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Commutation classes of reduced words and higher Bruhat orders for affine permutations

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Abstract. The higher Bruhat orders are partial orders that generalize the weak order on the symmetric group $\mathfrak{S}_n$, and the second higher Bruhat order is a poset on commutation classes of reduced words for the longest element in $\mathfrak{S}_n$, where covering relations correspond to braid relations. Constructing analogs in other settings is an area of recent interest, and we present an analog that generalizes any interval $[\mathrm{id}, w]$ in the weak order of the affine symmetric group $\widetilde{\mathfrak{S}}_n$. Paralleling the classical case, we show the second higher Bruhat order is a poset on commutation classes of reduced words for $w \in \widetilde{\mathfrak{S}}_n$. When $w \in \mathfrak{S}_n$, we also establish results for all higher Bruhat orders that are direct analogs of ones in the classical case.

Keywords: affine symmetric group, Coxeter group, weak order, reduced words, reflection orders, higher Bruhat orders

1 Introduction

The classical higher Bruhat orders $\mathcal{B}(n,k)$ are partial orders introduced by Manin and Schechtman [13] in their study of fundamental groups of certain hyperplane arrangements. They were further characterized by Ziegler [16] using single step inclusion of consistent sets. The first higher Bruhat order $\mathcal{B}(n,1)$ is isomorphic to the weak (Bruhat) order on the symmetric group $\mathfrak{S}_n$, and the second higher Bruhat order $\mathcal{B}(n,2)$ is a partial order on commutation classes of reduced words for the longest permutation. In the general case, $\mathcal{B}(n,k)$ can be described using pseudo-hyperplane arrangements [16], oriented matroids [15], and zonotopal tilings [9]. The higher Bruhat orders have applications to Bott-Samelson varieties [7] and Steenrod algebras [12].

Generalizing the higher Bruhat orders to other settings has been an area of recent interest [4, 14]. In 2022, Ben Elias generalized the second higher Bruhat orders to include nonreduced words to prove a generalized Bergman diamond lemma for Hecke-type algebras [8]. Daniel Hothem generalized the higher Bruhat orders beginning from intervals $[\mathrm{id}, w]$ in the weak order on $\mathfrak{S}_n$ [11]. Elias and Hothem posed the following question.

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Problem 1.1. *(Elias-Hothem, 2020) Generalize the higher Bruhat orders to all intervals* $[\mathrm{id}, w]$ *in the affine symmetric groups* $\widetilde{\mathfrak{S}}_n$.

For each affine permutation $w \in \widetilde{\mathfrak{S}}_n$, Elias proposed orienting edges between commutation classes of reduced words for w according to the directed braid relations $s_i s_{i+1} s_i \to s_{i+1} s_i s_{i+1}$, where each index is modulo n. This results in a directed graph, denoted $\mathcal{G}(w)$.

Conjecture 1.2. *(Elias, 2021) For any* $w \in \widetilde{\mathfrak{S}}_n$*, the directed graph* $\mathcal{G}(w)$ *is acyclic with a unique source vertex and a unique sink vertex.*

In this extended abstract, we prove Conjecture 1.2 and propose a construction for Problem 1.1. We begin with the set $\mathrm{Inv}_k(w)$ of k-inversions for $w \in \widetilde{\mathfrak{S}}_n$. These are the length k decreasing subsequences in the one-line notation of w, up to a congruence shift. Paralleling the classical constructions in [13], we define the set of admissible orders $\mathcal{A}_w(n,k)$ to be certain total orders of $\mathrm{Inv}_k(w)$, introduce a notion of commutation equivalence $\sim_w$, and construct the higher Bruhat orders $\mathcal{B}_w(n,k)$ as a poset on $\mathcal{A}_w(n,k)/\sim_w$. We associate to each admissible order a reversal set that is constant up to commutation equivalence. These satisfy conditions that motivate a notion of consistent sets that extends the one due to Ziegler [16], and we define a poset $\mathcal{C}_w(n,k)$ on the consistent subsets of $\mathrm{Inv}_k(w)$ ordered by single step inclusion. From our construction, $\mathcal{B}_w(n,1) \cong \mathcal{C}_w(n,2)$ is isomorphic to the interval $[\mathrm{id}, w]$ in the weak order on $\widetilde{\mathfrak{S}}_n$, and we establish the following theorem when $k = 2$. Conjecture 1.2 follows immediately from this result.

Theorem 1.3. *For any positive integer n and* $w \in \widetilde{\mathfrak{S}}_n$*, the following hold.*

(a) There are natural bijections between maximal chains of $\mathcal{C}_w(n,2)$*, reduced words for* w*, reflection orders for* w*, and* $\mathcal{A}_w(n,2)$.

(b) $\mathcal{B}_w(n,2) \cong \mathcal{C}_w(n,3)$ *is a ranked poset with unique minimal and maximal elements corresponding with reversal sets* $\varnothing$ *and* $\mathrm{Inv}_3(w)$.

(c) The Hasse diagram of $\mathcal{B}_w(n,2) \cong \mathcal{C}_w(n,3)$ *is isomorphic to* $\mathcal{G}(w)$ *as a directed graph. Furthermore, the diameter of* $\mathcal{G}(w)$ *as an undirected graph is* $|\mathrm{Inv}_3(w)|$.

Theorem 1.3 has connections to other work on graphs of reduced words. When $w \in \mathfrak{S}_n$, a comparison of our constructions shows $\mathcal{B}_w(n,2)$ is the quotient of a ranked poset on all reduced words for w defined by Assaf [1]. The diameter statement in Theorem 1.3(c) also extends a result of Gutierres, Mamede, and Santos [10] from $\mathfrak{S}_n$ to $\widetilde{\mathfrak{S}}_n$.

