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An Eventown Result for Permutations

Nathan Lindzey*[1]

[1]*Department of Mathematical Sciences, University of Memphis, Memphis, TN.*

Abstract. A family of permutations $\mathcal{F} \subseteq S_n$ is *even-cycle-intersecting* if $\sigma\pi^{-1}$ has an even cycle for all distinct $\sigma, \pi \in \mathcal{F}$. We show that if $\mathcal{F} \subseteq S_n$ is an even-cycle-intersecting family of permutations, then $|\mathcal{F}| \leq 2^{n-1}$, and that the bound is met with equality when n is a power of 2 and $\mathcal{F}$ is a double-translate of a Sylow 2-subgroup of the symmetric group. This result can be seen as an analogue of the classical "eventown problem" for subsets, and it also answers a conjecture of János Körner on reversing families of permutations.[2]

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1 Introduction

An *intersecting family* is a subset $\mathcal{F} \subseteq \mathcal{X}$ of some domain $\mathcal{X}$ such that any two elements of $\mathcal{F}$ intersect non-trivially. The Erdős–Ko–Rado theorem states that if $\mathcal{X}$ is the collection of all k-sets of an n-element set, then for $k < n/2$, an intersecting family $\mathcal{F} \subseteq \mathcal{X}$ has size no greater than $\binom{n-1}{k-1}$, and that equality holds if and only if $\mathcal{F}$ is a family obtained by taking all k-sets that contain some fixed element $i \in \{1, 2, \ldots, n\} =: [n]$. Many other seminal results in extremal combinatorics have since appeared under the umbrella of so-called *Erdős–Ko–Rado combinatorics* (see [16] for a comprehensive account).

Recently, the study of intersecting families in finite groups has led to several interesting developments in extremal combinatorics [21, 19, 13]. Many types of intersection have been studied (e.g., t-intersecting [9], setwise-intersection [10], λ-intersection [4]). In each of these cases, the notion of intersection is defined with respect to combinatorial properties of the *difference* gh^{-1} between two group elements $g, h \in G$. Groups of course possess more than just combinatorial structure, so one may try to further understand group structure by studying subsets that are extremal with respect to a pairwise algebraic condition. In this vein, we propose an algebraic notion of intersection that depends on the *order* of gh^{-1}, which we model as non-adjacency in the following natural graph.

Definition 1 (The p-regular graph of G)**.** *Let G be a finite group and p be a prime dividing the order of G. An element $g \in G$ is* p-regular *if its order $|g| \not\equiv 0 \mod p$. Let $G_{p'} \subseteq G$ be the*

*nathan.lindzey@memphis.edu

[2]For the full version of this extended abstract, see [23].

set of p-regular elements. We define the p-regular graph $\Gamma_{p'}(G) := \mathrm{Cay}(G, G_{p'} \backslash \{\mathrm{id}\})$ *to be the Cayley graph of G generated by its non-identity p-regular elements* $G_{p'} \backslash \{\mathrm{id}\}$.

Two elements $g, h \in G$ are non-adjacent in $\Gamma_{p'}(G)$ if gh^{-1} is *p-singular*, i.e. $|gh^{-1}| \equiv 0 \mod p$. The independent sets of $\Gamma_{p'}(G)$ are families of pairwise p-singular elements of G, which we call *(pairwise) p-singular families* of G. A *maximum p-singular family* of G is a p-singular family of G of largest size. By design, the *independence number* of $\Gamma_{p'}(G)$, denoted as $\alpha(\Gamma_{p'}(G))$, is the size of a maximum p-singular family of G.

Since $G_{p'}$ is closed under conjugation by G, the p-regular graph is a normal Cayley graph; therefore, its eigenvalues are determined by the character theory of G (see [5], for example). Erdős and Turán [12] obtained explicit formulas for the largest eigenvalue (i.e. degree) of $\Gamma_{p'}(G)$ when $G = S_n$ is the *symmetric group*. Siemons and Zalesski [28] consider the complement of $\Gamma_{p'}(G)$, i.e. the Cayley graph generated by its p-singular elements $G \setminus G_{p'}$. They were interested in the spectra of such graphs in relation to the problem of constructing Cayley graphs with a singular adjacency matrix. Ebrahimi [8] also investigated this graph for p-solvable groups in a similar vein.

