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On asymptotics of unitriangular groups

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Abstract. Enumerating irreducible representations of the group $\mathrm{UT}_n(q)$ of $n \times n$ upper unitriangular matrices over a finite field of order q has proven to be a difficult problem. When n is small, it has been computed that the number of irreducible representations of $\mathrm{UT}_n(q)$ is a polynomial in q, but it is not known if this holds true for all n. It is known however that their degrees are powers of q. We investigate the number $N_{n,e}(q)$ of irreducible representations of $\mathrm{UT}_n(q)$ of degree q^e when e is large and show that their asymptotics involve counting 321-avoiding permutations with a given number of inversions. We also propose a conjecture which entails similar results for $N_{n,e}(q)$ when e is small, and also a conjecture on the asymptotics of the number of conjugacy classes.

Keywords: representation theory of unitriangular groups, asymptotics, supercharacter theory, 321-avoiding permutation, 3-noncrossing partition

1 Introduction

For an integer $n \geq 1$ and prime power q, we let $\mathrm{UT}_n(q)$ denote the $n \times n$ *upper unitriangular group over* $\mathbb{F}_q$, consisting of $n \times n$ upper triangular matrices with entries in the finite field $\mathbb{F}_q$ and 1s on the diagonal.

The unitriangular group $\mathrm{UT}_n(q)$ is an instance of an *algebra group*, that is, a group of the form $1 + J$ for a finite-dimensional nilpotent $\mathbb{F}_q$-algebra J. Isaacs [13] proved that the irreducible complex representations of algebra groups have q-power degrees. In our case, by [11], the character degrees for $\mathrm{UT}_n(q)$ are precisely q^e for $0 \leq e \leq \mu(n)$, where

$$\mu(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We aim to study the number $N_{n,e}(q)$ of irreducible representations of $\mathrm{UT}_n(q)$ of degree q^e. We briefly mention some history of this problem. In 1974, Lehrer [15] conjectured that $N_{n,e}(q)$ is a polynomial in q, and later Isaacs [14] conjectured further that it has non-negative integer coefficients when expressed as a polynomial in $q-1$. Evseev [9] presented an algorithm to compute for algebra groups the number of irreducible representations of each degree. Using this, he verified these conjectures for $n \leq 13$. Tables of $N_{n,e}(q)$ for $n \leq 13$ can be found in [14, 9].

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In this paper, we do not aim to compute $N_{n,e}(q)$, but content with studying their asymptotics. For large degrees, we obtain the following asymptotics:

Theorem 1.1. *Let $n \geq 1$ and $0 \leq k \leq \frac{m^2}{4}$, where $m = \lceil \frac{n}{2} \rceil$. Then as $q \to \infty$,*

$$N_{n,\mu(n)-k}(q) \sim a_{m,k} q^{n-m+k},$$

where $a_{m,k}$ is the number of 321-avoiding permutations in $\mathfrak{S}_m$ with exactly k inversions.

This result accounts for the asymptotics for about 25% of all possible degrees. The range for k above guarantees that $a_{m,k} > 0$, see also OEIS [18] sequence A140717.

For small degrees, Marberg [16] has computed $N_{n,e}(q)$ for $e \leq 8$ and $n > 2e$. In this direction, we have the following conditional result:

Proposition 1.2. *Let $n \geq 1$ and $1 \leq e \leq \frac{n}{2} - 1$. Assuming that Conjecture 2.7 holds, then as $q \to \infty$,*

$$N_{n,e}(q) \sim (n - 2e - 1)q^{n+e-2}.$$

We remark that for intermediate degrees, the asymptotics of $N_{n,e}(q)$ seem more difficult to understand. For example, the degrees of the polynomials $N_{n,e}(q)$ (when $n \leq 13$) are less well-behaved. See also Remark 4.5.

Summing $N_{n,e}(q)$ over all $e \geq 0$, we obtain the total number $N_n(q)$ of irreducible representations (equivalently, conjugacy classes) of $\mathrm{UT}_n(q)$. In 1960, Higman [10] conjectured that $N_n(q)$ is a polynomial in q. Arregi and Vera-López [22] computed $N_n(q)$ for $n \leq 13$. More recently, Pak and Soffer [19] extended this to $n \leq 16$ by counting so-called coadjoint orbits. They also conjecture that Higman's conjecture is false for $n \geq 59$.

There is a gap between the current best upper [20] and lower [10] bounds for $N_n(q)$. The following conjecture, together with the discussion in Section 5, suggests a potential strategy towards closing this gap.