When $w \in \mathfrak{S}_n$, we also extend portions of Theorem 1.3 to arbitrary k, resulting in the following theorem that recovers numerous properties of the classical higher Bruhat orders in [13] and [16]. We conjecture that this theorem holds for arbitrary $w \in \widetilde{\mathfrak{S}}_n$. For $1 \leqslant k \leqslant n \leqslant 8$, we have computationally verified this for all $w \in \widetilde{\mathfrak{S}}_n$ of length up to and including 15. For $n = 6$, we have further verified through length 21, and for $n \leqslant 5$, we have further verified through length 27.

Theorem 1.4. *Let k, n be positive integers and $w \in \mathfrak{S}_n$. The following hold.*

(a) $\mathcal{B}_w(n,k)$ is isomorphic as a poset to $\mathcal{C}_w(n,k+1)$, and the isomorphism sends an equivalence class of admissible orders to the reversal set of the class.

(b) $\mathcal{B}_w(n,k) \cong \mathcal{C}_w(n,k+1)$ is a ranked poset with unique minimal and maximal elements corresponding with reversal sets $\varnothing$ and $\mathrm{Inv}_{k+1}(w)$.

(c) There is a natural bijection between maximal chains of $\mathcal{C}_w(n,k+1)$ and $\mathcal{A}_w(n,k+1)$.

We begin in Section 2 by giving preliminary information on the affine symmetric group. As our constructions in $\mathfrak{S}_n$ are simpler to understand, we will start in Section 3 by constructing our higher Bruhat orders for $w \in \mathfrak{S}_n$ and outlining our approach for proving Theorem 1.4. We then move to the affine higher Bruhat orders for $w \in \widetilde{\mathfrak{S}}_n$ in Section 4 and outline our approach for Theorem 1.3. We also state Conjecture 4.6, which is the missing link in a full generalization of Theorem 1.4 to all affine permutations.

2 Preliminaries

In this section, we give preliminaries on the affine symmetric group. See [3, Section 8.3] for additional background. Throughout, n is a positive integer.

A bijection $w : \mathbb{Z} \to \mathbb{Z}$ is called *n-periodic* if for all $x \in \mathbb{Z}$, we have $w(x+n) = w(x) + n$. Such a bijection is uniquely determined by its values on $[n] = \{1, 2, \ldots, n\}$. The *affine symmetric group* $\widetilde{\mathfrak{S}}_n$ is the group of all n-periodic bijections $w : \mathbb{Z} \to \mathbb{Z}$ that satisfy $\sum_{i \in [n]} w(i) = \binom{n+1}{2}$, where the group operation is function composition. An element in $\widetilde{\mathfrak{S}}_n$ is called an *affine permutation*, and in our work, we will express it in *window notation* $(w(1), w(2), \ldots, w(n))$. Note that $\widetilde{\mathfrak{S}}_n$ contains the usual symmetric group $\mathfrak{S}_n$ as the subgroup of permutations such that $w([n]) = [n]$.

When $n \geqslant 2$, the group $\widetilde{\mathfrak{S}}_n$ is generated by the *simple transpositions* $s_0, s_1, \ldots, s_{n-1}$ where s_i maps $s_i(i) = i+1$, $s_i(i+1) = i$, and fixes all $j \in \mathbb{Z}$ such that $j \not\equiv i, i+1 \pmod{n}$. We consider the indices of the simple transpositions modulo n, so s_0 and s_n denote the same simple transposition. The simple transpositions in $\widetilde{\mathfrak{S}}_n$ satisfy the following set of minimal relations: (1) $s_i^2 = 1$ for all $i \in \mathbb{Z}$, (2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i \in \mathbb{Z}$, and (3) $s_i s_j = s_j s_i$ for all $i, j \in \mathbb{Z}$ with $j \not\equiv i, i+1, i-1 \pmod{n}$. Relations of the second kind are termed *braid relations*, and relations of the third kind are termed *commutation relations*. The symmetric group $\mathfrak{S}_n$ is generated by $s_1, \ldots, s_{n-1}$, omitting s_0.

A *reduced expression* for an element $w \in \widetilde{\mathfrak{S}}_n$ is a product $s_{i_1} \cdots s_{i_\ell}$ of minimal length such that $s_{i_1} \cdots s_{i_\ell} = w$. For brevity, we will consider *reduced words*, which are the indices $i_1 \cdots i_\ell \in (\mathbb{Z}/n\mathbb{Z})$ of a reduced expression $s_{i_1} \cdots s_{i_\ell}$. All reduced words for an element $w \in \widetilde{\mathfrak{S}}_n$ have the same *length*, which is denoted $\ell(w)$. In general, an element $w \in \widetilde{\mathfrak{S}}_n$ can have many reduced words, and given a reduced word, one can generate additional

$$
\begin{tikzcd}[column sep=small]
 & & 2141232 \arrow[r, dash] \arrow[d, dash] & 2141323 \arrow[r, dash] \arrow[d, dash] & 2143123 \\
 & 2412432 \arrow[r, dash] \arrow[d, dash] & 2414232 \arrow[r, dash] \arrow[d, dash] & 2414323 \arrow[d, dash] & \\
4121432 \arrow[r, dash] & 4212432 \arrow[r, dash] & 4214232 \arrow[r, dash] & 4214323 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 2141232 \arrow[r] & \begin{array}{c} 2141323 \\ 2143123 \end{array} \\
4121432 \arrow[r] & \begin{array}{c} 2414232 \\ 2412432 \\ 4212432 \\ 4214232 \end{array} \arrow[u] \arrow[r] & \begin{array}{c} 2414323 \\ 4214323 \end{array} \arrow[u]
\end{tikzcd}
$$