Besides being a natural variation of the Erdős–Ko–Rado problem for groups, these questions seem to have strong connections to Sylow theory, as one might expect. For example, consider the group S_n and let $p = 2$. We say two permutations σ, π are *even-cycle-intersecting* if $\sigma\pi^{-1}$ has an even cycle, equivalently, $|\sigma\pi^{-1}| \equiv 0 \mod 2$. A family of permutations is *even-cycle-intersecting* if any two distinct members are even-cycle-intersecting. Our main result is a tight bound on the size of a maximum 2-singular family of S_n when $n = 2^\ell$.

Theorem 1 (Main Result). *If* $\mathcal{F} \subseteq S_n$ *is even-cycle-intersecting, then* $|\mathcal{F}| \leq 2^{n-1}$ *for all* $n \geq 2$. *The bound is sharp when* $\mathcal{F}$ *is a Sylow 2-subgroup and* $n = 2^\ell$ *for some* $\ell \in \mathbb{N}$.

The result implies a symmetric group analogue of the "eventown problem" for subsets. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be an *eventown* if $|\pi| \equiv 0 \mod 2$ for each $\pi \in \mathcal{F}$ and $|\sigma \cap \pi| \equiv 0 \mod 2$ for all $\sigma, \pi \in \mathcal{F}$ (for more details, see [1, Ch. 1], for example). Answering a question of Erdős, Berlekamp [2] showed that the maximum size of an eventown is $2^{\lfloor n/2 \rfloor}$, and that equality is attained by taking all subsets $S \in 2^{[n]}$ such that $2i \in S \Leftrightarrow 2i - 1 \in S$ for all $1 \leq i \leq n/2$. One can define a family $\mathcal{F} \subseteq S_n$ to be an *eventown* if $|\pi| \equiv 0 \mod 2$ for non-trivial $\pi \in \mathcal{F}$ and $|\sigma\pi^{-1}| \equiv 0 \mod 2$ for all $\sigma, \pi \in \mathcal{F}$ (size and intersection are naturally replaced by order and difference, respectively) in which case a Sylow 2-subgroup is an eventown of maximum size if n is a power of 2.

A family $\mathcal{F} \subseteq S_n$ is a *reversing* if the difference of any two distinct members contains a 2-cycle. It is clear that a reversing family is also even-cycle-intersecting; therefore, our main result proves the following conjecture of János Körner with $C = 2$ (see the full version [23] for a more detailed discussion).

Conjecture 1 (Körner's Conjecture [11, Conjecture 4]). *If $\mathcal{F} \subseteq S_n$ is a reversing family, then $|\mathcal{F}| \leq C^n$ for some constant $C > 1$.*

We find Theorem 1 somewhat surprising. Let S be the union of all Sylow 2-subgroups of S_n. The order of any element of S is a power of 2, a much more restrictive condition than just being 2-singular. Erdős–Turán [12] showed the probability of drawing a 2-singular element of S_n uniformly at random is at least $(1 - 3/\sqrt{n})$, i.e. a typical permutation is 2-singular. The foregoing shows that despite the preponderance of 2-singular elements, we have $\alpha(\Gamma_{p'}(S_{2^\ell})) = \alpha(\mathrm{Cay}(S_{2^\ell}, \bar{S}))$, even though $\Gamma_{p'}(S_{2^\ell}) \subset \mathrm{Cay}(S_{2^\ell}, \bar{S})$ is much sparser.

For primes $p \neq 2$, one can verify for small n that the Sylow p-subgroups of S_n are not maximum p-singular families, which might seem unexpected if it weren't for the fact that the Sylow 2-subgroups of symmetric groups often possess exceptional properties. For example, Diaconis, Giannelli, Guralnick, Law, Navarro, Sambale, and Spink [6] show that a pair of random Sylow p-subgroups of S_n almost always intersect trivially for $p \neq 2$ and $n \to \infty$, whereas if $p = 2$, then such pairs intersect non-trivially with probability at least $(1 - \sqrt{e} + o(1))$. Eberhard [7] later gave a matching upper bound and Renteln [27] found a good algorithm for counting Sylow p-subgroup double cosets of S_n.

As we prove our main result, we point out along the way some perhaps overlooked connections between bijective combinatorics and character theory. For example, we prove a new character-theoretic identity (Theorem 6) that is complementary to a result of Regev [26], leading to another extremal-combinatorial result of minor interest. Our results also have some connection to Steinberg-like representations somewhat recently introduced by Malle and Zalesski. We conclude with a few conjectures, open problems, and directions for future work.