Conjecture 1.3. *Let $n \geq 1$. Then as $q \to \infty$,*

$$N_n(q) \sim K_n q^{\lfloor n(n+6)/12 \rfloor},$$

where K_n is a positive integer, explicitly described in Section 5.

The exponent $\lfloor n(n+6)/12 \rfloor$ above was already noted in [22]. Very recently, Diaconis and Zhong [8] gave a statistical method to approximate $N_n(q)$. Their findings support the factor of $\frac{1}{12}$. We have verified that our K_n agrees with the data from [19] for all $n \leq 16$.

Our approach relies on the supercharacter theory for $\mathrm{UT}_n(q)$ developed by André [1, 2] and Yan [23]. We outline this theory in the next section. We shall see that the leading coefficients that arise in our results above count certain collections of set partitions of $[n] = \{1, 2, \ldots, n\}$.

2 Supercharacter theory

For this section, we fix an integer $n \geq 1$ and a prime power q. The supercharacter theory of André and Yan provides a useful coarsening of the set of irreducible characters of $\mathrm{UT}_n(q)$ that turns out to be amenable to computation.

2.1 Definitions

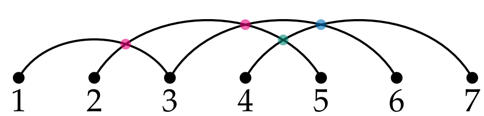
Our definitions and notation will mostly follow [3, 14, 16, 17].

Let Λ be a set partition of $[n]$; we denote this by $\Lambda \vdash [n]$. We call the elements of Λ its *parts*, and write $\ell(\Lambda)$ for the number of parts of Λ. For $1 \leq i < j \leq n$, we say that (i,j) is an *arc* of Λ if i and j are consecutive elements in the same part of Λ, with respect to the usual ordering on $[n]$. We let $\mathsf{Arc}(\Lambda)$ denote the set of arcs of Λ. Note that $|\mathsf{Arc}(\Lambda)| = n - \ell(\Lambda)$. If $(i,k),(j,l) \in \mathsf{Arc}(\Lambda)$, then we say that $((i,k),(j,l))$ is a *crossing* of Λ if $1 \leq i < j < k < l \leq n$. We let $\mathsf{Cr}(\Lambda) = \{(i,j) \mid ((i,k),(j,l)) \text{ is a crossing of } \Lambda \text{ for some } k,l\}$.

Example 2.1. Let $\Lambda = \{\{1,3,6\},\{2,5\},\{4,7\}\} \vdash [7]$. We have

$$\mathsf{Arc}(\Lambda) = \{(1,3),(2,5),(3,6),(4,7)\} \quad \text{and} \quad \mathsf{Cr}(\Lambda) = \{(1,2),(2,3),(2,4),(3,4)\}.$$

Its *standard representation* is shown below, with its crossings highlighted.



An $\mathbb{F}_q$*-labeled set partition of* $[n]$ is a set partition $\Lambda \vdash [n]$ together with a labeling of its arcs by elements of $\mathbb{F}_q^\times$, that is, a function $\mathsf{Arc}(\Lambda) \to \mathbb{F}_q^\times$. By abuse of notation, we will omit the labeling and simply say that $\Lambda \vdash [n]$ is an $\mathbb{F}_q$-labeled set partition.

Let $\Phi = \{(i,j) \mid 1 \leq i < j \leq n\}$ denote the set of *positive roots*. For $(i,j) \in \Phi$, we write e_{ij} for the $n \times n$ matrix with a 1 in row i and column j, and zeros elsewhere. We say that a subset $\mathcal{C} \subseteq \Phi$ is a *closed pattern* if $(i,j),(j,k) \in \mathcal{C}$ implies that $(i,k) \in \mathcal{C}$. For a closed pattern $\mathcal{C}$, we define its associated *pattern subgroup* to be

$$\mathsf{C} = \{x \in \mathrm{UT}_n(q) \mid x_{ij} = 0 \text{ for } (i,j) \in \Phi \setminus \mathcal{C}\},$$

which is in fact a subgroup of $\mathrm{UT}_n(q)$.

