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Diagonal supersymmetry for coinvariant rings

John Lentfer*[1]

[1]*Department of Mathematics, University of California, Berkeley, CA, USA*

Abstract. For finite groups G, we show that bosonic-fermionic coinvariant rings have a natural $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module structure. In particular, we show that their character series are a sum of super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ times irreducible characters of G with universal coefficients, which do not depend on k, j. In the case where G is the symmetric group with diagonal action, this proves the "Diagonal Supersymmetry" conjecture of Bergeron (2020).

Keywords: coinvariant rings, super Schur functions, diagonal supersymmetry

1 Introduction

A foundational object in algebraic combinatorics is the coinvariant ring

$$R_n^{(1,0)} = \mathbb{C}[x_1, \dots, x_n] / \langle \mathbb{C}[x_1, \dots, x_n]_+^{\mathfrak{S}_n} \rangle, \tag{1.1}$$

the quotient of a polynomial ring by symmetric functions without constant term. The symmetric group $\mathfrak{S}_n$ permutes the variables and gives $R_n^{(1,0)}$ a graded $\mathfrak{S}_n$-module structure, where the grading is by total degree. Borel [13] showed that $R_n^{(1,0)}$ is the cohomology ring of the flag variety, it affords the regular representation of the symmetric group, and Stanley [41] gave a formula for the graded structure of $R_n^{(1,0)}$ in terms of statistics on standard Young tableaux.

The diagonal coinvariant ring

$$R_n^{(2,0)} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] / \langle \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]_+^{\mathfrak{S}_n} \rangle \tag{1.2}$$

was introduced by Haiman in 1994 [25]. The defining ideal $\langle \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]_+^{\mathfrak{S}_n} \rangle$ is generated by all polynomials in $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ without constant term, which are invariant under the diagonal action of the symmetric group $\mathfrak{S}_n$. Haiman conjectured [25] and then later proved [26] formulas for the dimension, bigraded Hilbert series, and Frobenius series of $R_n^{(2,0)}$ using novel results on the Hilbert scheme of points in $\mathbb{C}^2$.

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The diagonal coinvariant ring $R_n^{(2,0)}$ has been involved in several breakthroughs in algebraic combinatorics, including the proofs of the longstanding Shuffle Conjecture [23, 36, 11] and the Loehr–Warrington Conjecture [34, 9]. It is connected to the elliptic Hall algebra [15, 12, 20] and to link homology [21, 22, 44].

There has been previous work on coinvariant rings with three or more sets of commuting variables [6, 3], however a proof of just the dimension for the case with three sets of variables has been elusive [8].

The study of the coinvariant ring with one set of commuting variables and one set of anticommuting variables $R_n^{(1,1)}$ was initiated by N. Bergeron, Colmenarejo, Li, Machacek, Sulzgruber, and Zabrocki in a working seminar in algebraic combinatorics at the Fields Institute in 2018. Eventually, the multiplicity of the sign character of $R_n^{(1,1)}$ [43], its dimension and Hilbert series [39], a monomial basis [40, 2] and its Frobenius series [37] were all determined. Zabrocki introduced a coinvariant ring with two sets of commuting variables and one set of anticommuting variables $R_n^{(2,1)}$ [45, 5], which is conjecturally related to a symmetric function appearing in the Delta theorem (conjectured by Haglund, Remmel, and Wilson [24] and proven by [19] and also [10]). Then D'Adderio, Iraci, and Vanden Wyngaerd [18] considered a coinvariant ring with two sets of commuting variables and two sets of anticommuting variables, related to their work on Theta operators.

The coinvariant ring with two sets of anticommuting variables $R_n^{(0,2)}$ was studied by Kim and Rhoades, where they determined its Frobenius series, Hilbert series, dimension, and the multiplicity of the sign character [29]. There has been some additional conjectural work done on the coinvariant ring with one set of commuting variables and two sets of anticommuting variables $R_n^{(1,2)}$ in [30, 28].

All of these are instances of coinvariant rings with k sets of n commuting (bosonic, even) variables $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(k)}$, where $\boldsymbol{x}^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$, and j sets of n anticommuting (fermionic, odd) variables $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(j)}$, where $\boldsymbol{\theta}^{(i)} = \{\theta_1^{(i)}, \dots, \theta_n^{(i)}\}$, for k, j nonnegative integers or infinity.