Figure 1: $\mathcal{R}(w)$ (left) and $\mathcal{G}(w)$ (right) for the affine permutation $w = (1, 7, 2, 0) \in \widetilde{\mathfrak{S}}_4$.

ones using relations among the generators of $\widetilde{\mathfrak{S}}_n$. Two reduced words are *commutation equivalent* if they differ by a sequence of commutation relations. If a reduced word for w has $i(i+1)i$ (resp. $(i+1)i(i+1)$) in consecutive entries, then a *braid* is the operation that replaces these three elements with $(i+1)i(i+1)$ (resp. $i(i+1)i$).

The *graph on reduced words of* w, denoted $\mathcal{R}(w)$, is the undirected graph whose vertex set consists of reduced words for w and edges between reduced words that can be obtained from one another using a commutation or braid. Letting $\sim_w$ denote commutation equivalence, one can construct the quotient graph $\mathcal{R}(w)/\sim_w$ with vertices given by the equivalence classes of reduced words. The *directed braid graph on commutation classes of reduced words of* w, denoted $\mathcal{G}(w)$, is obtained from $\mathcal{R}(w)/\sim_w$ by directing the edge between two commutation classes if suitable representatives from the classes differ by a braid in the direction $i(i+1)i \to (i+1)i(i+1)$. See Figure 1 for examples.

For $w \in \widetilde{\mathfrak{S}}_n$, the *inversion set* of w is

$$\operatorname{Inv}(w) = \left\{ (x, y) \in [n] \times \mathbb{Z} : x < y \text{ and } w^{-1}(x) > w^{-1}(y) \right\}. \tag{2.1}$$

Each affine permutation is uniquely determined by its inversion set, and the length of w can be found using $\ell(w) = |\operatorname{Inv}(w)|$. The *weak (Bruhat) order* is the partial order on $\widetilde{\mathfrak{S}}_n$ defined by $v \leqslant w$ whenever $\operatorname{Inv}(v) \subseteq \operatorname{Inv}(w)$. The covering relations in the weak order are all of the form $w \lessdot ws_i$ if $\ell(w) = \ell(ws_i) - 1$ for some $i \in [n]$, so one could also say the weak order is defined by *single step inclusion* on inversion sets. Furthermore, the set of maximal chains of the interval $[\mathrm{id}, w]$ is in bijection with the set of reduced words for w.

Reflection orders for Coxeter groups were originally studied by Dyer in his Ph.D. thesis [5] and now have been applied extensively in Coxeter groups and Kazhdan-Lusztig polynomials. We give a definition in the setting of $\widetilde{\mathfrak{S}}_n$.

Definition 2.1. Let $w \in \widetilde{\mathfrak{S}}_n$. For each reduced word $\mathbf{i} = i_1 \cdots i_\ell \in \mathcal{R}(w)$, associate affine permutations $w^{(j)} = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}$ for all $1 \leqslant j \leqslant \ell$, and define $\rho_j = (w^{(j)}(i_j), w^{(j)}(i_j + 1))$ for all $1 \leqslant j \leqslant \ell$. The *reflection order* associated to $\mathbf{i}$ is the total order on $\operatorname{Inv}(w)$ given by $\rho(\mathbf{i}) = (\rho_1, \ldots, \rho_\ell)$. The set of *reflection orders* of w is the set of all such total orders on $\operatorname{Inv}(w)$ associated to reduced words of w.

Example 2.2. If $w = (1, 7, 2, 0) \in \widetilde{\mathfrak{S}}_4$, then the reflection order for the reduced word 0121032 is $((0, 1), (0, 2), (0, 3), (2, 3), (1, 3), (0, 7), (2, 7))$

3 Higher Bruhat orders for permutations in $\mathfrak{S}_n$

In this section, we construct the higher Bruhat orders for arbitrary $w \in \mathfrak{S}_n$. When w is the longest element in $\mathfrak{S}_n$, these reduce to definitions for the classical higher Bruhat orders in [13] and [16]. Throughout this section, assume $1 \leqslant k \leqslant n$ and $w \in \mathfrak{S}_n$.

Let $\binom{[n]}{k}$ denote the collection of ordered subsets $[x_1, ..., x_k]$ of k distinct elements in $[n]$ listed in increasing order. For each $X = [x_1, \ldots, x_k] \in \binom{[n]}{k}$ and $i \in [k]$, define $X_i = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]$, and define the *packet* of X to be $P(X) = \{X_1, X_2, \ldots, X_k\}$. This packet has a natural *lexicographic* (lex) order $(X_k, X_{k-1}, \ldots, X_1)$ and *antilexicographic* (antilex) order $(X_1, X_2, \ldots, X_k)$. A *prefix* of $P(X)$ is a subset of the form $\{X_k, X_{k-1}, \ldots, X_i\}$, and a *suffix* of $P(X)$ is a subset of the form $\{X_i, X_{i-1}, \ldots, X_1\}$.

In this notation, $[i, j] \in \binom{[n]}{2}$ is an inversion for w whenever $w^{-1}(i) > w^{-1}(j)$. More generally, we consider the size k-subsets appearing in reverse order for w.