Now fix a nontrivial linear character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. Supercharacters are characters of $\mathrm{UT}_n(q)$ that are indexed by $\mathbb{F}_q$-labeled set partitions of $[n]$. Let $\Lambda \vdash [n]$ be an $\mathbb{F}_q$-labeled set partition. We let $a_{ij} \in \mathbb{F}_q^\times$ denote the label of an arc (i,j). Define the closed pattern

$$\mathcal{N} = \Phi \setminus \bigcup_{(i,j) \in \mathsf{Arc}(\Lambda)} \{(i,k) \mid i < k < j\}$$

and let N denote the associated pattern subgroup. Define the linear character $\lambda : N \to \mathbb{C}^\times$ by

$$\lambda(x) = \prod_{(i,j)\in \mathsf{Arc}(\Lambda)} \psi(a_{ij}x_{ij}). \tag{2.1}$$

The *supercharacter* associated to the $\mathbb{F}_q$-labeled set partition Λ is the induced character $\xi = \lambda^{\mathrm{UT}_n(q)}$. We call the underlying set partition Λ the *shape* of the supercharacter ξ.

Remark 2.2. In André's original works, supercharacters were referred to as *basic characters*. The first use of the term 'supercharacter' in the literature in this context appears to be in [4]. Yan's approach to the supercharacter theory of $\mathrm{UT}_n(q)$ was later generalized by Diaconis and Isaacs to more general algebra groups [7].

2.2 Basic properties

We now describe the key properties of supercharacters that allow us to study the asymptotics of $N_{n,e}(q)$. The most important among these is

Theorem 2.3 (André [2, Theorem 1])**.** *Let χ be an irreducible character of $\mathrm{UT}_n(q)$. Then there is a unique $\mathbb{F}_q$-labeled set partition $\Lambda \vdash [n]$ such that χ is a constituent of the supercharacter associated to Λ.*

As observed in [16], if two supercharacters have the same shape $\Lambda \vdash [n]$, then they are related by an automorphism of $\mathrm{UT}_n(q)$, hence decompose in the same way. Let $N_{\Lambda,e}(q)$ denote the number of irreducible constituents of degree q^e of any supercharacter of shape Λ. Since there are $(q-1)^{|\mathsf{Arc}(\Lambda)|}$ possible $\mathbb{F}_q$-labelings of Λ, Theorem 2.3 implies

$$N_{n,e}(q) = \sum_{\Lambda\vdash[n]} (q-1)^{|\mathsf{Arc}(\Lambda)|} N_{\Lambda,e}(q). \tag{2.2}$$

Thus, in order to study the asymptotics of $N_{n,e}(q)$, we need to identify the set partitions $\Lambda \vdash [n]$ which contribute, and prove that the rest do not. For the former, results of Marberg [16] allow us to compute $N_{\Lambda,e}(q)$ using Evseev's algorithm, at least when n is small, and hence conjecture the appropriate subset of set partitions. For proofs, we need some results on how supercharacters decompose.

Fix an $\mathbb{F}_q$-labeled set partition $\Lambda \vdash [n]$ and let λ be the linear character of N as in Section 2.1. Recall the corresponding closed pattern $\mathcal{N}$, and define $\mathcal{S} = \mathcal{N} \cup \mathsf{Cr}(\Lambda)$, which is a closed pattern by [3, Proposition 2]. Let S denote its associated pattern subgroup, which contains N as a normal subgroup. λ is fixed under conjugation by elements of S, so $\lambda^S|_N = [S : N]\lambda$. Let ϕ be a constituent of λ^S. Then $\phi|_N$ is a constituent of $\lambda^S|_N$, so $\phi|_N = \phi(1)\lambda$. It follows by Frobenius reciprocity that the multiplicity of any irreducible constituent ϕ of λ^S is $\langle \phi, \lambda^S\rangle = \langle \phi|_N, \lambda\rangle = \phi(1)$. Hence, λ^S decomposes as

$$\lambda^S = \sum_{\substack{\phi\in\mathrm{Irr}(S)\\ \langle\phi,\lambda^S\rangle>0}} \phi(1)\phi, \tag{2.3}$$

where $\operatorname{Irr}(S)$ denotes the set of irreducible characters of S.

The following result implies that it suffices to understand the irreducible constituents of λ^S instead of the supercharacter $\xi = \lambda^{\mathrm{UT}_n(q)}$.