2 Preliminaries and a Proof Sketch

We give a brief overview of some well known bounds on the independence numbers of graphs. For any graph $X = (V, E)$, the *Lovász ϑ-function* $\vartheta(X)$ is the following semidefinite program (SDP):

$$\vartheta(X) = \begin{cases} \min \theta \\ A \succeq 0, \quad A \in \mathrm{Mat}^{n \times n}(\mathbb{R}) \\ A_{i,i} = \theta - 1 \text{ for every } i \in V \\ A_{i,j} = -1 \text{ if } i \not\sim j \end{cases} \tag{2.1}$$

that famously gives an upper bound $\alpha(X) \leq \vartheta(X)$ on the size of a maximum independent set in X. To prove Theorem 1, we solve this SDP defined with respect to the 2-regular graph of S_n for all n that are powers of 2 (in fact, we give an integral extreme point of the associated spectahedron).

If X is a regular graph, then we can obtain a spectral upper bound on $\vartheta(X)$ using a well-known result of Delsarte and Hoffman.

Theorem 2 (Delsarte–Hoffman). *Let A be a weighted adjacency matrix of a regular graph $X = (V, E)$ with constant row and column sum. Let $\eta_{\max}$ and $\eta_{\min}$ be the greatest and least eigenvalue of A, respectively. Then $\alpha(X) \leq \vartheta(X) \leq |V|(-\eta_{\min}/(\eta_{\max} - \eta_{\min}))$. Moreover, if equality holds, then the characteristic vector $1_S \in \mathbb{R}^V$ of a maximum independent $S \subseteq V$ lives in the direct sum of the eigenspaces of A corresponding to $\eta_{\max}$ and $\eta_{\min}$.*

Furthermore, if $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph, i.e. its generating set $S \subseteq G$ is closed under conjugation, then by Schur's lemma, we can determine its eigenvalues by evaluating irreducible characters (see [5], for example).

Theorem 3. *Let G be a finite group and $\{\chi^{(i)}\}_{i=1}^m$ be the set of irreducible characters of G. Then the eigenvalues of the adjacency matrix of a normal Cayley graph $X = \mathrm{Cay}(G, S)$ are given by $\eta^{(i)} = \frac{1}{\chi^{(i)}(1)} \sum_{s \in S} \chi^{(i)}(s)$ for all $i = 1, \ldots, m$ where the multiplicity of $\eta^{(i)}$ is $\chi^{(i)}(1)^2$.*

When G is a symmetric group, its characters can be computed combinatorially as follows. Let χ^λ denote the character of S_n corresponding to the shape $\lambda \vdash n$. A skew shape λ/ν is a *border strip* if it is connected and does not contain the shape ⊞. The *leg-length* $ll(\lambda/\nu)$ of a border strip λ/ν is the number of rows minus 1.

Theorem 4 (Murnaghan–Nakayama). *Let $\sigma \in S_n$ be a permutation with cycle type $\mu \vdash n$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be an integer composition of n that is any reordering of μ. Let $\alpha' = (\alpha_2, \ldots, \alpha_\ell)$ and $\lambda \vdash n$. Then $\chi^\lambda(\sigma) = \chi^\lambda_\mu = \chi^\lambda_\alpha = \sum_\nu (-1)^{ll(\lambda/\nu)} \chi^\nu_{\alpha'}$, where the sum ranges over all ν such that λ/ν is a border strip on α_1 cells.*

Let $G = S_n$, $S = G_{2'} \setminus \{\mathrm{id}\}$, and recall that $\Gamma_{2'}(S_n) = \mathrm{Cay}(G, S) =: X$. The standard approach to prove Theorem 1 at this juncture would be to use the Murnaghan–Nakayama rule to determine the least eigenvalue of the adjacency matrix of X, and observe that the Delsarte–Hoffman bound is met with equality. For small n, one can verify computationally that this approach does not work, and that Delsarte–Hoffman in fact gives rather poor bounds on the independence number of the 2-regular graph of S_n. This leads one to consider the myriad of edge weightings of X, and the difficulty lies in finding a weighting such that the eigenvalues of the weighted adjacency matrix meet the Delsarte–Hoffman bound with equality.

We find this proverbial needle in a haystack by choosing an edge weighting of X informed by a curious bijection on permutations, that evidently has a strong connection to Sylow 2-subgroups of the symmetric group.