Definition 3.1. For $X = [x_1, \ldots, x_k] \in \binom{[n]}{k}$, we say X is a *k-inversion* for w provided $w^{-1}(x_1) > \cdots > w^{-1}(x_k)$. The *k-inversion set* of w is defined to be

$$\mathrm{Inv}_k(w) = \left\{ [x_1, \ldots, x_k] \in \binom{[n]}{k} : w^{-1}(x_1) > \cdots > w^{-1}(x_k) \right\}. \tag{3.1}$$

Note that for $k > n$, we have that $\mathrm{Inv}_k(w) = \varnothing$, and for $k = 0$, we have that $\mathrm{Inv}_0(w) = \{\varnothing\}$. For $k = 1$, every $\{i\} := [i] \in \binom{[n]}{1}$ is a 1-inversion of $w \in \mathfrak{S}_n$. For $k \geqslant 2$ and $X \in \binom{[n]}{k+1} \setminus \mathrm{Inv}_{k+1}(w)$ the intersection $P(X) \cap \mathrm{Inv}_k(w)$ will not be the entire packet. However, $P(X) \cap \mathrm{Inv}_k(w)$ can be characterized using the following four cases.

Lemma 3.2. *For any $X \in \binom{[n]}{k+1}$, the intersection $P(X) \cap \mathrm{Inv}_k(w)$ is one of the following: $\varnothing$, $\{X_i\}$ for some $i \in [k+1]$, $\{X_i, X_{i+1}\}$ for some $i \in [k]$, or the entire packet $P(X)$.*

Lemma 3.2 motivates another definition. For $X = [x_1, \ldots, x_k] \in \binom{[n]}{k}$, we say X is a *k-quasi-inversion* for w if all but one of the pairs in $\{[x_i, x_j] : 1 \leqslant i < j \leqslant k\}$ are 2-inversions of w. When this occurs, $P(X) \cap \mathrm{Inv}_{k-1}(w) = \{X_i, X_{i+1}\}$ for some $i \in [k-1]$.

Example 3.3. For $w = (6, 4, 5, 2, 3, 1) \in \mathfrak{S}_6$, we see $[1, 2, 4, 6]$ is a 4-inversion, $[1, 2, 3, 4]$ is a 4-quasi-inversion, and $[2, 3, 4, 5]$ is neither an inversion nor a quasi-inversion.

A 2-quasi-inversion is sometimes referred to as a co-inversion of w. Furthermore, permutations in the interval $[\mathrm{id}, w]$ in the weak order can be identified with the linear extensions of a poset on $[n]$ with relations $i < j$ if $[i, j]$ is a quasi-inversion for w. For larger k, we will construct a similar partial order based on quasi-inversions.

Definition 3.4. The *permanent poset* $\mathcal{P}_w(n, k)$ is the poset on $\mathrm{Inv}_k(w)$ with order relation $\leqslant_{\mathcal{P}_w(n,k)}$ given by the transitive closure of the *quasi-inversion relations*: if $X \in \binom{[n]}{k+1}$ is a quasi-inversion for w such that $P(X) \cap \mathrm{Inv}_k(w) = \{X_i, X_{i+1}\}$, then $X_i <_{\mathcal{P}_w(n,k)} X_{i+1}$ whenever $(k - i)$ is odd, and $X_{i+1} <_{\mathcal{P}_w(n,k)} X_i$ whenever $(k - i)$ is even.

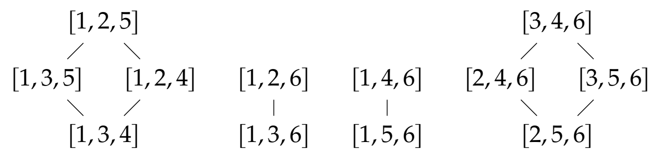

Figure 2: The Hasse diagram of $\mathcal{P}_w(6,3)$ for $w = (6,4,5,2,3,1) \in \mathfrak{S}_6$.

To give some further intuition on the relations, consider the reflection orders for w in Definition 2.1. One can show that for any quasi-inversion $[x,y,z]$, if $P([x,y,z]) \cap \mathrm{Inv}_2(w) = \{[x,y],[x,z]\}$ (resp. $\{[x,z],[y,z]\}$), then $[x,y]$ (resp. $[y,z]$) must appear before $[x,z]$. These aligns with the relations in $\mathcal{P}_w(n,2)$. See Figure 2 for an example of $\mathcal{P}_w(n,3)$.

Reflexivity and transitivity of $\leqslant_{\mathcal{P}_w(n,k)}$ follow by construction. To establish antisymmetry and show that $\mathcal{P}_w(n,k)$ actually is a poset, we construct functions that are monotone on $\mathcal{P}_w(n,k)$. These functions also allow us to describe the restriction of $\mathcal{P}_w(n,k)$ to $P(X)$ for any $X \in \mathrm{Inv}_{k+1}(w)$.

Lemma 3.5. *The permanent poset $\mathcal{P}_w(n,k)$ is a poset. Furthermore, if $X \in \mathrm{Inv}_{k+1}(w)$, then the elements in $P(X)$ form an antichain in $\mathcal{P}_w(n,k)$.*

We now consider admissible orders. The primary modification from the classical admissible orders is a separate condition for quasi-inversions, which is encoded in $\mathcal{P}_w(n,k)$.