Theorem 2.4 (André [3, Theorem 2]). *The map $\phi \mapsto \phi^{\mathrm{UT}_n(q)}$ defines a bijection between the set of irreducible constituents of λ^S and the set of irreducible constituents of $\xi = \lambda^{\mathrm{UT}_n(q)}$. Moreover, this bijection preserves multiplicities.*

Thus, $N_{\Lambda,e}(q)$ is the number of irreducible constituents of λ^S of degree equal to $q^e[\mathrm{UT}_n(q) : S]^{-1} = q^{e-d(\Lambda)+|\mathsf{Cr}(\Lambda)|}$, where

$$d(\Lambda) = \sum_{(i,j)\in\mathsf{Arc}(\Lambda)} (j-i-1).$$

In particular, $N_{\Lambda,e}(q) = 0$ if $e < d(\Lambda) - |\mathsf{Cr}(\Lambda)|$. This is however not sharp, that is, the minimum $e \geq 0$ such that $N_{\Lambda,e}(q) \neq 0$ is often greater than $d(\Lambda) - |\mathsf{Cr}(\Lambda)|$. We illustrate this in the next section.

2.3 Special cases

We now study a class of set partitions whose supercharacters have a particularly simple decomposition.

Following [5], we say that a set partition $\Lambda \vdash [n]$ is 3-*crossing* if there are $1 \leq i < j < k < i' < j' < k' \leq n$ such that $(i,i'), (j,j'), (k,k') \in \mathsf{Arc}(\Lambda)$, otherwise it is 3-*noncrossing*.

A *k-chain* in Λ is a sequence $1 \leq i_0 < i_1 < \ldots < i_{k+2} \leq n$ such that $(i_j, i_{j+2}) \in \mathsf{Arc}(\Lambda)$ for all $0 \leq j \leq k$. A *maximal chain* in Λ is a k-chain that cannot be extended to a $(k+1)$-chain. We call k the *length* of the maximal chain. Note that maximal chains partition the set of crossings of Λ, i.e. every crossing is contained in a unique maximal chain. Define $\mathsf{Cr}_{\mathrm{odd}}(\Lambda)$ to be the subset of $\mathsf{Cr}(\Lambda)$ consisting of those (i,j) that occur at odd positions in their maximal chain when counted from right to left. More precisely, if $1 \leq i_0 < i_1 < \ldots < i_{k+2} \leq n$ is a maximal chain in Λ, then for $0 \leq j \leq k-1$, $(i_j, i_{j+1}) \in \mathsf{Cr}_{\mathrm{odd}}(\Lambda)$ if and only if $k-j$ is odd.

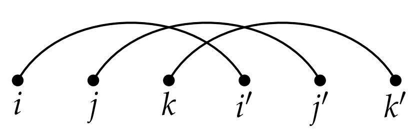


Figure 1: A 3-crossing.

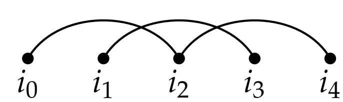


Figure 2: A 2-chain.

Let $\mathcal{P} = \mathcal{N} \cup \mathsf{Cr}_{\mathrm{odd}}(\Lambda)$, which is a closed pattern by [3, Lemma 3]. Let P denote the associated pattern subgroup, so that we have the inclusions $N \leq P \leq S \leq \mathrm{UT}_n(q)$.

Theorem 2.5. *If $\Lambda \vdash [n]$ is a 3-noncrossing set partition, then*

(i) λ^P decomposes as a sum of distinct linear characters;

(ii) $\phi \mapsto \phi^S$ defines a map from the set of irreducible constituents of λ^P to the set of irreducible constituents of λ^S; and

(iii) the map in (ii) is $[S : P]$-to-1 and surjective.

Proof sketch. The key observation is that the assumption that Λ being 3-noncrossing implies that S/N is abelian, see also the proof of [3, Proposition 2].

We know that λ extends to a linear character μ on P by the same definition (2.1), see the discussion before [3, Proposition 3]. Thus by Gallagher's theorem [6, Theorem 11.5], we have the decomposition

$$\lambda^P = \sum_{\omega \in \mathrm{Irr}(P/N)} \omega(1)\omega\mu = \sum_{\omega \in \mathrm{Irr}(P/N)} \omega\mu$$

into irreducible characters, where the second equality follows from P/N being abelian. This proves (i).

Let $\omega \in \mathrm{Irr}(P/N)$. We need to show that $(\omega\mu)^S$ is irreducible. Since S/N is abelian, ω can be extended to a linear character ω_1 of S/N, see [12, Corollary 5.5]. By [6, Corollary 10.20], $(\omega\mu)^S = \omega_1\mu^S$. By [3, Proposition 3], μ^S is irreducible, so $\omega_1\mu^S$ is also irreducible. We thus have the decomposition

$$\lambda^S = \sum_{\omega \in \mathrm{Irr}(P/N)} (\omega\mu)^S$$

into (not necessarily distinct) irreducible characters, which proves surjectivity.