We first describe the construction of the polynomial ring $\mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$. Let $G \subset \mathrm{GL}(V) = \mathrm{GL}(n)$ be a finite group acting on $V = \mathbb{C}^n$. Let $\mathbb{C}^{k|j}$ denote the complex superspace of even dimension k and odd dimension j. Then $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = (\mathrm{Sym}(V))^{\otimes k} \otimes (\bigwedge(V))^{\otimes j}$, and upon choosing a basis of V, we number the coordinates so that they correspond to the tensor factors. That is, $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}$ correspond to coordinates on k copies of the symmetric algebra $\mathrm{Sym}(V)$ and $\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}$ correspond to coordinates on j copies of the exterior algebra $\bigwedge(V)$. Thus we write $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$, where commuting variables commute with all variables; anticommuting variables anticommute with all anticommuting variables. That is, $\theta_i^{(\ell)}\theta_h^{(m)} = -\theta_h^{(m)}\theta_i^{(\ell)}$, which implies that $(\theta_i^{(\ell)})^2 = 0$.

We now define the *(k, j)-bosonic-fermionic coinvariant ring for a finite group $G \subset \mathrm{GL}(n)$*

by

$$R_G^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}] / \langle \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]_+^G \rangle, \tag{1.3}$$

where $\mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]_+^G$ denotes the polynomials without constant term, which are invariant under the diagonal action of G on each set of variables. In the case where G is the symmetric group $\mathfrak{S}_n$ acting diagonally by permuting the indices of the variables within each set, we denote this by $R_n^{(k,j)}$. Coinvariant rings have been studied in this more general setting (see [3, 7, 4, 46]).

A good understanding of $R_n^{(k,j)}$ remains elusive, especially once $k + j \geq 3$. In fact, the only character beyond the trivial character known for any choice of k, j subject to $k + j \geq 3$ is the sign character of $R_n^{(0,3)}$ [32]. Presently, this extended abstract of [31] provides one of the first structural results for arbitrary k, j. The extended abstract is organized as follows. In Section 2, we state the main results, which are:

- a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module structure for $R_G^{(k,j)}$ (Theorem 2.1);
- a multigraded character series for $R_G^{(k,j)}$ using super Schur functions (Theorem 2.2);
- in the case of the symmetric group, a proof of the "diagonal supersymmetry" conjecture of François Bergeron (Corollary 2.3);
- and a construction of the universal series (Proposition 2.5).

Definitions and additional background are presented in Section 3. Two applications are previewed: in Section 4, we discuss a combinatorial consequence of cancellation on the Hilbert series of $R_n^{(k,j)}$, while in Section 5 we summarize the results of [33], which are completely explicit in the case of the dihedral groups. Complete proofs, additional discussion and further applications can be found in the full paper [31].

2 Main Results

In this extended abstract, we propose that a natural setting for studying $R_G^{(k,j)}$ is as a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module, where $U(\mathfrak{gl}(k|j))$ is the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(k|j)$. Recall that a G-representation $G \to \mathrm{GL}(V)$ is equivalent to a $\mathbb{C}[G]$-module [14, §21.1]. In general, given any Lie superalgebra $\mathfrak{g}$, a $\mathfrak{g}$-module is equivalent to a $U(\mathfrak{g})$-module, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ [16, Chapter 1.5.1].

We describe the relevant actions on $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. First, $\mathfrak{gl}(k|j)$ acts on $\mathbb{C}^{k|j}$ as the defining representation. Since $\mathfrak{gl}(k|j)$ acts trivially on $V = \mathbb{C}^n$, then $\mathfrak{gl}(k|j)$ acts on $\mathbb{C}^{k|j} \otimes V$. As $\mathbb{C}^{k|j} \otimes V$ is the degree 1 (in total degree) part of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$, then the action of $\mathfrak{gl}(k|j)$ extends to the whole ring $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ by left superderivations. Next, since $G \subset \mathrm{GL}(V)$, then G acts in a natural way on V, in addition to trivially on $\mathbb{C}^{k|j}$. Combining these, G acts on $\mathbb{C}^{k|j} \otimes V$, which extends to $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. The actions of $\mathfrak{gl}(k|j)$ and G commute with each other (cf. [17, Section 5.2.1]).

