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Combinatorics of Leray models for permutahedral varieties and beyond

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Abstract. We study Bibby, Denham, and Feichtner's Leray model of a matroid for the maximum building set. Using the monomial basis they constructed, we prove the projective Leray model of a matroid is equivariant γ-positive in each graded component. For the Boolean matroid, which corresponds to the permutahedral variety, we show that the Hilbert series of its Leray model is governed by a Catalan-Stieltjes matrix studied in (Cheng, Eu, and Hsu, 2023), which naturally admits a combinatorial interpretation in terms of bicolored permutations. We further construct a bijection between these bicolored permutations and the monomial basis of the Leray model. Then we show that this combinatorial interpretation lifts to the $\mathfrak{S}_n$-module structure on the Leray model, thereby recovering the bicolored Eulerian quasisymmetric functions introduced by (Hyatt, 2012). Finally, we reveal an intriguing connection between the $\mathfrak{S}_n$-representation on the Leray model and the natural Weyl group representation on the cohomology of type B/C permutahedral variety.

Keywords: Leray model, Eulerian quasisymmetric function, bicolored permutations, γ-positivity, Catalan-Stieltjes matrix, Riordan array

1 Introduction

Given a matroid M, the (projective) Orlik-Solomon algebra and the Chow ring $A(M, \mathcal{G})$ with respect to a building set $\mathcal{G}$ are two matroid-invariant graded algebras associated to M. When M is realized by a hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^{\ell+1}$, the former arises as the cohomology of the complement $U(\mathcal{A}) = \mathbb{P}^\ell \setminus \bigcup_{H \in \mathcal{A}} \mathbb{P}H$, while the latter is isomorphic to the cohomology of the *wonderful compactification* $Y(\mathcal{A}, \mathcal{G})$ [7]. The cohomology algebras of $U(\mathcal{A})$ and $Y(\mathcal{A}, \mathcal{G})$ can be viewed within a common framework via the Leray spectral

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sequence associated with the inclusion $U(\mathcal{A}) \hookrightarrow Y(\mathcal{A}, \mathcal{G})$. Motivated by this geometric relationship, Bibby, Denham, and Feichtner [2] combined features of the Orlik-Solomon algebra and the Chow rings to introduce the *Leray model of a matroid*.

Given a loopless matroid M on a ground set E with lattice of flats $\mathcal{L}(M)$ and a building set $\mathcal{G} \subset \mathcal{L}(M)$, Bibby, Denham, and Feichtner define[1] the Leray model of M as the quotient algebra

$$\hat{B}(M, \mathcal{G}) := R(\mathcal{G})/(I + J) \tag{1.1}$$

where $R(\mathcal{G}) = \mathbb{Q}[e_F, x_F : F \in \mathcal{G}]$ is a commutative differential graded algebra with $\deg(e_F) = (0, 1)$, $\deg(x_F) = (2, 0)$, and differential $d(e_F) = x_F$, $d(x_F) = 0$; the ideals

- I is generated by $e_S x_T$ whenever $S \cup T$ is not $\mathcal{G}$-nested in the sense of [7, Def. 2];
- J is generated by $\sum_{F:\, i \in F} x_F$ for each $i \in E$.

When M is realized by a hyperplane arrangement $\mathcal{A}$, the algebra $\hat{B}(M, \mathcal{G})$ coincides with the E_2-page of the Leray spectral sequence associated with the inclusion above, and it also recovers the Morgan model of $\mathcal{A}$ due to De Concini–Procesi [6]. The Chow ring $A(M, \mathcal{G})$ appears naturally as the subalgebra of $\hat{B}(M, \mathcal{G})$ generated by x_F's. Furthermore, Bibby, Denham, and Feichtner [2, Theorem 5.3.1, Corollary 5.3.2] found a Gröbner basis for $I + J$, which restricts on the x-variables to the Feichtner–Yuzvinsky Gröbner basis for $A(M, \mathcal{G})$ [7, Theorem 2]. Consequently, they constructed a monomial basis for $\hat{B}(M, \mathcal{G})$ that extends the Feichtner–Yuzvinsky basis for $A(M, \mathcal{G})$.

This model was further generalized by Pagaria and Pezzoli [14] to polymatroids. Coron [4] employed the theory of Feynman categories to study the Koszulness of the operadic structures on Chow rings and Orlik–Solomon algebras of arbitrary building sets, providing a new interpretation of the Leray models. Beyond these works, little is known about the combinatorial aspect of Leray models of matroids, likely due to the technical nature of their construction.

In this work, we focus on the case where $\mathcal{G}$ is the maximum building set, whose associated Chow ring is commonly referred to as the Chow ring of a matroid. We set $\hat{B}(M) := \hat{B}(M, \mathcal{L}(M) \setminus \{\emptyset\})$ and refer to it as the *Leray model of M*. Using the monomial basis for $\hat{B}(M)$, we prove that the *projective Leray model* $B(M) := \hat{B}(M)/(e_E)$ is equivariant γ-positive (see section 3), which generalizes the work of Liao and others [11, 8]. Let $\mathfrak{S}_n$ denote the symmetric group, i.e., the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. We study the Hilbert series and the $\mathfrak{S}_n$-module structures of the Leray model for the permutahedral variety, namely $\hat{B}(B_n)$ for the Boolean matroid B_n. In this setting, rich enumerative combinatorics emerges (see Table 2), extending the combinatorial and the $\mathfrak{S}_n$-module structures of the Chow ring $A(B_n)$ and augmented Chow ring $\widetilde{A}(B_n)$ described in [12]. We begin by reviewing the relevant results in [12], before extending them to Leray models.