Definition 3.6. A total order ρ of $\mathrm{Inv}_k(w)$ is a *k-admissible order* for w if ρ is a linear extension of $\mathcal{P}_w(n,k)$, and for every $X \in \mathrm{Inv}_{k+1}(w)$, the restriction $\rho|_{P(X)}$ is the lex or antilex order on $P(X)$. Let $\mathcal{A}_w(n,k)$ be the *set of k-admissible orders* of $\mathrm{Inv}_k(w)$. Each admissible order has a *reversal set* defined by

$$\mathrm{Rev}_{n,k,w}(\rho) = \{X \in \mathrm{Inv}_{k+1}(w) : \rho|_{P(X)} = (X_1, X_2, \ldots, X_{k+1})\}. \tag{3.2}$$

One can show that the admissible orders in $\mathcal{A}_w(n,1)$ are exactly the permutations in the interval $[\mathrm{id}, w]$ in the weak order on $\mathfrak{S}_n$ if one equates each singleton set $\{i\}$ with the value i. Additionally, the admissible orders in $\mathcal{A}_w(n,2)$ are the reflection orders for w. Examples of admissible orders for $k = 3$ are shown in Table 1.

We next define two operations on admissible orders. Elements $X, Y \in \mathrm{Inv}_k(w)$ *commute with respect to* w if they are incomparable in $\mathcal{P}_w(n,k)$ and do not belong to a common packet. Two total orders of $\mathrm{Inv}_k(w)$ are *commutation equivalent* if they can be obtained from one another using commutations on adjacent elements that commute with respect to w. We will extend $\sim_w$ from reduced words to admissible orders to denote this equivalence relation, and the commutation class of $\rho \in \mathcal{A}_w(n,k)$ will be denoted $[\rho]$.

For any $\rho \in \mathcal{A}_w(n,k)$ and $X \in \mathrm{Inv}_{k+1}(w)$, the packet $P(X)$ is *flippable* in ρ if $P(X)$ forms a saturated chain in the total order ρ. If $\rho|_{P(X)}$ is the lex (resp. antilex) order on $P(X)$, then a *lex-to-antilex packet flip* (resp. *antilex-to-lex packet flip*) at $P(X)$ is the operation that

Admissible order	Reversal Set
$\rho^1 = (134, 124, 135, 136, 125, 126, 156, 256, 146, 246, 356, 346)$	$\varnothing$
$\rho^2 = (134, 256, 135, 136, 156, 356, 124, 126, 146, 246, 346, 125)$	$\{1256\}$
$\rho^3 = (134, 256, 135, 136, 156, 356, 246, 146, 126, 124, 346, 125)$	$\{1246, 1256\}$
$\rho^4 = (134, 256, 356, 156, 136, 135, 124, 126, 146, 246, 346, 125)$	$\{1256, 1356\}$
$\rho^5 = (134, 256, 356, 156, 136, 135, 246, 146, 126, 124, 346, 125)$	$\{1246, 1256, 1356\}$
$\rho^6 = (256, 246, 356, 156, 346, 146, 136, 134, 126, 124, 135, 125)$	$\mathrm{Inv}_4(w)$

Table 1: Six 3-admissible orders for the permutation $w = (6, 4, 5, 2, 3, 1) \in \mathfrak{S}_6$ from Figure 2. Square brackets and internal commas have been suppressed for clarity.

reverses the order of $P(X)$ in ρ. Similarly, the packet $P(X)$ is *flippable* for the equivalence class $[\rho] \in \mathcal{A}_w(n, k)/\sim_w$ if it is flippable for some representative in $[\rho]$.

Example 3.7. In the admissible order ρ^2 from Table 1, each of $P(1246)$ and $P(1356)$ form a saturated chain in lex order, so each packet is lex-to-antilex flippable. Flipping these respectively results in ρ^3 and ρ^4. Additionally, ρ^2 is commutation equivalent to

$$\sigma = (134, 124, 135, 136, 256, 156, 126, 125, 146, 246, 356, 346). \tag{3.3}$$

Observe that $P(1256)$ is antilex-to-lex flippable for σ, and flipping this results in ρ^1.

Lemma 3.8. *Let $\rho \in \mathcal{A}_w(n, k)$.*

(a) The reversal set $\mathrm{Rev}_{n,k,w}(\rho) \subseteq \mathrm{Inv}_{k+1}(w)$ is a (lower) order ideal of $\mathcal{P}_w(n, k+1)$, and its intersection with packet $P(X)$ for $X \in \binom{[n]}{k+2}$ is a prefix or suffix of $P(X)$.

(b) If $\sigma \sim_w \rho$, then $\sigma \in \mathcal{A}_w(n, k)$ and $\mathrm{Rev}_{n,k,w}(\rho) = \mathrm{Rev}_{n,k,w}(\sigma)$.

(c) If σ is obtained from ρ by a packet flip at $P(X)$, then we have that $\sigma \in \mathcal{A}_w(n, k)$ and $\mathrm{Rev}_{n,k,w}(\sigma) \triangle \mathrm{Rev}_{n,k,w}(\rho) = \{X\}$, where $\triangle$ denotes symmetric difference.

Lemma 3.8 shows that $\mathcal{A}_w(n, k)$ is closed under commutations and packet flips. With this in mind, we now define the higher Bruhat orders for any $w \in \mathfrak{S}_n$.

Definition 3.9. The k^{th} *higher Bruhat order for w*, denoted $\mathcal{B}_w(n, k)$, is the partial order on $\mathcal{A}_w(n, k)/\sim_w$ where $[\rho] \leqslant [\sigma]$ if σ can be obtained from ρ by some sequence of commutations and lex-to-antilex packet flips.