Now P is normal in S. By Clifford's theorem [6, Proposition 11.4], $\mu^S|_P$ decomposes as a sum of S-conjugates of μ with the same multiplicity $\langle \mu, \mu^S|_P \rangle = \langle \mu^S, \mu^S \rangle = 1$, so $\mu^S|_P$ is a sum of $[S : P]$ distinct linear characters. Let $\phi \in \mathrm{Irr}(P)$ be an irreducible constituent of $\mu^S|_P$. Restricting to N, we see that $\phi|_N$ is a constituent of $\mu^S|_N = [S : P]\lambda$. Thus $\phi|_N = \lambda$, and by Frobenius reciprocity, ϕ is a constituent of λ^P. We see that the set of irreducible constituents of $\mu^S|_P$ coincides with the set of irreducible constituents ϕ of λ^P such that $\phi^S = \mu^S$. Hence the fiber of μ^S under the map in (ii) has cardinality $[S : P]$.

Finally, if $(\omega\mu)^S$ is another irreducible constituent of λ^S, where $\omega \in \mathrm{Irr}(P/N)$, then the map $\phi \mapsto \omega\phi$ defines a bijection between the fibers of μ^S and $(\omega\mu)^S$, proving (iii). □

Consequently, $N_{\Lambda,e}(q)$ in the above situation is much simpler than the general case.

Corollary 2.6. *If $\Lambda \vdash [n]$ is a 3-noncrossing set partition, then*

$$N_{\Lambda,e}(q) = \begin{cases} q^{m_{\mathrm{odd}}(\Lambda)} & \text{if } e = d(\Lambda) - \left|\mathsf{Cr}_{\mathrm{odd}}(\Lambda)\right|, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_{\mathrm{odd}}(\Lambda)$ denotes the number of maximal chains in Λ of odd length.

Corollary 2.6 and computational evidence for the general case suggest the following:

Conjecture 2.7. *Let $\Lambda \vdash [n]$ and let $m(\Lambda)$ denote the number of maximal chains in Λ. If $e < d(\Lambda) - \frac{1}{2}(|\mathsf{Cr}(\Lambda)| + m(\Lambda))$, then $N_{\Lambda,e}(q) = 0$.*

It is not true in general that irreducible constituents of λ^P induce to irreducible constituents of λ^S, as the next example shows.

Example 2.8. The set partition Λ of Example 2.1 has three maximal chains, marked in different colors in its standard representation. We have $\mathsf{Cr}_{\mathrm{odd}}(\Lambda) = \{(2,3),(2,4),(3,4)\}$, so $P/N \cong \mathrm{UT}_3(q)$, which has irreducible characters of degrees 1 and q. Let ω be an irreducible character of P/N. If $\omega(1) = 1$, then $(\omega\mu)^S$ is irreducible. If $\omega(1) = q$, then $(\omega\mu)^S$ decomposes into q distinct irreducible characters of degree q. We leave it to the reader to verify these claims, and that

$$N_{\Lambda,e}(q) = \begin{cases} q^2 & \text{if } e = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.9. We followed André's [3] terminology for k-chains. In [16], these are referred to as k-crossings, but this has a different meaning in the literature [5] consistent with our definition of 3-crossing. In [3], $m_{\mathrm{odd}}(\Lambda)$ is the cardinality of the *derived set* of $\mathsf{Arc}(\Lambda)$.

3 Large degrees

Our goal for this section is to sketch a proof of Theorem 1.1. Fix $n \geq 1$ and let $m = \lceil \frac{n}{2} \rceil$ as in the statement of the theorem. We need to identify the set partitions $\Lambda \vdash [n]$ which contribute to the asymptotics in (2.2).

Definition 3.1. A *balanced matching* is a set partition $\Lambda \vdash [n]$ such that

(a) if $n = 2m$ is even, then for all parts $\lambda \in \Lambda$, $\lambda = \{i, j\}$ for some $i \leq m$ and $j > m$;

(b) if $n = 2m - 1$ is odd, then for all parts $\lambda \in \Lambda$, one of the following holds:

  (i) $\lambda = \{i, j\}$ for some $i < m$ and $j > m$; or

  (ii) $\lambda = \{m\}$; or

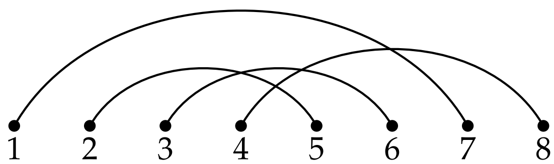
  (iii) $\lambda = \{i, m, j\}$ for some $i < m$ and $j > m$.