[1] Their definition is more general for *partial building sets*.

y^3	$e_1e_{12}e_{123}$, $e_1e_{13}e_{123}$, $e_2e_{12}e_{123}$, $e_2e_{23}e_{123}$, $e_3e_{13}e_{123}$, $e_3e_{23}e_{123}$ 6		
y^2	e_1e_{12}, e_1e_{13}, e_2e_{12}, e_2e_{23}, e_3e_{13}, e_3e_{23}, e_1e_{123}, e_2e_{123}, e_3e_{123} $e_{12}e_{123}$, $e_{13}e_{123}$, $e_{23}e_{123}$ 12	$e_{12}x_{12}e_{123}$, $e_{13}x_{13}e_{123}$, $e_{23}x_{23}e_{123}$, $e_1e_{123}x_{123}$, $e_2e_{123}x_{123}$, $e_3e_{123}x_{123}$ 6	
y	e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123} 7	e_1x_{123}, e_2x_{123}, e_3x_{123} $e_{12}x_{12}$, $e_{13}x_{13}$, $e_{23}x_{23}$, $x_{12}e_{123}$, $x_{13}e_{123}$, $x_{23}e_{123}$, $e_{123}x_{123}$ 10	$e_{123}x_{123}^2$ 1
y^0	1 1	x_{12}, x_{13}, x_{23}, x_{123} 4	x_{123}^2 1
$\deg(e,x)$	t^0	t^1	t^2

$n = 3$

Table 1: Monomial basis for $\hat{B}(B_3)$

1
1

$n = 1$

2	0
3	1
1	1

$n = 2$

6	0	0
12	6	0
7	10	1
1	4	1

$n = 3$

24	0	0	0
60	36	0	0
50	80	14	0
15	55	25	1
1	11	11	1

$n = 4$

120	0	0	0	0
360	240	0	0	0
390	660	150	0	0
180	630	360	30	0
31	236	276	56	1
1	26	66	26	1

$n = 5$

720	0	0	0	0	0
2520	1800	0	0	0	0
3360	5880	1560	0	0	0
2100	7140	4620	540	0	0
602	3892	4872	1372	62	0
63	889	2114	1134	119	1
1	57	302	302	57	1

$n = 6$

Table 2: Tables for $\dim_{\mathbb{Q}} \hat{B}^{2i,j}(B_n)$. Notice that the bottom rows of these tables are the Eulerian numbers.

1.1 Chow rings of Boolean matroids

If $A = \bigoplus_{(a_1,\ldots,a_k)\in\mathbb{N}^k} A_{(a_1,\ldots,a_k)}$ is a multi-graded algebra carrying a G-module structure which respects the grading for a finite group G, its *equivariant Hilbert series* is defined by

$$\mathsf{Hilb}_G(A; t_1, \ldots, t_k) := \sum_{(a_1,\ldots,a_k)\in\mathbb{N}^k} \left[A_{(a_1,\ldots,a_k)}\right] t_1^{a_1} \cdots t_k^{a_k}$$

where $\left[A_{(a_1,\ldots,a_k)}\right]$ is the isomorphism class of G-modules containing $A_{(a_1,\ldots,a_k)}$ in the representation ring $\mathrm{Rep}_{\mathbb{Q}}(G)$. In particular, when G is the trivial group and $\mathfrak{S}_n$, it is equivalent respectively to the *Hilbert series* and *Frobenius series*:

$$\mathsf{Hilb}(A; t_1, \ldots, t_k) := \sum_{(a_1,\ldots,a_k)\in\mathbb{N}^k} \dim A_{(a_1,\ldots,a_k)} t_1^{a_1} \cdots t_k^{a_k},$$

$$\mathsf{grFrob}(A; t_1, \ldots, t_k) := \sum_{(a_1,\ldots,a_k)\in\mathbb{N}^k} \mathrm{ch}\left(A_{(a_1,\ldots,a_k)}\right) t_1^{a_1} \cdots t_k^{a_k},$$

where ch denotes the Frobenius characteristic map, sending an $\mathfrak{S}_n$-module to a symmetric function. For background on representations of $\mathfrak{S}_n$ and symmetric functions, we refer the reader to [16, 18].

It is well-known that the Chow ring $A(B_n)$ is isomorphic to the cohomology ring of the permutahedral variety. Its Hilbert series $\mathsf{Hilb}(A(B_n); t)$ is given by the *Eulerian polynomial* $A_n(t)$, which may be interpreted as the distribution generating function of the *Eulerian statistics* on $\mathfrak{S}_n$. Two notable Eulerian statistics are the *excedance number* and the *ascent number*: for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$,

$$\mathsf{exc}(\sigma) := |\{i \in [n-1] : i < \sigma_i\}|, \quad \mathsf{asc}(\sigma) := |\{i \in [n-1] : \sigma_i < \sigma_{i+1}\}|.$$

Therefore, one has

$$\mathsf{Hilb}(A(B_n);t) = A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{exc}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{asc}(\sigma)}. \tag{1.2}$$

Equation (1.2) extends to the graded $\mathfrak{S}_n$-module structure on $A(B_n)$. The corresponding Frobenius series is given by the *Eulerian quasisymmetric function*

$$\mathsf{grFrob}(A(B_n);t) = Q_n(\mathbf{x};t) := \sum_{\sigma\in\mathfrak{S}_n} F_{\mathsf{DEX}(\sigma),n}(\mathbf{x}) t^{\mathsf{exc}(\sigma)}, \tag{1.3}$$

where DEX is a descent-type statistic introduced by Shareshian and Wachs [17] (to be defined more generally later), and $F_{S,n}(\mathbf{x})$ denotes the *Fundamental quasisymmetric function*. The generating function of $Q_n(\mathbf{x};t)$ admits the following closed form

$$1 + \sum_{n\geq 1} Q_n(\mathbf{x};t)z^n = \frac{(1-t)H(z)}{H(tz)-tH(z)} \tag{1.4}$$

where $H(z) := \sum_{n\geq 0} h_n(\mathbf{x})z^n$ can be regarded as the symmetric function analog of e^z.