From the definition, it is straightforward to show $B_w(n, 1)$ is isomorphic to the weak order interval $[\mathrm{id}, w]$. We will see later from Theorem 1.4 that the Hasse diagram of $\mathcal{B}_w(n, 2)$ can be viewed as the graph $\mathcal{G}(w)$. For an example with $k = 3$, one can use a computer to verify that there are 1228 3-admissible orders for $w = (6, 4, 5, 2, 3, 1)$, and the six admissible orders in Table 1 give a representative from each of the six commutation classes in $\mathcal{A}_w(6, 3)/\sim_w$. The poset $\mathcal{B}_w(6, 3)$ is shown on the left in Figure 3.

Lemma 3.8(a) also describes necessary conditions for reversal sets. Based on this, we extend the notion of consistent sets from [16]. See Figure 3 for examples.

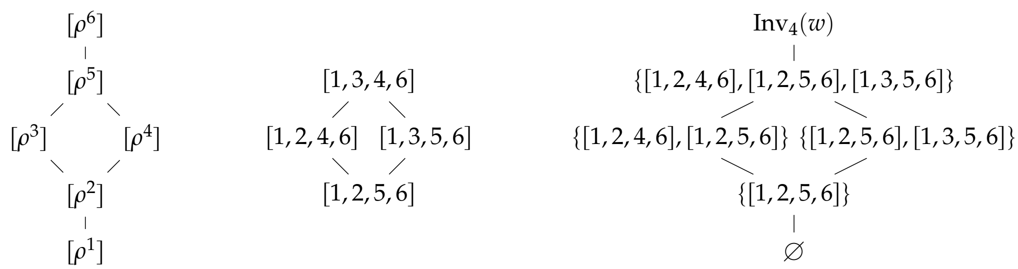

Figure 3: The Hasse diagrams for $\mathcal{B}_w(6,3)$ (left), $\mathcal{P}_w(6,4)$ (middle), and $\mathcal{C}_w(6,4)$ (right) for the permutation $w = (6,4,5,2,3,1)$ from Table 1.

Definition 3.10. A subset $R \subseteq \mathrm{Inv}_k(w)$ is *consistent with respect to* w if R is an order ideal of $\mathcal{P}_w(n,k)$ that satisfies the *Manin-Schechtman-Ziegler (MSZ) Condition*:

for any $X \in \mathrm{Inv}_{k+1}(w)$, the intersection $P(X) \cap R$ is a prefix or suffix of $P(X)$.

Define $\mathcal{C}_w(n,k)$ to be the poset on consistent subsets of $\mathrm{Inv}_k(w)$ with partial order generated by single step inclusion.

We will conclude this section by outlining our approach for proving Theorem 1.4. By Lemma 3.8, $\mathrm{Rev}_{n,k,w} : \mathcal{A}_w(n,k) \to \mathcal{C}_w(n,k+1)$ is well-defined and descends to the quotient $\mathcal{B}_w(n,k)$. To prove this is a bijection, we must show each $R \in \mathcal{C}_w(n,k+1)$ is the reversal set for some admissible order in $\mathcal{A}_w(n,k)$, and each $[\rho] \in \mathcal{B}_w(n,k)$ is uniquely determined by its reversal set. We do this by considering a directed graph G_R on $\mathrm{Inv}_k(w)$ constructed from $R \in \mathcal{C}_w(n,k+1)$. This graph has directed edges

- $X \to Y$ if $X <_{\mathcal{P}_w(n,k)} Y$ is a quasi-inversion relation,
- $X_i \to X_{i+1}$ for all $1 \leqslant i \leqslant k$ if $X \in R$, and
- $X_{i+1} \to X_i$ for all $1 \leqslant i \leqslant k$ if $X \in \mathrm{Inv}_{k+1}(w) \backslash R$.

An example is shown in Figure 4. Observe that the structure of G_R can be complex. Through a technical induction argument on n and k that decomposes G_R into well-behaved subgraphs, we establish the following structural result.

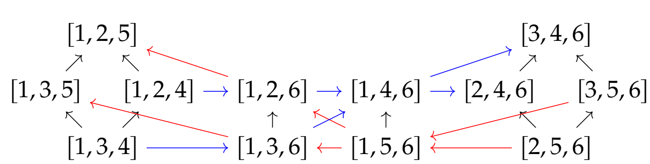

Figure 4: The graph G_R for $w = (6,4,5,2,3,1)$ and $R = \{[1,2,5,6],[1,3,5,6]\} \in \mathcal{C}_w(6,4)$.

Lemma 3.11. *For any $R \in \mathcal{C}_w(n, k+1)$, the directed graph G_R is acyclic.*

As G_R is acyclic, it induces a partial order $\leqslant_R$ on $\mathrm{Inv}_k(w)$. Letting $\mathscr{L}(\mathrm{Inv}_k(w), \leqslant_R)$ denote the set of linear extensions of $(\mathrm{Inv}_k(w), \leqslant_R)$, we establish the following result.

Theorem 3.12. *The map $\mathrm{Rev}_{n,k,w} : \mathcal{B}_w(n,k) \to \mathcal{C}_w(n, k+1)$ is a poset isomorphism with inverse given by $\mathrm{Rev}^{-1}_{n,k,w}(R) = \mathscr{L}(\mathrm{Inv}_k(w), \leqslant_R)$.*

To show that $\mathcal{C}_w(n,k)$ is ranked with unique minimal element $\varnothing$ and maximal element $\mathrm{Inv}_k(w)$, it suffices to show that a k-inversion can be removed or added whenever $R \in \mathcal{C}_w(n,k)$ is not $\varnothing$ or $\mathrm{Inv}_k(w)$, respectively. We do this by defining the *suffix set* for R as $S(R) = \{Y \in \mathrm{Inv}_{k+1}(w) : Y_1 \in R\}$. After showing $S(R) \in \mathcal{C}_w(n, k+1)$, we use $G_{S(R)}$ to establish the following result.