3 Bijections

We begin by recalling a complementary pair of identities on permutations that we attribute to Sam Hopkins, recorded here for posterity (see [3, 17, 32] for more discussion).

The first identity concerns elements of the symmetric group S_n that have only odd cycles:

$$n! = \sum_{\substack{\sigma \in S_n \\ |\sigma_i| \not\equiv 0 \bmod 2 \ \forall i}} 2^{\#\text{cyc}(\sigma)-1} \tag{3.1}$$

where $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ are the disjoint cycles that compose σ and $\#\text{cyc}(\sigma) = m$. To see this, we canonically order the permutations of S_n as follows. First, cyclically shift each cycle so that the largest element of each cycle appears first. Next, order the disjoint cycles in ascending order according to their first symbol. We say such a permutation is *canonically ordered*. To each canonically ordered $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in S_n$ with only odd cycles, we associate any bitstring $b = b_1b_2\cdots b_{m-1}$ of length $m-1$. We interleave the cycles and bits

$$\sigma_1 b_1 \sigma_2 b_2 \cdots \sigma_{m-1} b_{m-1} \sigma_m$$

and process them from left to right according to the following rule: if $b_i = 1$, then remove the last symbol from σ_i and place it at the end of σ_{i+1}; otherwise, we do nothing to σ_i. The resulting permutation $\pi \in S_n$ is canonically ordered. Moreover, given a canonically ordered permutation $\pi \in S_n$, one can reverse this process to obtain a unique permutation σ with m cycles, all of odd length, paired with a unique bitstring $b_1\cdots b_{m-1}$. This shows the procedure above gives a bijection, and so Equation (3.1) follows.

A similar identity holds for elements of S_n consisting of only even cycles, provided that n is even:

$$n! = \sum_{\substack{\sigma \in S_n \\ |\sigma_i| \equiv 0 \bmod 2 \ \forall i}} 2^{\#\text{cyc}(\sigma)}. \tag{3.2}$$

The bijection is the same as before, only now we associate an arbitrary bit to each cycle.

Equation (3.1) can be generalized at the level of characters. Let $h_{k,n} := \chi^{(n-k,1^k)}$ and define $H_n := \sum_{k=0}^{n-1} h_{k,n}$ to be the *sum of the hooks character*. The theorem below follows from the Murnaghan–Nakayama rule; however, it was first shown by Regev [26] in a more general form using the theory of Lie superalgebras. Taylor [29] later gave another proof of Regev's general result using skew characters of the symmetric group, and then Wildon [31] gave a short proof of the theorem below using exterior algebras. We give our elementary proof here for completeness, claiming no originality.

Theorem 5. *For all $\sigma \in S_n$, we have $H_n(\sigma) = 2^{\#\text{cyc}(\sigma)-1}$ if σ has no even cycle; 0 otherwise.*

Proof of Theorem 5. We proceed by induction on $m := \#\text{cyc}(\sigma)$, the number of cycles.

If σ is a single odd cycle, then by Theorem 4 we have $H_n(\sigma) = 1$. Similarly, if σ is a single even cycle, then $H_n(\sigma) = 0$.

Assume $\sigma = \sigma_1\cdots\sigma_m$ has no even cycle, $|\sigma_1| = l$, and $\sigma' := \sigma_2\cdots\sigma_m$. Theorem 4 gives

$$H_n(\sigma) = \sum_{k=0}^{n-1} h_{k,n}(\sigma) = \sum_{k=0}^{n-1} h_{k-l,n-l}(\sigma') + h_{k,n-l}(\sigma')$$

noting that, by default, if $k < l$, then $h_{k-l,n-l}(\sigma') = 0$, and if $n - k \leq l$, then $h_{k,n-l}(\sigma') = 0$. This gives

$$H_n(\sigma) = \sum_{k=l}^{n-1} h_{k-l,n-l}(\sigma') + \sum_{k=0}^{n-l-1} h_{k,n-l}(\sigma') = 2\sum_{j=0}^{n-l-1} h_{j,n-l}(\sigma') = 2H_{n-l}(\sigma') = 2^{\#\mathrm{cyc}(\sigma)-1}$$

where the last equality holds by induction.