1.2 Augmented Chow ring of Boolean matroids

Analogously, the augmented Chow ring $\widetilde{A}(B_n)$ is isomorphic to the cohomology ring of the stellohedral variety. Its Hilbert series is given by the *binomial Eulerian polynomial* $\widetilde{A}_n(t) := 1 + \sum_{k=1}^{n} \binom{n}{k} A_k(t)$. By considering the set $\widetilde{\mathfrak{S}}_n$ of *decorated permutations of* $[n]$, as defined in [12, Section 6], one obtains

$$\mathsf{Hilb}(\widetilde{A}(B_n);t) = \widetilde{A}_n(t) = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} t^{\mathsf{exc}(\sigma)+1} = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} t^{\mathsf{wex}(\sigma)}, \tag{1.5}$$

where $\mathsf{wex}(\sigma) := |\{i \in [n] : i \leq \sigma_i\}|$. Equation (1.5) extends to the graded $\mathfrak{S}_n$-module structure on $\widetilde{A}(B_n)$ once DEX is generalized to decorated permutations in $\widetilde{\mathfrak{S}}_n$. The Frobenius series is given by the *binomial Eulerian quasisymmetric function*

$$\mathsf{grFrob}(\widetilde{A}(B_n);t) = \widetilde{Q}_n(\mathbf{x};t) = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} F_{\mathsf{DEX}(\sigma),n}(\mathbf{x}) t^{\mathsf{exc}(\sigma)+1} = \sum_{\sigma\in\widetilde{\mathfrak{S}}_n} F_{\mathsf{DEX}(\sigma),n}(\mathbf{x}) t^{\mathsf{wex}(\sigma)}, \tag{1.6}$$

whose generating function has the closed form

$$1 + \sum_{n\geq 1} \widetilde{Q}_n(\mathbf{x};t)z^n = \frac{(1-t)H(z)H(tz)}{H(tz)-tH(z)}. \tag{1.7}$$

$n\backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	$1+t$	$3+t$	2			
3	$1+4t+t^2$	$7+10t+t^2$	$12+6t$	6		
4	$1+11t+11t^2+t^3$	$15+55t+25t^2+t^3$	$50+80t+14t^2$	$60+36t$	24	
5	$1+26t+66t^2+26t^3+t^4$	$31+236t+276t^2+56t^3+t^4$	$180+630t+360t^2+30t^3$	$390+660t+150t^2$	$360+240t$	120

Table 3: $\mathsf{Hilb}(\hat{B}^{\bullet,k}(B_n);t)$ for $n=0,\dots,5$

2 Leray model of permutahedral varieties

For a loopless matroid M with the maximum building set $\mathcal{G}=\mathcal{L}(M)\setminus\{\varnothing\}$, the monomial basis for $\hat{B}(M,\mathcal{G})$ given in [2, Corollary 5.3.2] simplifies considerably as follows.

Lemma 2.0.1. The following set forms a basis for $\hat{B}(M)$

$$FY_{\mathrm{Ler}}(M) := \left\{ e_{F_1}^{\delta_1}x_{F_1}^{a_1}e_{F_2}^{\delta_2}x_{F_2}^{a_2}\cdots e_{F_k}^{\delta_k}x_{F_k}^{a_k} : \substack{\varnothing=F_0\subsetneq F_1\subsetneq F_2\subsetneq\cdots\subsetneq F_k\in\mathcal{L}(M)\\ 0\le a_i\le \mathrm{rk}(F_i)-\mathrm{rk}(F_{i-1})-1}, \substack{(\delta_i,a_i)\neq(0,0)\}\\ \delta_i\in\{0,1\}} \right\}.$$

Let $\hat{B}^{2m,n}(M)=\mathbb{Q}\left\{\prod_i e_{F_i}^{\delta_i}x_{F_i}^{a_i}\in FY_{\mathrm{Ler}}(M)\middle|\ \sum_i a_i=m,\ \sum_i\delta_i=n\right\}$. When $M=B_n$, $\mathcal{L}(M)$ is the Boolean lattice $(2^{[n]},\subseteq)$ and $\mathrm{rk}(F)=|F|$. See Table 1 for examples of the basis elements; Table 2 and 3 for $\dim_{\mathbb{Q}}\hat{B}^{2i,j}(B_n)$ and the corresponding Hilbert series.

2.1 Hilbert series for $\hat{B}(B_n)$

The bigraded Hilbert series of $\hat{B}(B_n)$ is governed by an infinite *(generalized) Catalan matrix*, which is closely related to combinatorics of orthogonal polynomials, lattice paths, and continued fraction expressions of a formal power series (See [1, Ch. 7],[10]).