With this action in hand, we then show that $R_G^{(k,j)}$ is a quotient module of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. Since $\mathfrak{gl}(k|j)$ is not semisimple, we are not a priori guaranteed that $R_G^{(k,j)}$ will decompose into simple $U(\mathfrak{gl}(k|j))$-modules. Our first main result is that the desired $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module structure exists for $R_G^{(k,j)}$, and we describe how it decomposes into simple modules. The key ingredient in the proof is super Howe duality (Theorem 3.1). Since $R_G^{(k,j)}$ is a quotient module of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$, the irreducible representations of $R_G^{(k,j)}$ are the same as those of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$, just with possibly reduced multiplicity.

Denote the set of partitions λ with length $\ell(\lambda) \leq n$ which satisfy $\lambda_{k+1} \leq j$ by $P(k,j,n)$. Let μ index the irreducible characters of G. Let $s_\lambda(\mathbf{q}/\mathbf{u})$ denote a super Schur function (see Section 3 for definitions), where here and throughout, $\mathbf{q}$ denotes $q_1, \ldots, q_k$, and $\mathbf{u}$ denotes $u_1, \ldots, u_j$.

Theorem 2.1. *Fix a positive integer n and a finite group $G \subset \mathrm{GL}(n)$. There is an isomorphism of $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-modules:*

$$R_G^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu} \left(U_{k|j}^{\lambda} \otimes N^{\mu} \right)^{\oplus c_{\lambda\mu}}, \tag{2.1}$$

where the $U_{k|j}^{\lambda}$ are simple $U(\mathfrak{gl}(k|j))$-modules with character $s_\lambda(\mathbf{q}/\mathbf{u})$ and the N^μ are simple $\mathbb{C}[G]$-modules, for some nonnegative integer coefficients $c_{\lambda\mu}$.

Let $\mathrm{IrrChar}(G)$ denote the set of irreducible characters of G. From the module structure in Theorem 2.1, taking $\mathfrak{gl}(k|j)$ and G-characters yields a character series in terms of super Schur functions and G-characters. A priori, for fixed $G \subset \mathrm{GL}(V)$, these coefficients are functions of k, j. However, in our second main result we show that they are independent of k, j.

Theorem 2.2. *Fix a positive integer n and a finite group $G \subset \mathrm{GL}(n)$. For partitions λ with $\ell(\lambda) \leq n$, and indices μ of irreducible G-characters, there exist nonnegative integer coefficients $c_{\lambda\mu}$ such that for any k, j, the multigraded character series of $R_G^{(k,j)}$ is*

$$\mathrm{Char}(R_G^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu. \tag{2.2}$$

When G is the symmetric group acting by permuting variables, we may apply the Frobenius characteristic map to the symmetric group characters, which are indexed by partitions of n. Let $\mathbf{z}$ be an (infinite) auxiliary set of variables for the Frobenius characters. Then the following result is an immediate consequence of Theorem 2.2. Thus we prove a conjecture of Bergeron.

Corollary 2.3 (Diagonal Supersymmetry [7, Conjecture 1])**.** *Fix a positive integer n. For partitions λ with $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist nonnegative integer coefficients $c_{\lambda\mu}$ such that for any (k, j), the multigraded Frobenius series of $R_n^{(k,j)}$ is*

$$\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}). \tag{2.3}$$

Remark 2.4. Given fixed n, k, j, it is important to note that (2.3) does not determine $c_{\lambda\mu}$ for all partitions λ of length at most n, but only for those which satisfy $\lambda_{k+1} \leq j$.

Previously, Bergeron showed [7, equation (2.1)] that $R_n^{(k,j)}$ is a $\mathrm{GL}(k) \times \mathrm{GL}(j) \times \mathfrak{S}_n$-module, without taking advantage of the relationship between the even and odd parts, which in the present setting comes from the Lie superalgebra structure. The irreducible characters of polynomial representations of $\mathrm{GL}(k)$ are Schur functions s_λ where $\ell(\lambda) \leq k$ (see for example [35, I.A.8]). Taking $\mathrm{GL}(k) \times \mathrm{GL}(j)$ characters and Frobenius characters, one obtains a series which is a product of Schur functions in three sets of variables. That is, for partitions λ with $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist nonnegative integers $c_{\lambda\nu\mu}$ such that for any k, j

$$\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\substack{\lambda,\nu, \\ \ell(\lambda) \leq k, \ell(\nu) \leq j}} \sum_{\mu \vdash n} c_{\lambda\nu\mu} s_\lambda(\mathbf{q}) s_\nu(\mathbf{u}) s_\mu(\mathbf{z}). \tag{2.4}$$

Since a super Schur function $s_\lambda(\mathbf{q}/\mathbf{u})$ can always be written as a sum of products of Schur functions $s_\lambda(\mathbf{q}) s_\nu(\mathbf{u})$, but the converse does not hold, equation (2.3) implies equation (2.4), but equation (2.4) does not imply equation (2.3).

