly convex rational polyhedral cone in  $N_{\mathbf{R}}$  of dimension  $\dim \sigma_{J,K} = \#J + \#K$ . Moreover, given  $\sigma_{J,K}, \sigma_{P,Q} \in \Sigma_{n,q}$ , we have  $\sigma_{J,K} \cap \sigma_{P,Q} = \sigma_{J \cap P, K \cap Q}$ .*

**Corollary 3.3.** *The set  $\Sigma_{n,q}$  is a simplicial fan. In other words, the toric variety  $X_{\Sigma_{n,q}}$  associated to the fan  $\Sigma_{n,q}$  is  $\mathbf{Q}$ -factorial.*

*Proof.* Firstly, we check that  $\Sigma_{n,q}$  is a fan. Indeed, given a cone  $\sigma_{J,K} \in \Sigma_{n,q}$ , every nonempty face is given by the cone  $\sigma_{J',K'} \in \Sigma_{n,q}$  for  $J' \subseteq J$  and  $K' \subseteq K$ . Secondly, by the second assertion of Lemma 3.2, the intersection of any two cones  $\sigma_{J,K}$  and  $\sigma_{P,Q}$  in  $\Sigma_{n,q}$  is a face of both, namely,  $\sigma_{J,K} \cap \sigma_{P,Q}$ . Finally, by the first assertion of Lemma 3.2, the fan  $\Sigma_{n,q}$  is simplicial.  $\square$

Next, we show the toric variety  $X_{\Sigma_{n,q}}$  is projective. The argument goes by first establishing completeness and then detecting amplitude *via* the toric version of Kleiman’s criterion.

**Lemma 3.4.** *The fan  $\Sigma_{n,q}$  is complete.*

*Proof.* The proof here is the same as the one in [1, Proposition 4.3]. The proof in *loc.cit.* goes through upon replacing references to [1, Lemma 3.3 and Lemma 3.5] by the first and second assertion of Lemma 3.2.  $\square$

Considering the orbit-cone correspondence for toric varieties [5, §3.2], we denote by  $D_{-\alpha_i}$  the torus invariant Weil divisor on  $X_{\Sigma_{n,q}}$  corresponding to the ray  $\sigma_{\emptyset,i} = \text{cone}(-\alpha_i)$  for  $i \in [n]$ . Similarly, denote by  $D_{e_k}$  the torus invariant Weil divisor that corresponds to the ray  $\sigma_{k,\emptyset} = \text{cone } e_k$ .

**Proposition 3.5.** *For a torus invariant irreducible curve  $C \subseteq X_{\Sigma_{n,q}}$ , and an invariant Weil divisor  $D_{-\alpha_i}$  that corresponds to the ray generator  $-\alpha_i$ , the intersection number  $(D_{-\alpha_i} \cdot C)$  is nonnegative. Moreover, given an invariant curve  $C$ , there exists  $\ell \in [n]$  such that the corresponding Weil divisor  $D_{-\alpha_\ell}$  satisfies  $(D_{-\alpha_\ell} \cdot C) > 0$ .*

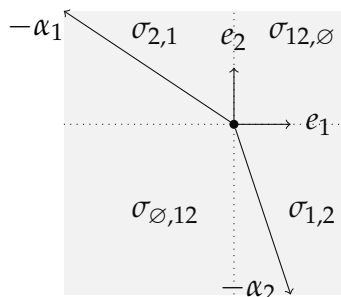
*Proof.* Under the orbit-cone correspondence, a torus invariant irreducible curve  $C$  corresponds to a codimension-one cone of the form  $\sigma_{J,K}$  with  $J \cup K = [n] \setminus \ell$  for some  $\ell$ . The cone  $\sigma_{J,K}$  is contained in the maximal cones  $\sigma_{J \cup \ell, K}$  and  $\sigma_{J, K \cup \ell}$ . By [5, Proposition 6.4.4], there exists a unique up to homothety wall relation among the  $n + 1$  ray generators in  $\sigma_{J \cup \ell, K}$  and  $\sigma_{J, K \cup \ell}$ , given by the linear relation