Now suppose $\sigma = \sigma_1 \cdots \sigma_m$ has an even cycle, say σ_1 without loss of generality. Let $|\sigma_1| = l$ and $\sigma' := \sigma_2\sigma_3 \cdots \sigma_m$. By Theorem 4 we have

$$H_n(\sigma) = \sum_{k=0}^{n-1} h_{k,n-l}(\sigma') - h_{k-l,n-l}(\sigma').$$

Similarly, this gives $H_n(\sigma) = H_{n-l}(\sigma') - H_{n-l}(\sigma') = 0$, which completes the proof. □

We are unaware of any previous work that generalizes Equation (3.2) at the level of characters, so we now prove the result complementary to the theorem above. Here, we let $b_{k,n} := \chi^{(n-k,k)}$ be our shorthand for a two-row character, and let $B_n := \sum_{k=0}^{n/2}(-1)^k b_{k,n}$.

Theorem 6. *Let n be even. For all $\sigma \in S_n$, we have $B_n(\sigma) = 2^{\#\mathrm{cyc}(\sigma)}$ if σ has no odd cycle; 0 otherwise.*

Proof. We proceed by induction on $m := \#\mathrm{cyc}(\sigma)$, the number of cycles. Let σ be a single even cycle. By Theorem 4, $B_n(\sigma) = b_{0,n}(\sigma) - b_{1,n}(\sigma) + \sum_{k=2}^{n/2-2} b_{k,n}(\sigma) = 1 - (-1) + 0 = 2$.

Assume that $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ has no odd cycle and $m \geq 2$. Let $|\sigma_1| = l \leq n/2$ without loss of generality, and let $\sigma' := \sigma_2 \cdots \sigma_m$.

For each summand $(-1)^k b_{k,n}$ of B_n such that $l \leq n - 2k$, remove l cells from the first row to obtain $b_{k,n-l}$. This gives a rim hook of leg length 0, together giving a total contribution of $B_{n-l}(\sigma')$. For each summand $(-1)^k b_{k,n}$ of B_n such that $l \leq k$, remove l cells from the second row to obtain $b_{k-l,n-l}$. This gives a rim hook of leg length 0, which in turn gives a contribution of $\sum_{k=0}^{n/2-l}(-1)^k b_{k,n-l}(\sigma')$. The only remaining rim hooks to consider are those of leg length 1. The shapes that admit such rim hooks are the $b_{k,n}$'s such that $n - 2k + 2 \leq l$. Since the leg length is 1, after removing the rim hook, each such summand $(-1)^k b_{k,n}$ of B_n gives $(-1)^{k+1} b_{k',n-l}$ for some $k' \not\equiv k \mod 2$. In particular, we obtain a total contribution of $\sum_{k=n/2-l+1}^{(n-l)/2}(-1)^k b_{k,n-l}(\sigma')$. Summing up all three cases gives $2B_{n-l}(\sigma')$, which equals 2^m by induction.

Assume $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ has an odd cycle σ_1. Since n is even, we can assume $l := |\sigma_1| \leq n/2$. Let $\sigma' := \sigma_2 \cdots \sigma_m$. A similar argument shows that $B_n(\sigma) = 0$. For convenience, define $B'_n(\sigma) := B_n(\sigma) - (-1)^{n/2} b_{n/2,n}$.

Similar to the argument above, one has that the rim hooks of leg length 0 on the top row together contribute $B'_{n-l}(\sigma')$, but since l is odd, the rim hooks of leg length 0 on the bottom row now collectively contribute $-\sum_{k=0}^{n/2-l}(-1)^k b_{k,n-l}(\sigma')$. Similarly, the remaining rim hooks of leg length 1 now contribute $-\sum_{k=n/2-l+1}^{(n-l)/2-1}(-1)^k b_{k,n-l}(\sigma')$. Altogether, this gives $B'_{n-l}(\sigma') - B'_{n-l}(\sigma') = 0$, as desired. □

Note that Equation (3.1) and Equation (3.2) are recovered by the orthogonality relations, i.e. $\langle 1_{S_n}, H_n\rangle_{S_n} = n! = \langle 1_{S_n}, B_n\rangle_{S_n}$ where 1_{S_n} denotes the trivial representation of S_n. In Section 4 we use these characters to prove Theorem 1 along with another new, but less surprising combinatorial result concerning odd-cycle-intersecting permutations.

4 Proof of Theorem 1

We say that a family of permutations of S_n is *odd-cycle-intersecting* if the difference of any two elements in the family has an odd cycle. If n is odd, then S_n itself is odd-cycle-intersecting, so we assume n is even.