Bergeron showed [7] that there is a well-defined universal series

$$\mathrm{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda,\nu} \sum_{\mu \vdash n} c_{\lambda\nu\mu} s_\lambda(\mathbf{q}) s_\nu(\mathbf{u}) s_\mu(\mathbf{z}), \tag{2.5}$$

which determines all $c_{\lambda\nu\mu}$ which could appear in the expansion in equation (2.4) for any k, j. In equation (2.5), $(k, j) = (\infty, \infty)$ means $\mathbf{q} = q_1, q_2, \ldots$ and $\mathbf{u} = u_1, u_2, \ldots$ are infinite formal alphabets. We strengthen equation (2.5) by showing that there is a well-defined universal series in terms of super Schur functions.

Proposition 2.5. *Fix a positive integer n. There is a well-defined universal series*

$$\mathrm{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(\infty,\infty,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}) \tag{2.6}$$

which determines all $c_{\lambda\mu}$ which could appear in the expansion in equation (2.3) for any (k, j).

In general, $c_{\lambda\mu}$ may be zero or nonzero. For some finite k or j in the expansion of $\mathrm{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u})$ (equation (2.3)), not all nonzero $c_{\lambda\mu}$ from the universal series (equation (2.6)) need to appear, since they are multiplied by $s_\lambda(\mathbf{q}/\mathbf{u})$, which equals 0 when k or j is too small.

Example 2.6. Let $n = 3$. Then one may compute the universal series

$$\begin{aligned}\mathrm{Frob}(R_3^{(\infty,\infty)};\mathbf{q};\mathbf{u}) = &\Big(s_{(3)}(\mathbf{q}/\mathbf{u}) + s_{(1,1)}(\mathbf{q}/\mathbf{u})\Big)\, s_{(1,1,1)}(\mathbf{z}) \\ &+ \Big(s_{(2)}(\mathbf{q}/\mathbf{u}) + s_{(1)}(\mathbf{q}/\mathbf{u})\Big)\, s_{(2,1)}(\mathbf{z}) + s_{\varnothing}(\mathbf{q}/\mathbf{u}) s_{(3)}(\mathbf{z}).\end{aligned} \tag{2.7}$$

An equivalent way to write this using the universal series coefficients is that $c_{(3),(1,1,1)}$, $c_{(1,1),(1,1,1)}, c_{(2),(2,1)}, c_{(1),(2,1)}, c_{\varnothing,(3)}$ are all 1 and $c_{\lambda\mu} = 0$ for all other λ of length at most 3 and for all $\mu \vdash 3$.

Consider if $(k, j) = (1, 0)$. Then the universal series simplifies to

$$\begin{aligned}\mathrm{Frob}(R_3^{(1,0)};q) = &\, s_{(3)}(q) s_{(1,1,1)}(\mathbf{z}) \\ &+ \Big(s_{(2)}(q) + s_{(1)}(q)\Big)\, s_{(2,1)}(\mathbf{z}) + s_{\varnothing}(q) s_{(3)}(\mathbf{z}),\end{aligned} \tag{2.8}$$

since $s_{(1,1)}(q) = 0$. This means that, while $c_{(1,1),(1,1,1)} = 1$ regardless of k, j, we would not be able to determine that from the Frobenius series of $R_3^{(1,0)}$ alone.

We desire some criteria for when computing a Frobenius series of $R_n^{(k,j)}$ for finite k, j allows us to conclude that we have computed enough $c_{\lambda\mu}$ to completely determine the universal series. Here we present one such criteria: once k, j have been fixed, in the special case that $n \le k + j$, then the purely bosonic case or the purely fermionic case completely determines the mixed bosonic-fermionic case. Then we can apply the fact that the Frobenius series of $R_n^{(k,0)}$ stabilizes at $k = n - 1$ [6].