A Catalan matrix $(a_{n,k})_{n,k\ge0}$ is defined from two nonnegative *seed sequences* $\mathbf{s}=(s_i)_{i=0}^{\infty}$, $\mathbf{t}=(t_j)_{j=1}^{\infty}$ via the recurrence

$$\begin{cases} a_{0,0}=1, a_{0,j}=0, a_{i,-1}=0\ (i\ge0, j>0)\\ a_{n,k}=a_{n-1,k-1}+s_k a_{n-1,k}+t_{k+1}a_{n-1,k+1}\end{cases}.$$

Cheng, Eu, and Hsu [3, after a slight revision of Theorem 3.1] generalized the Eulerian statistics from permutations in $\mathfrak{S}_n$ to partial permutations $\mathfrak{P}_n$ (see Definition 2.1.2) using Catalan matrices. In particular, they showed that the Catalan matrix $(a_{n,k})$ for $(s_k,t_k)=(1+(1+\ t)k, tk^2)$ admits the interpretation

$$a_{n,0}=A_n(t)=\sum_{\sigma\in\mathfrak{S}_n}t^{\mathsf{exc}(\sigma)},\text{ and } a_{n,k}=\sum_{\pi\in\mathfrak{P}_{n,k}}t^{\mathsf{exc}(\pi)}, \tag{2.1}$$

where $\mathfrak{P}_{n,k}$ denotes the set of partial permutations with k zeros (they use Xs instead of zeros in [3]). The first few entries of this Catalan matrix are displayed in Table 4.

Comparing Table 4 and Table 3, we are led to our first main result.

$n\backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	$1+t$	$3+t$	1			
3	$1+4t+t^2$	$7+10t+t^2$	$6+3t$	1		
4	$1+11t+11t^2+t^3$	$15+55t+25t^2+t^3$	$25+40t+7t^2$	$10+6t$	1	
5	$1+26t+66t^2+26t^3+t^4$	$31+236t+276t^2+56t^3+t^4$	$90+315t+180t^2+15t^3$	$65+110t+25t^2$	$15+10t$	1

Table 4: $(a_{n,k})$ with $(s_k, t_k) = (1+(1+\ t)k, tk^2)$

Theorem 2.1.1. Let $(a_{n,k})$ be the Catalan matrix defined by $(s_k, t_k) = (1+(1+t)k, tk^2)$. Then the bigraded Hilbert series of $\hat{B}(B_n)$ is given by

$$\mathsf{Hilb}\left(\hat{B}(B_n); y, t\right) = \sum_{k=0}^{n} \mathsf{Hilb}(\hat{B}^{\bullet,k}(B_n), t) y^k = \sum_{k=0}^{n} k! a_{n,k} y^k.$$

Theorem 2.1.1 and (2.1) together yield a natural interpretation for the Hilbert series of $\hat{B}(B_n)$. Rather than working directly with partial permutations, we will consider partial permutations with all zeros labeled, which are essentially *bicolored permutations*.

Definition 2.1.2. A *bicolored permutation* π of length n with k bars denoted $\mathsf{neg}(\pi) = k$, is a permutation of $[n]$ in which some k entries, say $i_1, i_2, \ldots, i_k$, in the one-line notation are replaced by their barred counterparts $\bar{i_1}, \bar{i_2}, \ldots, \bar{i_k}$. A *partial permutation* can be obtained by replacing the entries with 0's instead. Denote by $\mathfrak{P}^{\text{lab}}_{n,k}$ the set of such bicolored permutations and $\mathfrak{P}^{\text{lab}}_{n} := \bigcup_{k=0}^{n} \mathfrak{P}^{\text{lab}}_{n,k}$.

We extend the usual order on $[n]$ to a partial order on bicolored letters by setting all barred letters to be less than unbarred letters: $\bar{i} < j$ for $i, j \in [n]$. Then we define, for $\pi = \pi_1 \ldots \pi_n \in \mathfrak{P}^{\text{lab}}_{n}$, the *partial excedance set* and *number* of π to be $\mathsf{EXC}(\pi) = \{i \in [n] : i < \pi_i\}$ and $\mathsf{exc}(\pi) = |\mathsf{EXC}(\pi)|$.

The partial excedance on $\mathfrak{P}^{\text{lab}}_{n,k}$ is essentially the pullback of exc on partial permutations $\mathfrak{P}_{n,k}$ along the projection $\mathfrak{P}^{\text{lab}}_{n,k} \to \mathfrak{P}_{n,k}$ which replaces each barred entry $\bar{i}$ as 0.

Corollary 2.1.3. For integers $n \geq 1$, and $k \geq 0$,

$$\mathsf{Hilb}(\hat{B}^{\bullet,k}(B_n); t) = k! \sum_{\pi \in \mathfrak{P}_{n,k}} t^{\mathsf{exc}(\pi)} = \sum_{\pi \in \mathfrak{P}^{\text{lab}}_{n,k}} t^{\mathsf{exc}(\pi)}.$$

Although the introduction of bicolored permutations may appear somewhat artificial at first, it becomes essential when extending the interpretation of the Hilbert series to the $\mathfrak{S}_n$–module structure in Section 2.3.

2.2 Bijection between monomial basis and bicolored permutations

In this section, we establish a bijection between the basis $FY_{\text{Ler}}(B_n)$ and $\mathfrak{P}^{\text{lab}}_{n}$. The restriction of this map on the Feichtner–Yuzvinsky basis $FY(B_n)$ of $A(B_n)$ yields a graded bijection between $(FY(B_n), \deg)$ and $(\mathfrak{S}_n, \mathsf{asc})$, which is also new.

For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{P}_n^{\text{lab}}$, if $\pi_i = j$ or $\bar{j}$, we write $|\pi_i| = j$. The *partial ascent number* of $\pi \in \mathfrak{P}_n^{\text{lab}}$ is defined to be

$$\mathsf{asc}(\pi) := |\{i \in [n-1] : |\pi_i| < |\pi_{i+1}| \text{ and } \pi_{i+1} \text{ is unbarred}\}|.$$

Example 2.2.1. If $\pi = 9\,8\,\bar{1}\,4\,2\,\bar{5}\,\bar{6}\,3\,7 \in \mathfrak{P}_9^{\text{lab}}$, then $\mathsf{asc}(\pi) = |\{3,8\}| = 2$.