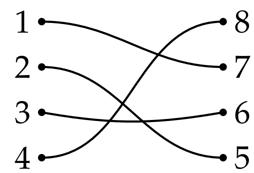
We associate a permutation $\sigma \in \mathfrak{S}_m$ to each balanced matching $\Lambda \vdash [n]$ as follows: For $1 \leq i \leq m$, if there is some $j \in [n]$ such that $(i, j) \in \mathsf{Arc}(\Lambda)$, we set $\sigma(i) = n + 1 - j$. If no such j exists, then $i = m$, and we set $\sigma(m) = m$. Note that

$$\mathsf{Arc}(\Lambda) = \{(i, n+1-\sigma(i)) \mid 1 \leq i \leq m, i \neq n+1-\sigma(i)\},$$

so that we have a bijection between balanced matchings of $[n]$ and permutations in $\mathfrak{S}_m$.

Example 3.2. $\Lambda_1 = \{\{1,6\},\{2,4,7\},\{3,5\}\}$ and $\Lambda_2 = \{\{1,7\},\{2,5\},\{3,6\},\{4,8\}\}$ are balanced matchings with the same associated permutation $\sigma = 2431 \in \mathfrak{S}_4$, written in one-line notation. We draw the standard representation of Λ_2 below. Upon rearranging the vertices, we obtain a braid diagram of σ.





The proofs of the next two lemmas are rather straightforward, so we omit them. The first characterizes balanced matchings:

Lemma 3.3. *Let $\Lambda \vdash [n]$ be a set partition. Then $d(\Lambda) \leq \ell(\Lambda) + \mu(n) - m$, with equality if and only if Λ is a balanced matching.*

Lemma 3.4. *Let $\Lambda \vdash [n]$ be a balanced matching with associated permutation $\sigma \in \mathfrak{S}_m$. Then $|\mathsf{Cr}(\Lambda)| = \ell(\Lambda) + \ell(\sigma) - m$, where $\ell(\sigma)$ denotes the number of inversions of σ.*

These lemmas help show the following result, which together with (2.2) directly shows Theorem 1.1.

Lemma 3.5. *Let $\Lambda \vdash [n]$ be a set partition and let $k \geq 0$. Then*

(i) if Λ is a balanced matching associated to a 321-avoiding permutation σ with k inversions, then $N_{\Lambda,\mu(n)-k}(q) = q^{\ell(\Lambda)+k-m}$;

(ii) otherwise, $N_{\Lambda,\mu(n)-k}(q) \leq q^{\ell(\Lambda)+k-m-1}$.

Proof. Recall the definitions of N, S, and λ in Section 2. $N_{\Lambda,\mu(n)-k}(q)$ is the number of irreducible constituents of λ^S of degree q^f, where $f = \mu(n) - k - d(\Lambda) + |\mathsf{Cr}(\Lambda)|$.

If $f < 0$, then $N_{\Lambda,\mu(n)-k}(q) = 0$. In this case, if Λ is a balanced matching with associated permutation $\sigma \in \mathfrak{S}_m$, then the previous two lemmas imply that $f = \ell(\sigma) - k$, so σ does not have k inversions, and we are in case (ii).

Now assume that $f > 0$ or Λ is not a balanced matching, so that we are again in case (ii). By (2.3), $q^{2f} N_{\Lambda,e}(q) \leq \lambda^S(1) = q^{|\mathsf{Cr}(\Lambda)|}$. It suffices to show that $|\mathsf{Cr}(\Lambda)| - 2f < \ell(\Lambda) + k - m$. By using the definition of f and rearranging, this is equivalent to $d(\Lambda) < \ell(\Lambda) + \mu(n) - m + f$, which holds by Lemma 3.3.

For the remainder of the proof, we assume that $f = 0$ and Λ is a balanced matching with associated permutation $\sigma \in \mathfrak{S}_m$. Since $f = 0$, we have $\ell(\sigma) = k$ and $\mu(n) - k = d(\Lambda) - |\mathsf{Cr}(\Lambda)|$. If σ is 321-avoiding, then we are in case (i), Λ is 3-noncrossing and does not have 2-chains. Hence, Corollary 2.6 implies $N_{\Lambda,\mu(n)-k}(q) = q^{|\mathsf{Cr}(\Lambda)|} = q^{\ell(\Lambda)+k-m}$.