Corollary 2.7. *For fixed $n \le k + j$, the multigraded Frobenius series of $R_n^{(k,j)}$ may be calculated from the multigraded Frobenius series of $R_n^{(k+j,0)}$ or $R_n^{(0,k+j)}$. In particular, the coefficients $c_{\lambda\mu}$ in equation (2.3) are determined by*

$$\mathrm{Frob}(R_n^{(k+j,0)};q_1,\dots,q_{k+j}) = \sum_{\lambda\in P(k+j,0,n)} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(q_1,\dots,q_{k+j}) s_\mu(\mathbf{z}), \tag{2.9}$$

or by

$$\mathrm{Frob}(R_n^{(0,k+j)};u_1,\dots,u_{k+j}) = \sum_{\lambda\in P(0,k+j,n)} \sum_{\mu\vdash n} c_{\lambda\mu} s_{\lambda'}(u_1,\dots,u_{k+j}) s_\mu(\mathbf{z}). \tag{2.10}$$

For more analysis of what information the multigraded Frobenius series of $R_n^{(k,j)}$ tells us about the multigraded Frobenius series of $R_n^{(k',j')}$ for some other choice of k' and j', see the full paper [31].

3 Background and definitions

We assume some familiarity with symmetric functions (see [42, Chapter 7] or [35, Chapter I]), and representation theory of finite groups and Lie algebras (see [14]). All tensor products are taken over $\mathbb{C}$. If a ring R decomposes as a direct sum of multihomogenous $\mathbb{C}[G]$-modules, for G a finite group:

$$R = \bigoplus_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} (R)_{r_1,\ldots,r_k,s_1,\ldots,s_j}, \tag{3.1}$$

then its *multigraded character series* is

$$\mathrm{Char}(R;\mathbf{q};\mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} \mathrm{char}\left((R)_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}, \tag{3.2}$$

where $\mathrm{char}(S)$ denotes the character of a G-module S.

Recall that the Frobenius characteristic map F is an isomorphism between the ring of virtual $\mathfrak{S}_n$-characters and the ring of symmetric functions $\Lambda_{\mathbb{Z}}$, given by $F(\chi^\mu) = s_\mu(\mathbf{z})$, where χ^μ is the irreducible $\mathfrak{S}_n$-character indexed by μ, and $s_\mu(\mathbf{z})$ is a Schur function (see [42, Chapter 7.18]). In the special case where G is the symmetric group $\mathfrak{S}_n$, such as when $R = R_n^{(k,j)}$, we can apply the Frobenius characteristic map F to the character of each multigraded component, and obtain the *multigraded Frobenius series:*

$$\mathrm{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} F\,\mathrm{char}\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}. \tag{3.3}$$

We may use q for q_1 and u for u_1 when $k, j \leq 1$.

Define a *super Schur function* (see [17, Appendix A.2.2]) by

$$s_\lambda(\mathbf{q}/\mathbf{u}) = \sum_{\nu\subseteq\lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}). \tag{3.4}$$

Super Schur functions can also be defined in terms of super tableaux (see [38, Chapter 12.3]). While we will not otherwise use it, we note that in [7] the plethystic notation $s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}]$ is used. Since (see for example [7, Section 2])

$$s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}] = \sum_{\nu\subseteq\lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}), \tag{3.5}$$

then $s_\lambda(\mathbf{q}/\mathbf{u}) = s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}]$. We will exclusively use $s_\lambda(\mathbf{q}/\mathbf{u})$.

Super Schur functions satisfy the following key properties [35, I.3 Example 23] of cancellation:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=-q_k} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_{j-1}), \tag{3.6}$$

and restriction:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_j)|_{q_k=0} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_j), \tag{3.7}$$
$$s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=0} = s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1}). \tag{3.8}$$

The super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ are characters of the simple $\mathfrak{gl}(k|j)$-modules $U^\lambda_{k|j}$ which appear in the decomposition of $\mathrm{Sym}(\mathbb{C}^{k|j}\otimes V)$ via Howe duality (see [27, 17]).

Theorem 3.1 (($\mathfrak{gl}(k|j)$, $\mathrm{GL}(n)$)-Howe Duality)**.** *For all $d \geq 0$, we have that*

$$\mathrm{Sym}^d(\mathbb{C}^{k|j}\otimes V) \cong \bigoplus_{\substack{\lambda\in P(k,j,n)\\ \lambda\vdash d}} U^\lambda_{k|j}\otimes U^\lambda_n, \tag{3.9}$$

where for each $\lambda \in P(k,j,n)$, $U^\lambda_{k|j}$ is a simple $\mathfrak{gl}(k|j)$-module with character the super Schur function $s_\lambda(\mathbf{q}/\mathbf{u})$, and U^λ_n is a simple $\mathrm{GL}(n)$-module with character the Schur function s_λ.