Theorem 7. *Let n be even. If $\mathcal{F} \subseteq S_n$ is an odd-cycle-intersecting family, then $|\mathcal{F}| \leq (n-1)!$. Moreover, equality holds if and only if $\mathcal{F}$ is a double translate of the Young subgroup $S_{n-1} \times S_1$.*

Proof. Let H_λ be the hook product of $\lambda \vdash n$, and let $E_\lambda := (H_\lambda^{-1}\chi^\lambda(\sigma\pi^{-1}))_{\pi,\sigma}$ denote the orthogonal projection onto the λ-isotypic component of the space of real-valued functions on S_n. Let $\mathcal{E}_n \subseteq S_n$ be the set of permutations that have no odd cycle. Any odd-cycle-intersecting family is an independent set of $\mathrm{Cay}(S_n, \mathcal{E}_n)$. By Theorem 6, the $n! \times n!$ matrix $\mathrm{EC}_n := \sum_{k=0}^{n/2}(-1)^k H_{(n-k,k)} E_{(n-k,k)}$ is a weighted adjacency matrix for $\mathrm{Cay}(S_n, \mathcal{E}_n)$ with constant row sum and column sum. The hook products $H_{(n)}$ and $H_{(n-1,1)}$ are the largest and second largest hook products arising in the summation, thus the greatest eigenvalue is $n!$ and the least eigenvalue is $-n!/(n-1)$. Delsarte–Hoffman gives $|S| \leq n!\frac{n!/(n-1)}{n!+n!/(n-1)} = (n-1)!$. The Young subgroups $S_{n-1} \times S_1$ all have an odd (singleton) cycle, so the bound is met with equality. The latter implies that the characteristic vector of any maximum odd-cycle-intersecting family lies in the direct sum of the (n) and $(n-1,1)$-isotypic components. Ellis, Friedgut, and Pilpel [9] show that the only Boolean functions with $(n-1)!$-many 1s that lie in this subspace are double translates of $S_{n-1} \times S_1$, completing the proof. □

Here, the combinatorial result is a bit unsurprising, as EC_n is also a weighted subgraph of *the derangement graph*, i.e. the Cayley graph on S_n generated by its derangements. It is well known that its maximum independent sets are the double translates of $S_{n-1} \times S_1$ (see [16], for example), and it has been observed that its independence number does not rise when passing to many natural conjugacy-closed subgraphs [20, 14]. (We also note that this is not an analogue of the "oddtown problem", despite appearances.)

Before we begin the proof of Theorem 1, we recall some basic facts about Sylow 2-subgroups of the symmetric group. We refer the reader to [27, 30] for more details. By definition, the order of each element in a Sylow p-subgroup P_n is a power of p. For $p = 2$, it is well known that P_n can be identified with the automorphism group of a forest F of m complete binary trees on n leaves, i.e. $P_n \cong \mathrm{Aut}(F) \cong \mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_m) \subseteq S_n$. In particular, write $n = 2^{h_1} + \cdots + 2^{h_m}$ where $h_1 > \cdots > h_m \geq 0$. Each h_i denotes the height of T_i, and $\mathrm{Aut}(T_i)$ is the h_i-fold iterated wreath product of the form $C_2 \wr \cdots \wr C_2$. Legendre's formula gives the exponent of the largest power of a prime p that divides $n!$, which for $p = 2$ gives us $|P_n| = 2^{\sum_{i\geq 1} \lfloor n/2^i \rfloor}$. When n is a power of 2, say $n = 2^\ell$, we have $|P_n| = 2^{1+2+\cdots+2^{\ell-1}} = 2^{2^\ell - 1} = 2^{n-1}$. We collect the foregoing observations into the following proposition.

Proposition 1. *Any Sylow 2-subgroup $P_n \leq S_n$ is a 2-singular family of permutations such that $|P_n| \leq 2^{n-1}$, and equality holds if and only if n is a power of 2.*

We are now in a position to give a proof of Theorem 1.