$$-(D_{-\alpha_\ell} \cdot C)\alpha_\ell - \sum_{j \in J} (D_{-\alpha_j} \cdot C)\alpha_j + \sum_{k \in K} (D_{e_k} \cdot C)e_k + (D_{e_\ell} \cdot C)e_\ell = 0; \quad (3.1)$$

moreover, the intersection numbers  $(D_{-\alpha_\ell} \cdot C)$  and  $(D_{e_\ell} \cdot C)$  are strictly positive—this proves the second assertion. Put  $J' = J \cup \ell$ . Pairing Eq. (3.1) against  $e_i$  for  $i \in J'$ , we obtain a system of linear equations

$$\sum_{j \in J'} A_{ij}(D_{-\alpha_j} \cdot C) = \delta_{i\ell} (D_{e_i} \cdot C) \text{ for all } i \in J'.$$

By Lemma 3.1, the submatrix  $A_{J'}$  is nonsingular, and its inverse  $A_{J'}^{-1}$  has nonnegative entries. Solving for the linear equation and noting that  $(D_{e_i} \cdot C) > 0$ , we have  $(D_{-\alpha_j} \cdot C) \geq 0$  for all  $j \in J'$ . This yields the first assertion.  $\square$



**Figure 2:** Illustration of the fan  $\Sigma_{2,2}$ , maximal cones labelled by two-part partitions of  $[2]$ .

By Lemma 3.4, the toric variety  $X_{\Sigma_{n,q}}$  is complete. Recall that the toric Kleiman criterion asserts that if  $D$  is a Cartier divisor on a complete toric variety  $X$ , then  $D$  is ample if and only if  $(D \cdot C) > 0$  for all torus invariant irreducible curves  $C$ . Combining this with Proposition 3.5 and the fact that  $X_{\Sigma_{n,q}}$  is  $\mathbf{Q}$ -factorial (Corollary 3.3), one immediately deduces

**Corollary 3.6.** *Given positive rational numbers  $a_i \in \mathbf{Q}_{>0}$  for  $1 \leq i \leq n$ , the  $\mathbf{Q}$ -divisor  $D = \sum_{i=1}^n a_i D_{-\alpha_i}$  is  $\mathbf{Q}$ -ample. That is, a sufficiently large multiple of  $D$  is Cartier and ample.*

**Example 3.7** (Case  $n = q = 2$ ). In Figure 2, we visualise the fan  $\Sigma_{2,2}$  in  $N_{\mathbf{R}} \simeq \mathbf{R}^2$ . The 4 maximal cones of  $\Sigma_{2,2}$  are the labelled regions. The standard basis vectors  $e_i$  generate the cone  $\sigma_{i,\emptyset}$ , for  $i = 1, 2$ . The remaining two vectors are  $-\alpha_1 = -3e_1 + 2e_2$  and  $-\alpha_2 = e_1 - 3e_2$ , which gives the cones  $\sigma_{\emptyset,1}$  and  $\sigma_{\emptyset,2}$  respectively. One observes that  $\Sigma_{2,2}$  is the normal fan of a lattice polytope that is combinatorially equivalent to a 2-dimensional cube.

### 3.3 Cohomology of the toric orbifold $X_{\Sigma_{n,q}}$

This subsection is devoted to proving Theorem 1.4. Since the toric variety  $X_{\Sigma_{n,q}}$  is  $\mathbf{Q}$ -factorial and projective, the rational Betti cohomology ring  $H^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q})$  can be computed by Stanley–Reisner theory [5, Chap. 12].

*Proof of Theorem 1.4.* By [5, Section 12.4], we have the ring isomorphism

$$\begin{aligned} \text{SR}: \mathbf{Q}[X_i, Y_i : i \in [n]] / (\mathcal{I}_{n,q} + \mathcal{J}_{n,q}) &\longrightarrow H^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q}); \\ X_i &\longmapsto [D_{-\alpha_i}], \quad Y_i \longmapsto [D_{e_i}], \quad \text{for all } i \in [n], \end{aligned}$$

where the ideal  $\mathcal{I}_{n,q}$  is generated by quadrics  $X_i Y_i$  for all  $i \in [n]$ , and the ideal  $\mathcal{J}_{n,q}$  is generated by linear forms  $-\sum_{1 \leq j \leq n} A(n, q)_{ij} X_j + Y_i$ , for all  $i \in [n]$ . By our definition of  $A(n, q)$ , the ring map SR factors through an isomorphism