Theorem 2.2.2. There is a bijection $\varphi : FY_{\text{Ler}}(B_n) \longrightarrow \mathfrak{P}_n^{\text{lab}}$ such that $e\text{-deg}(\mathbf{m}) = \mathsf{neg}(\varphi(\mathbf{m}))$ and $x\text{-deg}(\mathbf{m}) = \mathsf{asc}(\varphi(\mathbf{m}))$. Moreover, when restricted to the Feichtner–Yuzvinsky basis of $A(B_n)$, $\varphi\,|_{FY(B_n)} : FY(B_n) \longrightarrow \mathfrak{S}_n$ is a bijection such that $\deg(\mathbf{m}_{FY}) = \mathsf{asc}(\varphi(\mathbf{m}_{FY}))$.

For $S \subseteq [n]$, let B_S be the Boolean matroid on the ground set S. We let $\mathfrak{P}_S^{\text{lab}}$ denote the set of bicolored permutations on S equipped with the order inherited from $[n]$; in particular, $\mathfrak{P}_n^{\text{lab}} = \mathfrak{P}_{[n]}^{\text{lab}}$. Below, we define the bijection $\varphi_S :\ FY_{\text{Ler}}(B_S) \ \longrightarrow\ \mathfrak{P}_S^{\text{lab}}$.

Starting with a monomial $\mathbf{m} = e_{F_1}^{\delta_1} x_{F_1}^{a_1} e_{F_2}^{\delta_2} x_{F_2}^{a_2} \cdots e_{F_k}^{\delta_k} x_{F_k}^{a_k} \in FY_{\text{Ler}}(B_n)$ and $S = [n]$, we will construct the bicolored permutation $\varphi(\mathbf{m})$ recursively, starting by determining the position of the largest label in $S = [n]$ in $\varphi(\mathbf{m})$.

1. If n does not appear in the chain $(F_1, \ldots, F_k)$, set $\varphi(\mathbf{m}) = n\varphi_{[n-1]}(\mathbf{m}_{[n-1]})$, where $\mathbf{m}_{[n-1]}$ is the monomial obtained by restricting $\mathbf{m}$ to $[n-1]$.

2. Otherwise, let F_s be the first set in the chain containing n. Split both $[n] \setminus n$ and the monomial $\mathbf{m}$ into two parts:

   - the lower set $L := F_s \setminus n$ and the lower monomial
   $$\mathbf{m}_L = \begin{cases} e_{F_1}^{\delta_1} x_{F_1}^{a_1} \cdots e_{F_{s-1}}^{\delta_{s-1}} x_{F_{s-1}}^{a_{s-1}} e_{F_s \setminus n}^{\delta_s} x_{F_s \setminus n}^{a_s - 1} & \text{if } a_s \neq 0 \\ e_{F_1}^{\delta_1} x_{F_1}^{a_1} \cdots e_{F_{s-1}}^{\delta_{s-1}} x_{F_{s-1}}^{a_{s-1}} & \text{if } a_s = 0,\ \delta_s = 1 \end{cases};$$

   - the upper set $U := [n] \setminus F_s$ and the upper monomial
   $$\mathbf{m}_U = \quad e_{F_{s+1} \setminus F_s}^{\delta_{s+1}} x_{F_{s+1} \setminus F_s}^{a_{s+1}} \cdots e_{F_k \setminus F_s}^{\delta_k} x_{F_k \setminus F_s}^{a_k}.$$

   In the degenerate cases, we adopt the convention that $\mathbf{m}_L = 1$ when $s = 1$ with $F_s = \{n\}$, and $\mathbf{m}_U = 1$ when $s = k$ with $F_s = [n]$. Notice that $\mathbf{m}_L \in FY_{\text{Ler}}(B_L)$ and $\mathbf{m}_U \in FY_{\text{Ler}}(B_U)$. Then set
   $$\varphi(\mathbf{m}) = \begin{cases} \varphi_L(\mathbf{m}_L) \underset{|F_s|^{\text{th}}}{n} \varphi_U(\mathbf{m}_U) & \text{if } a_s \neq 0 \\[2ex] \varphi_L(\mathbf{m}_L) \underset{|F_s|^{\text{th}}}{\bar{n}} \varphi_U(\mathbf{m}_U) & \text{if } a_s = 0,\ \delta_s = 1 \end{cases}.$$

We then apply the same procedure recursively to $\mathbf{m}_S$, where S is the relevant subset ($[n-1]$, L, or U).

Example 2.2.3. Let $\mathbf{m} = x_{12}e_{1269}e_{123679}x^2_{123456789} \in FY_{\mathrm{Ler}}(B_9)$.
Step 1: Insert 9. Since $9 \in e_{1269}$ we decompose with respect to $F_s = \{1,2,6,9\}$:

$$\underbrace{\_\ \_\ \_}_{L}\ \bar{9}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{U}, \qquad L = \{1,2,6\},\ \mathbf{m}_L = x_{12}, \qquad U = \{3,4,5,7,8\},\ \mathbf{m}_U = e_{37}x^2_{34578}.$$

Step 2: Insert 6 *and* 8. Since $6 \notin \mathbf{m}_L$ and $8 \in x^2_{34578}$:

$$6\ \underbrace{\_\ \_}_{LL}\ \bar{9}\ \underbrace{\_\ \_\ \_\ \_}_{UL}\ 8, \qquad LL = \{1,2\},\ \mathbf{m}_{LL} = x_{12}, \qquad UL = \{3,4,5,7\},\ \mathbf{m}_{UL} = e_{37}x_{3457}.$$