Proof of Theorem 1. Let H_λ be the hook product of $\lambda \vdash n$, and let $E_\lambda = (H_\lambda^{-1} \chi^\lambda(\sigma\pi^{-1}))_{\pi,\sigma}$ be the orthogonal projection onto the λ-isotypic component of the space of real-valued functions on S_n. Define the $n! \times n!$ matrix $\mathrm{OC}_n := \sum_{k=0}^{n-1} H_{(n-k,1^k)} E_{(n-k,1^k)}$. Theorem 5 implies that $\mathrm{OC}_n' := \mathrm{OC}_n - 2^{n-1} I$ is a weighted adjacency matrix for the 2-regular graph of S_n with constant row and column sum. In particular, we have

$$\mathrm{OC}_n' = \sum_{k=0}^{n-1} \left(n \cdot (n-k-1)! \cdot k! - 2^{n-1}\right) E_{h_{k,n}} + \sum_{\substack{\lambda \vdash n \\ \lambda \text{ not a hook}}} -2^{n-1} E_\lambda;$$

thus -2^{n-1} is the least eigenvalue and $n! - 2^{n-1}$ is the largest eigenvalue of OC_n'. Delsarte–Hoffman gives $|S| \leq n! \cdot \frac{2^{n-1}}{n! - 2^{n-1} + 2^{n-1}} = 2^{n-1}$. Now let $n = 2^\ell$ and let $S = P_n \leq S_n$ be a Sylow 2-subgroup. Proposition 1 implies that the Sylow 2-subgroups and their double-translates, i.e. $\sigma P_n \pi$ for some $\sigma, \pi \in S_n$, are largest independent sets of the 2-regular graph of S_n. This completes the proof. □

We now collect a few corollaries of Theorem 1. Letting J be the $n! \times n!$ all-ones matrix, it is easy to see that $\mathrm{OC}_n - J$ is, remarkably, an *integral* feasible solution to the SDP (2.1) that minimizes the objective function when n is a power of 2. Since $\Gamma_{2'}(S_n)$ is a normal Cayley graph, Theorem 1 and a result of Lovász [24] implies that $\vartheta(\overline{\Gamma_{2'}(S_n)}) = n!/2^{n-1}$ when n is a power of 2. Another corollary of our main result is a simple proof of a result of Giannelli on the induced representation of the trivial representation of a Sylow 2-subgroup (e.g., see [22, Prop 3.5]).

Corollary 1 (Giannelli [15])**.** *Let n be a power of 2 and let $P_n \leq S_n$ be a Sylow 2-subgroup of S_n. Then the non-trivial hook-shaped irreducible representations of S_n are not irreducible constituents of the permutation representation of S_n acting on S_n/P_n.*

Proof. Equality is met in the Delsarte–Hoffman bound, so the characteristic vector of any Sylow 2-subgroup P_n is orthogonal to the direct sum of the irreducibles indexed by non-trivial hooks, i.e. no non-trivial hook appears in the induced representation $1 \uparrow_{P_n}^{S_n}$. □

5 Steinberg-like Characters

The significance of the sum of hooks character has not gone unnoticed by group theorists, as it shares some remarkable similarities with a distinguished irreducible character of $\mathrm{GL}_{n,q} := \mathrm{GL}_n(\mathbb{F}_q)$ known as *the Steinberg character*. For any finite field $\mathbb{F}_q$ of characteristic p, it is well-known that the Steinberg character vanishes on the p-singular elements of $\mathrm{GL}_{n,q}$ and its degree is the size of a largest Sylow p-subgroup of $\mathrm{GL}_{n,q}$ (see [18], for example). The sum of hooks character indeed mimics properties of the Steinberg character in characteristic 2, which raises the question of whether there are other characters that mirror the Steinberg character for various primes. In pursuit of this, Malle and Zalesski [25] proposed the following.

Definition 2 (Syl_p-regular, p-vanishing, Steinberg-like [25])**.** *Let G be a finite group, let p be a prime, and let $S \leq G$ be its Sylow p-subgroup. A character χ is* Syl_p-regular *if the restriction of χ to its Sylow p-subgroup S is the character of the regular representation of S. If, in addition, χ is* p-vanishing, *i.e. $\chi(g) = 0$ for all p-singular elements $g \in G$, then we say that χ is* Steinberg-like*.*

As noted in [25], the sum of hooks character H_n is Steinberg-like with respect to $p = 2$. It is not hard to see that Theorem 1 generalizes to any Steinberg-like character of a group.