Suppose that σ is not 321-avoiding, then we are in case (ii). Either Λ is 3-crossing or it has a 2-chain. Suppose that Λ is 3-crossing. Define a map

$$\{\phi \in \operatorname{Irr}(S) \mid \langle \phi, \lambda^S \rangle > 0 \text{ and } \phi(1) = 1\} \to \operatorname{Irr}(\mathbb{F}_q)^{\mathsf{Cr}(\Lambda)}$$

by $\phi \mapsto (\phi_{ij})_{(i,j)\in\mathsf{Cr}(\Lambda)}$, where $\phi_{ij}(x) = \phi(1 + xe_{ij})$. This map is injective. Indeed by Frobenius reciprocity, $\phi \in \operatorname{Irr}(S)$ is in the domain if and only if $\phi|_N = \lambda$. Note that S is generated by N and $1 + \mathbb{F}_q e_{ij}$ for $(i,j) \in \mathsf{Cr}(\Lambda)$. Thus ϕ is determined by $\phi|_N$ and ϕ_{ij} for $(i,j) \in \mathsf{Cr}(\Lambda)$. If $1 \le i < j < k < i' < j' < k' \le n$ are such that $(i,i'), (j,j'), (k,k') \in \mathsf{Arc}(\Lambda)$, then $H = 1 + \operatorname{span}_{\mathbb{F}_q}\{e_{ij}, e_{jk}, e_{ik}\}$ is a subgroup of S isomorphic to $\mathrm{UT}_3(q)$. Since $1 + \mathbb{F}_q e_{ik}$ is the commutator subgroup of H, we see that ϕ_{ik} is the trivial character, so $N_{\Lambda,\mu(n)-k}(q) \le q^{|\mathsf{Cr}(\Lambda)|-1} = q^{\ell(\Lambda)+k-m-1}$.

Finally, if Λ is 3-noncrossing but has a 2-chain, then Corollary 2.6 implies that $N_{\Lambda,\mu(n)-k}(q) = 0$. □

4 Small degrees

In this section, we sketch a proof of Proposition 1.2.

Lemma 4.1. *For a set partition $\Lambda \vdash [n]$, we have $m(\Lambda) \le d(\Lambda) - |\mathsf{Cr}(\Lambda)|$.*

Lemma 4.2. *Assume that the conclusion of Conjecture 2.7 holds for the set partition $\Lambda \vdash [n]$. Then for $e \ge 0$, we have $N_{\Lambda,e}(q) \le q^e$.*

Set partitions $\Lambda \vdash [n]$ with at most 2 parts correspond bijectively to compositions of n as follows: Let $\lambda_1 \in \Lambda$ be the part containing 1. Then there is a unique composition $\rho = (\rho_1, \dots, \rho_k)$ of n, such that

$$\lambda_1 = \bigcup_{i\ge 0} \{\rho_1 + \ldots + \rho_{2i} + 1, \ldots, \rho_1 + \ldots + \rho_{2i+1}\}.$$

We call ρ the *type* of Λ and write $\mathsf{type}(\Lambda) = \rho$.

The next lemma characterizes the set partitions which contribute to the asymptotics.

Lemma 4.3. *Let $\Lambda \vdash [n]$ be a set partition with $\ell(\Lambda) = 2$, and let $1 \le e \le \frac{n}{2} - 1$. Then the following are equivalent:*

(i) *Λ does not contain 2-chains and $|\mathsf{Cr}(\Lambda)| = d(\Lambda) - |\mathsf{Cr}(\Lambda)| = e$; and*

(ii) $\mathsf{type}(\Lambda) = (k, 1, \underbrace{2, \ldots, 2}_{e-1}, 1, n - 2e - k)$ *for some $1 \le k \le n - 2e - 1$.*

Proposition 1.2 follows from (2.2) and the following lemma:

Lemma 4.4. *Assume that Conjecture 2.7 holds. Let $\Lambda \vdash [n]$ and $1 \leq e \leq \frac{n}{2} - 1$. Then*

(i) if Λ is of the form in Lemma 4.3 (ii), then $N_{\Lambda,e}(q) = q^e$;

(ii) otherwise, $N_{\Lambda,e}(q) \leq q^{e+\ell(\Lambda)-3}$.

Proof. If $\ell(\Lambda) = 1$, then a supercharacter of shape Λ is linear, so $N_{\Lambda,e}(q) = 0$. If $\ell(\Lambda) \geq 3$, then by Lemma 4.2, $N_{\Lambda,e}(q) \leq q^e \leq q^{e+\ell(\Lambda)-3}$.