Step 3: Insert 2 *and* 7. Since $2 \in \mathbf{m}_{LL}$ and $7 \in e_{37}$:

$$6\ \underbrace{\_}_{LLL}\ 2\,\bar{9}\ \underbrace{\_}_{ULL}\ \bar{7}\ \underbrace{\_\ \_}_{ULU}\ 8, \qquad \mathbf{m}_{ULL} = x_3^0, \qquad \mathbf{m}_{ULU} = x_{45}.$$

Step 4: Insert 3 *and* 5. Since $3 \notin \mathbf{m}_{ULL}$ and $5 \in x_{45}$ we obtain $6\,1\,2\,\bar{9}\,3\,\bar{7}\,4\,5\,8 = \varphi(\mathbf{m})$. This permutation has $\mathsf{neg}(\varphi(\mathbf{m})) = 2$ (from $\bar{9}$ and $\bar{7}$), $\mathsf{asc}(\varphi(\mathbf{m})) = 3$, and $\deg(\mathbf{m}) = 3$.

Theorem 2.2.2 and Corollary 2.1.3 together imply the following enumerative result.

Corollary 2.2.4. The joint distributions $(\mathsf{neg}, \mathsf{exc})$, $(\mathsf{neg}, \mathsf{asc})$ are equidistributed over $\mathfrak{P}_n^{\mathrm{lab}}$.

Remark 2.2.5. While this work was in preparation, we learned from David Speyer that Calvin Yost-Wolff independently discovered what appears to be an inverse formulation of our bijection in the special case of the Chow ring.

2.3 The $\mathfrak{S}_n$-representations on $\hat{B}(B_n)$

To extend Corollary 2.1.3 to the $\mathfrak{S}_n$-module structure on $\hat{B}(B_n)$, it is natural to seek an analogue of DEX in (1.3) for $\mathfrak{P}_n$ and $\mathfrak{P}_n^{\mathrm{lab}}$.

Definition 2.3.1. For $\pi \in \mathfrak{P}_n^{\mathrm{lab}}$, let π^D be a new word obtained from π by replacing π_i with π_i' whenever $i \in \mathsf{EXC}(\pi)$ for all $i \in [n]$. Then under the total order,

$$1' <_D 2' <_D \cdots <_D n' <_D \bar{1} <_D \bar{2} <_D \ldots <_D \bar{n} <_D 1 <_D 2 <_D \ldots <_D n, \tag{2.2}$$

define $\mathsf{DEX}(\pi) := \mathsf{DES}(\pi^D) = \{i \in [n-1] : \pi^D_{i+1} <_D \pi^D_i\}$.

Then one can define a generalization of the Eulerian quasisymmetric function by

$$Q^{\mathrm{lab}}_{n,k}(\mathbf{x}; t) := \sum_{\pi \in \mathfrak{P}^{\mathrm{lab}}_{n,k}} F_{\mathsf{DEX}(\pi),n}(\mathbf{x}) t^{\mathsf{exc}(\pi)}, \tag{2.3}$$

which turns out to be symmetric for all k, hence serves as a natural candidate to the Frobenius series of $\hat{B}^{\bullet,k}(B_n)$.

Remark 2.3.2. Removing all $\bar{i}$ s in (2.2) recovers Shareshian–Wachs definition of DEX on $\mathfrak{S}_n$. Alternatively, replacing all $\bar{i}$ s in (2.2) with a 0 yields a natural definition of DEX on $\mathfrak{P}_n$. However, the analogue of the sum in (2.3) over $\mathfrak{P}_{n,k}$ is not symmetric for all k. This asymmetry motivates the use of the labeled version $\mathfrak{P}^{\mathrm{lab}}_{n,k}$ in Corollary 2.1.3.

On the other hand, for any subgroup $G \leq \mathrm{Aut}(M)$, consider the G-action on $\hat{B}(M)$ induced by G acting on the indices of its variables. The basis $FY_{\mathrm{Ler}}(M)$ in Lemma 2.0.1 forms a permutation basis under this action. This yields an explicit description of the corresponding permutation module structure.

Proposition 2.3.3. Let M be a loopless matroid of rank r. Then $\mathsf{Hilb}_G(\hat{B}(M); y, t)$ equals

$$(1+y) \sum_{S \subseteq [r-1]} \prod_{i=1}^{\ell} \left(t[s_i - s_{i-1} - 1]_t + y[s_i - s_{i-1}]_t\right) [r - s_\ell]_t \ \left[\alpha_{\mathcal{L}(M)}(S)\right].$$

where $\alpha_{\mathcal{L}(M)}(S)$ is the *equivariant flag f-vector* for $S = \{s_1 < s_2 < \cdots < s_\ell\}$, i.e. the permutation module generated by chains in $\mathcal{L}(M)$ having rank set S.

When $M = B_n$ and $G = \mathfrak{S}_n$, note that $\mathrm{ch}(\alpha_{B_n}(S)) = h_{s_1, s_2 - s_1, \ldots, n - s_\ell}$. we obtain from Proposition 2.3.3 the following theorem that generalizes (1.4).

Theorem 2.3.4. We have

$$1 + \sum_{n \geq 1} \mathsf{grFrob}(\hat{B}(B_n); y, t) z^n = \frac{(1-t)H(z)}{(1+y)H(tz) - (y+t)H(z)}.$$

We find the closed formula in Theorem 2.3.4 is a specialization of the N-colored Eulerian quasisymmetric functions ($N = 2, r_0 = r_1 = 1, s_1 = y$) introduced by Hyatt [9]. This identification provides us with the F-expansion of the Frobenius series.