4 Cancellation and the Hilbert series for the symmetric group

The multigraded Hilbert series of $R_n^{(k,j)}$ is defined by

$$\mathrm{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0} \dim\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k}u_1^{s_1}\cdots u_j^{s_j}. \tag{4.1}$$

Recall that we can recover the Hilbert series from the Frobenius series via:

$$\langle \mathrm{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u}), h_{(1^n)}(\mathbf{z})\rangle = \mathrm{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u}). \tag{4.2}$$

We give a Hilbert series version of Corollary 2.3.

Proposition 4.1. *Fix a positive integer n. For partitions λ with $\ell(\lambda)\leq n$, there exist nonnegative integer coefficients c_λ such that for any k, j, the multigraded Hilbert series of $R_n^{(k,j)}$ is*

$$\mathrm{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u}) = \sum_{\lambda\in P(k,j,n)} c_\lambda s_\lambda(\mathbf{q}/\mathbf{u}). \tag{4.3}$$

Now we show a general result on Frobenius and Hilbert series involving certain evaluations.

Proposition 4.2. *For any $n \geq 1$ and for any $0 \leq m \leq \min(k,j)$, we have*

$$\mathrm{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u})|_{q_k=-u_j,\ldots,q_{k-m+1}=-u_{j-m+1}} = \mathrm{Frob}(R_n^{(k-m,j-m)};q_1,\ldots,q_{k-m},u_1,\ldots,u_{j-m}),$$
$$\mathrm{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u})|_{q_k=-u_j,\ldots,q_{k-m+1}=-u_{j-m+1}} = \mathrm{Hilb}(R_n^{(k-m,j-m)};q_1,\ldots,q_{k-m},u_1,\ldots,u_{j-m}).$$

Sagan–Swanson conjectured [40] and Rhoades–Wilson proved [39] that

$$\operatorname{Hilb}(R_n^{(1,1)}; q; u) = \sum_{d=0}^{n} [d]_q! \operatorname{Stir}_q(n,d) u^{n-d}, \tag{4.4}$$

where $[d]_q!$ is a q-analogue of the factorial $d!$ and $\operatorname{Stir}_q(n,d)$ is a q-analogue of the Stirling number of the second kind $\operatorname{Stir}(n,d)$.

As an application of Proposition 4.1, we give a short proof of the following result, originally proven by Sagan–Swanson with a sign-reversing involution.

Corollary 4.3 ([40, Theorem 5.5])**.** *For $n \geq 0$, we have*

$$\sum_{d=0}^{n} [d]_q! \operatorname{Stir}_q(n,d)(-q)^{n-d} = 1. \tag{4.5}$$

Proof. By Proposition 4.2, $\operatorname{Hilb}(R_n^{(1,1)}; q; u)|_{q=-u} = \operatorname{Hilb}(R_n^{(0,0)}) = 1$. The result follows by equation (4.4). □

5 Dihedral groups

For groups with non-constant rank, such as the symmetric group $\mathfrak{S}_n$, it is quite difficult to determine the coefficients $c_{\lambda\mu}$ in general. In the case of dihedral groups $\mathfrak{I}_2(n)$, since the rank of these groups is fixed at 2, their associated coinvariant algebras are easier to describe. As an application of Theorem 2.2, along with a theorem of Alfano [1], we can derive an explicit multigraded character series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$.

Theorem 5.1 ([33])**.** *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded character series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ is given by*

$$\begin{aligned}\operatorname{Char}(R^{(k,j)}_{\mathfrak{I}_2(n)}; \mathbf{q}; \mathbf{u}) = \chi_1 + s_{(1,1)}(\mathbf{q}/\mathbf{u})\chi_2 + s_{(n)}(\mathbf{q}/\mathbf{u})\chi_2 + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (s_{(i)}(\mathbf{q}/\mathbf{u}) + s_{(n-i)}(\mathbf{q}/\mathbf{u}))\chi^i \\ + \begin{cases} s_{(\frac{n}{2})}(\mathbf{q}/\mathbf{u})(\chi_3 + \chi_4) & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}\end{aligned} \tag{5.1}$$

Here $\chi_1, \chi_2, \chi_3, \chi_4$ are irreducible 1-dimensional $\mathfrak{I}_2(n)$-characters and the χ^i are irreducible 2-dimensional $\mathfrak{I}_2(n)$-characters.