$$\text{SR}' : \mathbf{Q}[X_1, \dots, X_n] / \mathcal{I}'_{n,q} \longrightarrow \mathbf{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q}), \quad X_i \longmapsto [D_{-\alpha_i}],$$

where the ideal  $\mathcal{I}'_{n,q} \subseteq \mathbf{Q}[X_1, \dots, X_n]$  is generated by quadrics

$$X_i ((q+1)X_i - X_{i+1} - qX_{i-1}), \quad \text{for } 1 \leq i \leq n,$$

where we set  $X_0 = X_{n+1} = 0$ . But the  $q$ -Klyachko algebra  $\text{Kly}_{n,q}$  is isomorphic to the source  $\mathbf{Q}[X_1, \dots, X_n] / \mathcal{I}'_{n,q}$  of  $\text{SR}'$  by sending  $u_i$  to  $X_i$ . The second assertion of the theorem follows from Corollary 3.6.  $\square$

### 3.4 A Kähler package for the $q$ -Klyachko algebra

In this subsection, we use the Hodge theory for simplicial polytopes to prove Corollary 1.5.

**Definition 3.8.** Let  $A^\bullet = \bigoplus_{i=0}^d A^i$  be a graded Artinian  $\mathbf{Q}$ -algebra of finite type, in addition to an isomorphism  $\text{deg} : A^d \rightarrow \mathbf{Q}$ . We say that  $(A^\bullet, \text{deg})$  satisfies the Kähler package with respect to an element  $\ell \in A^1$  if the following conditions are satisfied,

- (i) (Poincaré duality) For  $0 \leq k \leq d/2$ , the bilinear form  $A^k \times A^{d-k} \rightarrow \mathbf{Q}$  sending  $(\eta_1, \eta_2) \mapsto \text{deg}(\eta_1 \eta_2)$  is a perfect pairing.
- (ii) (Hard Lefschetz theorem) For every integer  $0 \leq k \leq d/2$ , the multiplication map  $\ell^{d-2k} : A^k \rightarrow A^{d-k}$ , sending  $\eta \mapsto \ell^{d-2k} \eta$  is an isomorphism.
- (iii) (Hodge index theorem) For every integer  $0 \leq k \leq d/2$ , the bilinear form on  $A^k$  sending  $(\eta_1, \eta_2)$  to  $(-1)^k \text{deg}(\ell^{d-2k} \eta_1 \eta_2)$  is positive definite upon restricting to the kernel of multiplication by  $\ell^{d-2k+1}$ .

*Proof of Corollary 1.5.* For a simplicial lattice polytope  $P$  with normal fan  $\Sigma_P$ , Stanley and McMullen [15, 13] showed that on the toric variety  $X_{\Sigma_P}$  associated to the fan  $\Sigma_P$ , the rational Betti cohomology ring  $\mathbf{H}^\bullet(X_{\Sigma_P}, \mathbf{Q})$  satisfies Poincaré duality and the hard Lefschetz theorem with respect to any ample divisor class  $\ell \in \mathbf{H}^2(X_{\Sigma_P}, \mathbf{Q})$ . Moreover, in *loc.cit.*, McMullen also established the Hodge index theorem for any ample class  $\ell$ . Since the fan  $\Sigma_{n,q}$  is projective and simplicial, Item (i) follows. Concerning Item (ii), by [5, Lemma 12.5.2], we have  $\int_{X_{\Sigma_{n,q}}} D_{-\alpha_1} \cdots D_{-\alpha_n} = |\det A(n, q)|^{-1} = (n)_q!^{-1}$ . Therefore we have

$$\int_{X_{n,q}} \prod_i D_{-\alpha_i}^{\eta(i)} = \frac{1}{[(n)_q!]^2} \text{deg}_{n,q} \prod_i u_i^{\eta(i)} = \frac{p([n]; \eta)}{(n)_q!}$$

for any  $\eta : [n] \rightarrow \mathbf{Z}_{\geq 0}$  with  $\sum_i \eta(i) = n$ .  $\square$

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