Theorem 2.3.5. For any positive integer n, we have

$$\mathsf{grFrob}\left(\hat{B}(B_n); y, t\right) = \sum_{k=0}^{n} Q^{\mathrm{lab}}_{n,k}(\mathbf{x}; t) y^k = \sum_{\pi \in \mathfrak{P}^{\mathrm{lab}}_n} F_{\mathsf{DEX}(\pi)}(\mathbf{x}) t^{\mathsf{exc}(\pi)} y^{\mathsf{neg}(\pi)}$$

Corollary 2.3.6. For any integer $k \geq 0$, we have

$$\sum_{n \geq 0} \mathsf{grFrob}(\hat{B}^{\bullet,k}(B_n); t) z^n = \sum_{n \geq 0} Q^{\mathrm{lab}}_{n,k}(\mathbf{x}; t) z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)} \left(\frac{(1-t)H(z)}{H(tz) - tH(z)} - 1\right)^k$$

Corollary 2.3.6 specializes to the following corollary for Hilbert series.

Corollary 2.3.7. For any integer $k \geq 0$, we have

$$\sum_{n \geq 0} \mathsf{Hilb}(\hat{B}^{\bullet,k}(B_n); t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z} \left(\frac{(1-t)e^z}{e^{tz} - te^z} - 1\right)^k$$

Remark 2.3.8. Corollary 2.3.7 implies that the Catalan matrix $(a_{n,k})$ defined by $(s_k, t_k) = (1 + (1+t)k, tk^2)$ is an *exponential Riordan array* $\left[\frac{(1-t)e^z}{e^{tz}-te^z}, \frac{(1-t)e^z}{e^{tz}-te^z} - 1\right]$. This is lifted to $\mathfrak{S}_n$-module level in Corollary 2.3.6 and forms a symmetric function analogue of a Riordan array (see [5] for definitions and notations of Riordan array):

$$\left(\mathsf{grFrob}(\hat{B}^{\bullet,k}(B_n); t)\right) = (Q^{\mathrm{lab}}_{n,k}(\mathbf{x}; t))_{n,k\geq 0} = \left(\frac{(1-t)H(z)}{H(tz) - tH(z)}, \frac{(1-t)H(z)}{H(tz) - tH(z)} - 1\right).$$

2.4 Connection to permutahedral variety of type B/C

Since our combinatorial model for $\hat{B}(B_n)$ is given by the bicolored permutations $\mathfrak{P}^{\mathrm{lab}}_n$, which also serves as the combinatorial model of the *hyperoctahedral group* $W(\mathrm{C}_n)$. It is natural to ask whether the $\mathfrak{S}_n$-representation on $\hat{B}(B_n)$ is related to the natural $W(C_n)$-representation on the cohomology ring of the type B/C permutahedral variety X_{C_n}, studied by Stembridge [19]. The answer is affirmative. After forgetting the gradings on both $\hat{B}(B_n)$ and $H^\bullet(X_{C_n})$, we obtain the following result.

Theorem 2.4.1. For any integer $n \geq 1$, we have

$$\hat{B}(B_n) \cong_{\mathfrak{S}_n} H^\bullet(X_{C_n}) \downarrow^{W(C_n)}_{\mathfrak{S}_n} \qquad \text{as } \textbf{ungraded } \mathfrak{S}_n\text{-modules}$$

We proved Theorem 2.4.1 by showing that the generating function in [19, Theorem 7.6] after some specialization is equal to the closed formula in Theorem 2.3.4 after letting $y, t \to 1$. The equality involves tedious computation.

3 γ-positivity for projective Leray model

Let M be a loopless matroid of rank r on the ground set E and $G \leq \mathrm{Aut}(M)$. It is evident from Table 3 that the Hilbert series of $\hat{B}^{\bullet,k}(M)$ is not always palindromic. However, if we consider the *projective Leray model* $B(M) := \hat{B}(M)/(e_E)$ of M, then by [2, paragraph after Def. 5.1.1], one has $\hat{B}(M) \cong B(M) \otimes \mathbb{Q}[e_E]$, and consequently,

$$\mathsf{Hilb}(\hat{B}(M); y, t) = (1+y)\mathsf{Hilb}(B(M); y, t). \tag{3.1}$$

Then $B(M)$ inherits both the bigrading and the graded G-module structure from $\hat{B}(M)$. Its graded components $B^{\bullet,k}(M)$ turn out to have palindromic Hilbert series for all k. Moreover, we prove a stronger result that each $B^{\bullet,k}(M)$ is *G-equivariant γ-positive*.

Definition 3.1. Let $A = \bigoplus_{i=0}^d A_i$ be a graded G-module. We say that A is *(G-)equivariant γ-positive* if its equivariant Hilbert series admits an expansion of the form

$$\mathsf{Hilb}_G(A; t) = \sum_{i=0}^{d} [A_i] t^i = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k t^k (1+t)^{d-2k}$$

where each coefficient $\gamma_k \in \mathrm{Rep}_{\mathbb{Q}}(G)$ is uniquely determined and represents a genuine (i.e. non-virtual) representation of G over $\mathbb{Q}$; equivalently, $\gamma_k \geq_{\mathrm{Rep}_{\mathbb{Q}}(G)} 0$ for all k.

For a subset T of $[r-1]$, let $\mathsf{Stab}(T)$ be the collection of subsets of T that contain no consecutive integers. For any $\mathcal{P} \subseteq 2^{[r-1]}$, define $\mathbb{1}_{\mathcal{P}}(S) = 1$ if $S \in \mathcal{P}$ and 0, otherwise.