Theorem 8. *If χ is a Steinberg-like character of a finite group G with respect to some prime p and $\langle \chi, 1 \rangle = 1$, then its Sylow p-subgroups are maximum p-singular families of G.*

This gives a combinatorial way to check whether there exists a Steinberg-like character with respect to p such that $\langle \chi, 1 \rangle = 1$. In particular, if one can show there exists a p-singular family that is larger than a Sylow p-subgroup, then such characters cannot exist. From the SDP (2.1), one can easily deduce the following more general theorem.

Theorem 9. *For any union S of conjugacy classes of G, if $\mathrm{Cay}(G, \bar{S} \setminus \{\mathrm{id}\})$ has an independent set of size $d+1$, then there exists no S-vanishing character χ such that $\langle \chi, 1 \rangle = 1$ and $\chi(1) = d$.*

Building off a series of papers, Malle and Zalesski [25] essentially classified the Steinberg-like characters of finite simple groups. They do not exist for many natural classes of groups for primes $p \neq 2$, and in fact, the existence of the sum of hooks character for $n = 2^\ell$ is what stymied their classification. It is an open question whether Steinberg-like characters of S_n exist for $p = 2$ and $n \neq 2^\ell, 2^\ell + 1$.

Finally, it is not difficult to show that the Steinberg character St itself gives a non-trivial bound on the maximum size of a p-singular family of $\mathrm{GL}_{n,q}$ when q is a power of p. For this, we require another bound, incomparable to the Lovász-ϑ and Delsarte–Hoffman, based on matrix rank.

Theorem 10 (Haemers)**.** *For any graph $X = (V, E)$ and a field K, a $|V| \times |V|$ matrix M with entries in K is said to* fit *X if $m_{i,i} = 1$ for all $i \in V$ and $m_{i,j} = m_{j,i} = 0$ whenever $ij \notin E$. Let $M_K(X)$ to be the set of all matrices over K that fit X. Then $\alpha(X) \leq \mathcal{H}_K(X) := \min\{\mathrm{rank}_K(M) : M \in M_K(X)\}$.*

Theorem 11. *Let $q = p^a$. Then for all $n \geq 2$, we have $\alpha(\Gamma_{p'}(\mathrm{GL}_{n,q})) \leq q^{n^2-n} = o(|\mathrm{GL}_{n,q}|)$.*

Proof. It is well-known that the absolute value of the character value of St on an element g equals the order of a Sylow subgroup of the centralizer of g if g has order prime to p, and is zero if the order of g is divisible by p (see [18], for example). Therefore, the orthogonal projection matrix E_{St} onto the St-isotypic component is a $|\mathrm{GL}_n(q)| \times |\mathrm{GL}_n(q)|$ matrix that fits the graph $\Gamma_{p'}(\mathrm{GL}_n(q))$ after scaling so that the diagonal entries are 1. Then the degree of the Steinberg character is the order of a Sylow p-subgroup, which is isomorphic to its subgroup of unitriangular matrices. Thus we have $\mathrm{St}(1) = q^{\binom{n}{2}}$ which shows $\mathrm{rank}(E_{\mathrm{St}}) = \chi(1)^2 = q^{n^2-n}$. Haemers' bound gives $\alpha(\Gamma_{p'}(\mathrm{GL}_n(q))) \leq q^{n^2-n} = o(|\mathrm{GL}_n(q)|)$ for fixed q and $n \to \infty$. Note $|\mathrm{GL}_n(q)| = q^{\binom{n}{2}} \prod_{i=1}^{n}(q^i - 1)$ and $\prod_{i=1}^{n}(q^i - 1) \approx q^{\binom{n+1}{2}}$. □

Similar bounds also hold for other finite classical groups with a Steinberg representation.

6 Conjectures and Future Work

We believe the Sylow 2-subgroups of S_n are precisely the extremal families for all $n \geq 2$.

Conjecture 2. *The Sylow 2-subgroups and their double-translates are the extremal even-cycle-intersecting families of S_n for all $n \geq 2$.*

For odd primes, it is unclear what the bound should be for the size of a maximum p-singular family of S_n, but for finite general linear groups, we conjecture the following.

Conjecture 3. *Let $p \in \mathbb{N}$ be a prime and q be a power of p. Then the Sylow p-subgroups and their double-translates are the extremal p-singular families of $\mathrm{GL}_{n,q}$ for all $n \geq 2$.*

Finally, it may be interesting to generalize Theorem 6 to other families of representations of S_n along the lines of Regev's generalization of Theorem 5 [26].

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