As a consequence of Theorem 5.1, we can obtain an explicit multigraded Hilbert series, the dimension, and a monomial basis for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ using straightening relations [33].

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References

[1] J. A. Alfano. "The module of diagonal harmonic polynomials". PhD thesis. University of California, San Diego, 1994.

[2] R. Angarone, P. Commins, T. Karn, S. Murai, and B. Rhoades. "Superspace coinvariants and hyperplane arrangements". *Adv. Math.* **467** (2025), Paper No. 110185, 36 pp. DOI.

[3] F. Bergeron. "$(\mathrm{GL}_k \times \mathbb{S}_n)$-modules of multivariate diagonal harmonics". *Open problems in algebraic combinatorics*. Vol. 110. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, [2024] ©2024, pp. 1–22. DOI.

[4] F. Bergeron, J. Haglund, A. Iraci, and M. Romero. "Bosonic-fermionic diagonal coinvariants and Theta operators". *preprint* (2023), https://www2.math.upenn.edu/~jhaglund/preprints/BF2.pdf.

[5] F. Bergeron, J. Haglund, and M. Zabrocki. "Fianl report from BIRS Workshop 19w5131: Representation Theory Connections to (q,t)-Combinatorics". https://www.birs.ca/workshops/2019/19w5131/report19w5131.pdf. 2019.

[6] F. Bergeron. "Multivariate diagonal coinvariant spaces for complex reflection groups". *Adv. Math.* **239** (2013), pp. 97–108. DOI.

[7] F. Bergeron. "The Bosonic-Fermionic Diagonal Coinvariant Modules Conjecture". *preprint* (2020), https://arxiv.org/abs/2005.00924.

[8] F. Bergeron and L.-F. Préville-Ratelle. "Higher trivariate diagonal harmonics via generalized Tamari posets". *J. Comb.* **3**.3 (2012), pp. 317–341. DOI.

[9] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "Dens, nests and the Loehr-Warrington conjecture". *J. Amer. Math. Soc.* **38**.4 (2025), pp. 1049–1106. DOI.

[10] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "A proof of the extended delta conjecture". *Forum Math. Pi* **11** (2023), Paper No. e6, 28. DOI.

[11] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "A shuffle theorem for paths under any line". *Forum Math. Pi* **11** (2023), Paper No. e5, 38. DOI.

[12] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "LLT polynomials in the Schiffmann algebra". *J. Reine Angew. Math.* **811** (2024), pp. 93–133. DOI.

[13] A. Borel. "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts". *Ann. Math. (2)* **57** (1953), pp. 115–207. DOI.

[14] N. Bourbaki. *Elements of mathematics. Algebra. Chapter 8. Translated from the 2nd French edition by Reinie Erné*. Cham: Springer, 2023. DOI.

[15] I. Burban and O. Schiffmann. "On the Hall algebra of an elliptic curve. I." *Duke Math. J.* **161**.7 (2012), pp. 1171–1231. DOI.

[16] S.-J. Cheng and W. Wang. "Howe duality for Lie superalgebras". *Compos. Math.* **128**.1 (2001), pp. 55–94. DOI.

[17] S.-J. Cheng and W. Wang. *Dualities and representations of Lie superalgebras*. Vol. 144. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xviii+302. DOI.

[18] M. D'Adderio, A. Iraci, and A. Vanden Wyngaerd. "Theta operators, refined Delta conjectures, and coinvariants". *Adv. Math.* **376** (2021). Id/No 107447, p. 60. DOI.

[19] M. D'Adderio and A. Mellit. "A proof of the compositional delta conjecture". *Adv. Math.* **402** (2022), Paper No. 108342, 17 pp. DOI.

[20] N. Gonzalez, E. Gorsky, and J. Simental. "The elliptic Hall algebra and the double Dyck path algebra". `https://arxiv.org/abs/2502.16113`. 2025. arXiv:2502.16113.

[21] E. Gorsky and A. Neguţ. "Refined knot invariants and Hilbert schemes". *J. Math. Pures Appl. (9)* **104**.3 (2015), pp. 403–435. DOI.