Theorem 3.2. For all $0 \leq k \leq r-1$, the projective Leray model $B^{\bullet,k}(M)$ is G-equivariant γ-positive with the γ-expansion of $\mathsf{Hilb}_G(B^{\bullet,k}(M), t)$ given by

$$\sum_{B \in \binom{[r-1]}{k}} \sum_{T \subseteq [r-1]} \mathbb{1}_{\mathsf{Stab}([2,r-1])}(T \setminus B) \left[\beta_{\mathcal{L}(M)}(T)\right] t^{|T \setminus B|}(1+t)^{r-1-\ell-2|T \setminus B|},$$

where $\beta_{\mathcal{L}(M)}(T) := \sum_{S \subseteq T}(-1)^{|T \setminus S|}\alpha_{\mathcal{L}(M)}(S) \cong_G \tilde{H}_{|T|-1}(\mathcal{L}(M)_T)$ is the *equivariant flag h-vector* of $\mathcal{L}(M)$. In particular, $B^{2i,k}(M) \leq_{\mathrm{Rep}_{\mathbb{Q}}(G)} B^{2i+2,k}(M)$ and $B^{2i,k}(M) \cong_G B^{2n-2-2i,k}(M)$.

When $k = 0$, Theorem 3.2 recovers the equivariant γ-positivity of the Chow ring $A(M)$ in [13, 11]. Our proof of Theorem 3.2 is parallel to those in [11, 13], it involves a combinatorial model that generalizes the one in the proof of Lemma 4.1 in [11].

Letting G be the trivial group, then Theorem 3.2 reduces to the usual γ-positivity, unimodality, and palindromicity of $\mathsf{Hilb}(B^{\bullet,k}(M), t)$.

4 Further works

In the full version of this extended abstract, Theorem 2.2.2 is generalized to a bijection between the monomial basis associated with a chordal building set $\mathcal{G}$ and the $\mathcal{G}$-trees of Postnikov-Reiner-Williams [15]. We further study the "augmented" counterpart $\hat{B}^{\mathrm{aug}}(M)$ of the Leray model $\hat{B}(M)$. In particular, for $\hat{B}^{\mathrm{aug}}(B_n)$, which corresponds to the stellohedral variety, we obtain "augmented analogues" of Theorem 2.1.1, Corollary 2.1.3, Theorems 2.3.4 and 2.3.5, with "*decorated bicolored permutations*" providing the corresponding permutation model. These results extend equations (1.5), (1.6), and (1.7).

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References

[1] M. Aigner. *A course in enumeration*. Vol. 238. Graduate Texts in Mathematics. Springer, Berlin, 2007, pp. x+561.

[2] C. Bibby, G. Denham, and E. M. Feichtner. "A Leray model for the Orlik-Solomon algebra". *Int. Math. Res. Not. IMRN* 24 (2022), pp. 19105–19174. DOI.

[3] Y.-J. Cheng, S.-P. Eu, and H.-C. Hsu. "Statistics of partial permutations via Catalan matrices". *Adv. in Appl. Math.* **143** (2023), Paper No. 102451, 30. DOI.

[4] B. Coron. "Matroids, Feynman categories, and Koszul duality". *Duke Math. J.* **174**.11 (2025), pp. 2303–2381. DOI.

[5] D. E. Davenport, S. K. Frankson, L. W. Shapiro, and L. C. Woodson. "An invitation to the Riordan group". *Enumer. Comb. Appl.* **4**.3 (2024), Paper No. S2S1, 26. DOI.

[6] C. De Concini and C. Procesi. "Wonderful models of subspace arrangements". *Selecta Math. (N.S.)* **1**.3 (1995), pp. 459–494. DOI.

[7] E. M. Feichtner and S. Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". *Invent. Math.* **155**.3 (2004), pp. 515–536. DOI.

[8] L. Ferroni, J. P. Matherne, M. Stevens, and L. Vecchi. "Hilbert–Poincaré series of matroid Chow rings and intersection cohomology". *Advances in Mathematics* **449** (2024), p. 109733. DOI.

[9] M. Hyatt. "Eulerian quasisymmetric functions for the type B Coxeter group and other wreath product groups". *Adv. in Appl. Math.* **48**.3 (2012), pp. 465–505. DOI.

[10] M. E. H. Ismail and J. Zeng. "Addition theorems via continued fractions". *Trans. Amer. Math. Soc.* **362**.2 (2010), pp. 957–983. DOI.

[11] H.-C. Liao. "Equivariant γ-positivity for Chow rings and augmented Chow rings of matroids". *arXiv preprint arXiv:2408.00745* (2024). Link.

[12] H.-C. Liao. "Stembridge codes, permutahedral varieties, and their extensions". *arXiv preprint arXiv:2403.10577* (2024). Link.

[13] H.-C. Liao. "Equivariant γ-positivity for Chow rings and augmented Chow rings of matroids". *Sém. Lothar. Combin.* **93B** (2025), Art. 72, 12.

[14] R. Pagaria and G. M. Pezzoli. "Hodge theory for polymatroids". *Int. Math. Res. Not. IMRN* 23 (2023), pp. 20118–20168. DOI.

[15] A. Postnikov, V. Reiner, and L. Williams. "Faces of generalized permutohedra". *Doc. Math.* **13** (2008), pp. 207–273.

[16] B. E. Sagan. *The symmetric group*. Second. Vol. 203. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xvi+238. DOI.

[17] J. Shareshian and M. L. Wachs. "Eulerian quasisymmetric functions". *Adv. Math.* **225**.6 (2010), pp. 2921–2966. DOI.

[18] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI.

[19] J. R. Stembridge. "Some permutation representations of Weyl groups associated with the cohomology of toric varieties". *Adv. Math.* **106**.2 (1994), pp. 244–301. DOI.