Now assume that $\ell(\Lambda) = 2$. We necessarily have that Λ is 3-noncrossing, so Corollary 2.6 applies. If $N_{\Lambda,e}(q) > q^{e-1}$, then

$$m(\Lambda) \geq m_{\mathrm{odd}}(\Lambda) \geq e = d(\Lambda) - \left|\mathsf{Cr}_{\mathrm{odd}}(\Lambda)\right| \geq d(\Lambda) - \left|\mathsf{Cr}(\Lambda)\right|$$

By Lemma 4.1, all the inequalities are equalities. In particular, $\left|\mathsf{Cr}_{\mathrm{odd}}(\Lambda)\right| = \left|\mathsf{Cr}(\Lambda)\right|$ implies that Λ does not contain 2-chains, so $\left|\mathsf{Cr}(\Lambda)\right| = m(\Lambda) = e$. By Lemma 4.3, we are in case (i), and we see that $N_{\Lambda,e}(q) = q^e$ by Corollary 2.6. □

Remark 4.5. It is not true in general that for intermediate e, i.e. those not covered by either Theorem 1.1 or Proposition 1.2, that the only set partitions Λ which contribute to the asymptotics of $N_{n,e}(q)$ in (2.2) are 3-noncrossing. This already fails for $n = 6$ and $e = 3$ with the 3-crossing $\Lambda = \{\{1,4\},\{2,5\},\{3,6\}\}$.

5 Conjugacy classes

As we have seen in previous sections, in the case of large degrees, only 3-noncrossing set partitions contribute to the asymptotics in (2.2), and the same appears to hold for small degrees. We conjecture that this phenomenon continues to hold for the total number of irreducible representations $N_n(q)$. For convenience, we set

$$\nu(n) = \left\lfloor \frac{n(n+6)}{12} \right\rfloor.$$

Conjecture 5.1. *For $n \geq 1$, the maximum of $\left|\mathsf{Arc}(\Lambda)\right| + m_{\mathrm{odd}}(\Lambda)$ over the collection of 3-noncrossing set partitions $\Lambda \vdash [n]$ is equal to $\nu(n)$.*

In fact, the above conjecture seems to hold true with $m(\Lambda)$ in place of $m_{\mathrm{odd}}(\Lambda)$. The next proposition shows that the maximum above is at least $\nu(n)$.

Proposition 5.2. *For $n \geq 1$, there exists a set partition $\Lambda \vdash [n]$ such that Λ is 3-noncrossing, does not have 2-chains, and $\left|\mathsf{Arc}(\Lambda)\right| + m_{\mathrm{odd}}(\Lambda) = \nu(n)$.*

Proof sketch. We assume $n \geq 3$. Set $m = \lceil \frac{n}{2} \rceil$ and $k = m - \frac{n-r}{3}$, where $|r| \leq 1$ is an integer with $r \equiv n \pmod 3$. Define $\Lambda \vdash [n]$ to be the set partition with the following set of arcs:

(i) $(i, m+1-i)$ for $1 \leq i \leq k$,

(ii) $(i, 2m+1-i)$ for $k+1 \leq i \leq m$, and

(iii) $(i, n+m+1-i)$ for $m+1 \leq i \leq n-m+k$.

Then it can be shown that Λ has the required properties. □

Define K_n to be the number of 3-noncrossing set partitions $\Lambda \vdash [n]$ with $|\mathsf{Arc}(\Lambda)| + m_{\mathrm{odd}}(\Lambda) = \nu(n)$, that is, the number of those that attain the maximum of Conjecture 5.1. This is the conjectured coefficient in Conjecture 1.3. Summing (2.2) over $e \geq 0$, we get

$$N_n(q) = \sum_{\Lambda \vdash [n]} (q-1)^{|\mathsf{Arc}(\Lambda)|} N_\Lambda(q),$$

where $N_\Lambda(q)$ is the number of irreducible constituents of a supercharacter of shape Λ. Proposition 5.2 and Corollary 2.6 then imply that for $n \geq 1$,

$$N_n(q) \geq K_n (q-1)^{\nu(n)}.$$

See also [22, Theorem 12]. In order to prove Conjecture 1.3, we also need to show that if $\Lambda \vdash [n]$ is 3-crossing, then $N_\Lambda(q) = o(q^{\nu(n)-|\mathsf{Arc}(\Lambda)|})$.

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