Corollary 3.13. *The poset $\mathcal{C}_w(n,k)$ is a ranked poset with unique minimal element $\varnothing$ and a unique maximal element $\mathrm{Inv}_k(w)$. The rank of $R \in \mathcal{C}_w(n,k)$ is $|R|$.*

For Theorem 1.4, it remains to show the natural bijection between maximal chains of $\mathcal{C}_w(n, k+1)$ and $\mathcal{A}_w(n, k+1)$ is given by using the single step inclusions

$$(\varnothing \lessdot R_1 \lessdot R_2 \lessdot \cdots \lessdot R_{|\mathrm{Inv}_k(w)|}) \mapsto (R_1, R_2 \backslash R_1, \ldots, R_{|\mathrm{Inv}_k(w)|} \backslash R_{|\mathrm{Inv}_k(w)|-1}). \tag{3.4}$$

While proving injectivity is straightforward, proving surjectivity requires a technical argument that utilizes consequences of Theorem 3.12 and Corollary 3.13.

4 Higher Bruhat orders for permutations in $\widetilde{\mathfrak{S}}_n$

In this section, we will generalize the definitions from Section 3 to elements of the affine symmetric group. Throughout, assume $2 \leqslant n$, $1 \leqslant k \leqslant n$, and $w \in \widetilde{\mathfrak{S}}_n$.

In order to generalize the notion of k-inversions to affine permutations accounting for n-periodicity, let $\binom{\mathbb{Z}}{k}$ denote the k-subsets of $\mathbb{Z}$. Define an equivalence relation $\sim_n$ on the elements of $\binom{\mathbb{Z}}{k}$ where $\{x_1, x_2, \ldots, x_k\} \sim_n \{y_1, y_2, \ldots, y_k\}$ if $k > 1$ and there exists an integer m such that $\{x_1, x_2, \ldots, x_k\} = \{y_1 + mn, y_2 + mn, \ldots, y_k + mn\}$. If $k = 1$, then $\{x\} \sim_n \{y\}$ if and only if $x = y$. Denote the affine equivalence classes of size k subsets of $\mathbb{Z}$ with distinct elements modulo n listed in increasing order as

$$\binom{\mathbb{Z}}{k}_n = \big\{[x_1, \ldots, x_k] : x_i \not\equiv x_j \pmod{n} \text{ for } 1 \leqslant i < j \leqslant k\big\}. \tag{4.1}$$

For each $X \in \binom{\mathbb{Z}}{k}_n$, the packet of X is defined similarly as in Section 3.

Adapting the definition from Section 3, $X = [x_1, \ldots, x_k] \in \binom{\mathbb{Z}}{k}_n$ is a *k-inversion* for w provided $w^{-1}(x_1) > \cdots > w^{-1}(x_k)$. We then define the *$k$-inversion set* of w to be

$$\mathrm{Inv}_k(w) = \left\{[x_1, \ldots, x_k] \in \binom{\mathbb{Z}}{k}_n : w^{-1}(x_1) > \ldots > w^{-1}(x_k)\right\}. \tag{4.2}$$

Using the notation of affine equivalences classes in $\binom{\mathbb{Z}}{k}_n$, the following statement is a characterization of 2-inversion sets for affine permutations. The proof follows from [2, Prop 2.1] and [6, Lemma 4.1(d)].

Theorem 4.1. *Let $R \subseteq \binom{\mathbb{Z}}{2}_n$. Then, R is the inversion set for some affine permutation in $\widetilde{\mathfrak{S}}_n$ if and only if for all $[x,y,z] \in \binom{\mathbb{Z}}{3}_n$, we have*

- *$[x,z] \in R$ implies $[x,y] \in R$ or $[y,z] \in R$,*
- *$[x,y] \in R$ and $[y,z] \in R$ implies $[x,z] \in R$, and*
- *$[x,y] \in R$ implies $[x, y-en] \in R$ for all $e \in \mathbb{Z}_{\geqslant 0}$ such that $x \leqslant y - en \leqslant y$.*

The reflection orders for $w \in \widetilde{\mathfrak{S}}_n$ have the property that each ordered prefix must also be a reflection order since each ordered prefix of a reduced word is still a reduced word. Therefore, one can use Theorem 4.1 to characterize reflection orders.

Corollary 4.2. *Let $w \in \widetilde{\mathfrak{S}}_n$. A total order $\rho = (\rho_1, \rho_2, \ldots, \rho_\ell)$ of $\mathrm{Inv}_2(w)$ is a reflection order if and only if*

- *for each $[x,y,z] \in \binom{\mathbb{Z}}{3}_n$ the total order ρ restricted to $P([x,y,z]) \cap \mathrm{Inv}_2(w)$ is a prefix of $P([x,y,z])$ ordered in lex order or a suffix of $P([x,y,z])$ ordered in antilex order, and*
- *for each pair $[x,y], [x,y+n] \in \mathrm{Inv}_2(w)$, the pair $[x,y]$ appears before $[x,y+n]$ in ρ.*

The definition of quasi-inversion in Section 3 carries over similarly from $\mathfrak{S}_n$ to $\widetilde{\mathfrak{S}}_n$. As in $\mathfrak{S}_n$, we define a permanent poset for $w \in \widetilde{\mathfrak{S}}_n$. This is motivated by the weak order on $\widetilde{\mathfrak{S}}_n$, computer experimentation, Theorem 4.1, and Corollary 4.2.