[22] E. Gorsky, A. Oblomkov, J. Rasmussen, and V. Shende. "Torus knots and the rational DAHA". *Duke Math. J.* **163**.14 (2014), pp. 2709–2794. DOI.

[23] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. "A combinatorial formula for the character of the diagonal coinvariants". *Duke Math. J.* **126**.2 (2005), pp. 195–232. DOI.

[24] J. Haglund, J. B. Remmel, and A. T. Wilson. "The delta conjecture". *Trans. Amer. Math. Soc.* **370**.6 (2018), pp. 4029–4057. DOI.

[25] M. Haiman. "Conjectures on the quotient ring by diagonal invariants". *J. Algebr. Comb.* **3**.1 (1994), pp. 17–76. DOI.

[26] M. Haiman. "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane". *Invent. Math.* **149**.2 (2002), pp. 371–407. DOI.

[27] R. Howe. "Remarks on classical invariant theory". *Trans. Amer. Math. Soc.* **313**.2 (1989), pp. 539–570. DOI.

[28] A. Iraci, P. Nadeau, and A. Vanden Wyngaerd. "Smirnov words and the delta conjectures". *Adv. Math.* **452** (2024), Paper No. 109793, 41 pp. DOI.

[29] J. Kim and B. Rhoades. "Lefschetz theory for exterior algebras and fermionic diagonal coinvariants". *Int. Math. Res. Not.* **2022**.4 (2022), pp. 2906–2933. DOI.

[30] J. Lentfer. "A conjectural basis for the $(1,2)$-bosonic-fermionic coinvariant ring". *Algebr. Comb.* **8**.3 (2025), pp. 711–743.

[31] J. Lentfer. "Diagonal supersymmetry for coinvariant rings". *preprint* (2025), https://arxiv.org/abs/2505.14885. arXiv:2505.14885.

[32] J. Lentfer. "The sign character of the triagonal fermionic coinvariant ring". *preprint* (2025), https://arxiv.org/abs/2501.09920. arXiv:2501.09920.

[33] J. Lentfer. "Universal series for dihedral group coinvariant rings". *preprint* (2025), https://arxiv.org/abs/2509.15054. arXiv:2509.15054.

[34] N. A. Loehr and G. S. Warrington. "Nested quantum Dyck paths and $\nabla(s_\lambda)$". *Int. Math. Res. Not.* **2008** (2008). Id/No rnm157, p. 29. DOI.

[35] I. G. Macdonald. *Symmetric functions and Hall polynomials.* 2nd ed. Oxford: Clarendon Press, 1995.

[36] A. Mellit. "Toric braids and (m, n)-parking functions". *Duke Math. J.* **170**.18 (2021), pp. 4123–4169. DOI.

[37] S. Murai, B. Rhoades, and A. Wilson. "A proof of the Fields Conjectures". *preprint* (2025), https://arxiv.org/abs/2505.24027. arXiv:2505.24027.

[38] I. M. Musson. *Lie superalgebras and enveloping algebras*. Vol. 131. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2012.

[39] B. Rhoades and A. T. Wilson. "The Hilbert series of the superspace coinvariant ring". *Forum Math. Pi* **12** (2024). Id/No e16, p. 35. DOI.

[40] B. E. Sagan and J. P. Swanson. "q-Stirling numbers in type B". *Eur. J. Comb.* **118** (2024). Id/No 103899, p. 35. DOI.

[41] R. P. Stanley. "Invariants of finite groups and their applications to combinatorics". *Bull. Am. Math. Soc., New Ser.* **1** (1979), pp. 475–511. DOI.

[42] R. P. Stanley. *Enumerative combinatorics. Volume 2*. Vol. 62. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 1999.

[43] J. P. Swanson and N. R. Wallach. "Harmonic differential forms for pseudo-reflection groups I. Semi-invariants". *J. Combin. Theory Ser. A* **182** (2021), Paper No. 105474, 30. DOI.

[44] A. T. Wilson. "Torus link homology and the nabla operator". *J. Comb. Theory, Ser. A* **154** (2018), pp. 129–144. DOI.

[45] M. Zabrocki. "A module for the Delta conjecture". *preprint* (2019), https://arxiv.org/abs/1902.08966.

[46] M. Zabrocki. "Coinvariants and harmonics". https://realopacblog.wordpress.com/2020/01/26/coinvariants-and-harmonics/. 2020.