Definition 4.3. Let $v_i^{(n,k)}$ denote the vector $(0, \ldots, 0, n, \ldots, n)$ consisting of i copies of 0 followed by $(k-i)$ copies of n. The *congruence poset* $\mathrm{Cr}(n,k)$ is the poset on $\binom{\mathbb{Z}}{k}_n$ generated by the *congruence relations* for all $X \in \binom{\mathbb{Z}}{k}_n$ and $0 \leqslant i < k$: $X \lessdot_{\mathrm{Cr}(n,k)} X + v_i^{(n,k)}$ if $k-i$ is odd, and $X + v_i^{(n,k)} \lessdot_{\mathrm{Cr}(n,k)} X$ if $k-i$ is even. The *permanent poset* $\mathcal{P}_w(n,k)$ is the set $\mathrm{Inv}_k(w)$ with order relations given by the transitive closure of the quasi-inversion relations of Definition 3.4 and the congruence relations of $\mathrm{Cr}(n,k)$ restricted to $\mathrm{Inv}_k(w)$.

When $k = 1$, we define $\mathcal{A}_w(n,1)$ in such a way that 1-admissible orders for w are in bijection with the interval $[\mathrm{id}, w]$ in weak order. A total order ρ on $\mathrm{Inv}_1(w) \cong \mathbb{Z}$ is a 1-*admissible order* if ρ is a linear extension of $\mathcal{P}_w(n,1)$ with a well-defined finite reversal set in $\binom{\mathbb{Z}}{2}_n$. So $[z_1, z_2] \in \mathrm{Rev}(\rho)$ implies every pair $\{z_1 + mn, z_2 + mn\}$ for all $m \in \mathbb{Z}$ appears in antilex order in ρ. With these definitions in mind, one can also show that $\mathcal{B}_w(n,1) \cong \mathcal{C}_w(n,2)$ is isomorphic to $[\mathrm{id}, w]$ in the weak order on $\widetilde{\mathfrak{S}}_n$.

The k-admissible orders $\mathcal{A}_w(n,k)$ for $k \geqslant 2$ are defined as linear extensions of $\mathcal{P}_w(n,k)$ that satisfy the MSZ condition as in Definition 3.6, but using the affine notion of $\mathcal{P}_w(n,k)$.

Reversal sets, commutations, and packet flips for admissible orders are all defined as in Section 3, using $\binom{\mathbb{Z}}{k}_n$ in place of $\binom{[n]}{k}$. Definitions for $\mathcal{B}_w(n,k)$ as a poset on $\mathcal{A}_w(n,k)/\sim_w$ and $\mathcal{C}_w(n,k)$ as a poset on consistent subsets of $\mathrm{Inv}_k(w)$ carry over mutatis mutandis. The following result establishes the connections between $\mathcal{A}_w(n,2)$, reduced words, and reflection orders.

Lemma 4.4. *For any $w \in \widetilde{\mathfrak{S}}_n$, the admissible orders in $\mathcal{A}_w(n,2)$ are in natural bijection with reflection orders and reduced words of w. Under this bijection, two reduced words*

(a) are commutation equivalent if and only if their corresponding two admissible orders are commutation equivalent, and

(b) differ by a braid $s_i s_{i+1} s_i \to s_{i+1} s_i s_{i+1}$ if and only if their corresponding admissible orders differ by a lex-to-antilex packet flip.

Example 4.5. Let $w = (-3,-2,8,7) \in \widetilde{\mathfrak{S}}_4$. An admissible order in $\mathcal{A}_w(4,2)$ corresponding to the reduced word 232124134 is

$$\rho = ([2,3],[2,4],[3,4],[1,4],[1,3],[2,8],[2,7],[1,8],[1,7]) \tag{4.3}$$

with reversal set $\mathrm{Rev}_{n,k,w}(\rho) = \{[1,3,4],[2,7,8],[1,7,8]\}$. Observe that $\mathrm{Rev}_{n,k,w}(\rho)$ is an order ideal of $\mathcal{P}_w(4,3)$ and is a consistent subset of $\mathrm{Inv}_3(w)$. Applying the braid move $232124134 \to 323124134$ corresponds to a lex-to-antilex packet flip at $P([2,3,4])$.

One can show generalizations of Lemmas 3.5 and 3.8 hold for $w \in \widetilde{\mathfrak{S}}_n$. However, for $R \in \mathcal{C}_w(n,k)$, we must extend the definition of the graph G_R by including edges $X \to Y$ when $X <_{\mathcal{P}_w(n,k)} Y$ is a congruence relation. The key obstruction to proving a complete analog of Theorem 1.4 is our proof that G_R is acyclic, which cannot be directly adapted to affine permutations due to the congruence relations. However, we conjecture that an analog of Lemma 3.11 does hold. As noted in the Introduction, we have computationally verified this conjecture for $1 \leqslant k \leqslant 8$ and affine permutations $w \in \widetilde{\mathfrak{S}}_n$ up to certain lengths.

Conjecture 4.6. *For any $R \in \mathcal{C}_w(n,k)$, the graph G_R is acyclic.*

Through a technical argument, we are able to prove the special case of $k = 3$. Combining this with Lemma 4.4 and generalizations of results in Section 3, we establish Theorem 1.3. We note that if Conjecture 4.6 can be resolved for general k, then this may lead to a full generalization of Theorem 1.4 to $w \in \widetilde{\mathfrak{S}}_n$.

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