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On the Combinatorics of Schubert Calculus in Elliptic Cohomology

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Abstract. The main goal of this paper is to extend two fundamental combinatorial results in Schubert calculus on flag manifolds from equivariant cohomology and K-theory to equivariant elliptic cohomology. We derive a Billey-type formula for the localization of elliptic Schubert classes (for partial flag manifolds of arbitrary type) and a pipe dream model for their polynomial representatives in the case of type A flag manifolds. The latter extends the pipe dream model for double Schubert and Grothendieck polynomials.

Keywords: elliptic Schubert calculus, Billey formula, generic pipe dream

1 Introduction

Classes in torus equivariant elliptic cohomology are elliptic functions in line bundles, which makes elliptic cohomology less understood than cohomology and K-theory. By the seminal work of Aganagic and Okounkov [1], elliptic cohomology provides a framework to control the generating functions in the K-theoretic curve countings. It gains importance due to its close connection to an important duality called "3D mirror symmetry" from mathematical physics; see [15] for an introduction.

Cohomological and K-theoretic Schubert calculus, including their equivariant versions, have been extensively studied from various perspectives for many years. In contrast, elliptic Schubert calculus is a relatively recent subject. Its foundations were laid in [8, 11, 12, 13, 14, 18]; they include the recursive construction of elliptic Schubert classes via generalizations of the cohomology and K-theory push-pull operators, and the study of the corresponding Demazure algebra. However, elliptic Schubert calculus remains underdeveloped, particularly in terms of its combinatorial aspects. In this paper, which is an extended abstract of the full paper [9], we derive the analogues in equivariant el-

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liptic cohomology of two fundamental combinatorial formulas in Schubert calculus on partial flag manifolds: the *Billey-type formula* for the localization of Schubert classes, and the *pipe dream model* for their polynomial representatives in type A.

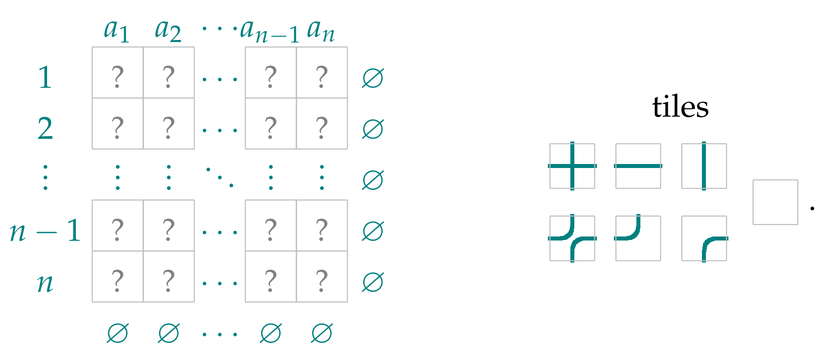
As noticed by Rimányi and Weber [13], the failure of well-definedness of fundamental classes leads to a remarkable new feature of elliptic Schubert calculus: the dependence on *dynamical parameters*. This feature tremendously increases the complexity, but at the same time unifies the corresponding formula in a more systematic way.

The first part of the paper concerns the restrictions of the elliptic Schubert classes. Let G be a reductive group. We use z_α (resp. $\lambda_{\alpha^\vee}$) to represent the equivariant parameter for a root α (resp., the dynamical parameter for a coroot $\alpha^\vee$) of G. Our first main result (Theorem 7) gives the Billey-type formula for the elliptic Schubert class $\mathbf{E}_w$. We also generalize it to the partial flag variety case in Theorem 9. If we restrict to type A, our Billey-type formula admits a wiring diagram presentation (Theorem 10). Using that, we can give a combinatorial proof of an identity of 3D mirror symmetry by constructing a simple sign-reversing involution; see [9] for the precise statement and its proof.

Our formula generalizes the original one for cohomology Schubert classes in [3, 4] and its K-theory analogue (for structure sheaves of Schubert varieties and their duals) in [5, 19], while it extends the Billey-type formula in hyperbolic cohomology derived in [10]. In fact, by taking the limit to K-theory (which is explained in detail in [9]), our formula recovers the one for Segre motivic Chern (SMC) classes of Schubert cells [17], which implies the mentioned formula for structure sheaves of Schubert varieties by specialization (see [2]). In this sense, our formula also generalizes the Billey-type formula for the Segre-Schwartz-MacPherson (SSM) classes of Schubert cells in cohomology [16].

Our second result (Theorem 16) is a combinatorial formula for polynomial representatives of equivariant elliptic Schubert classes (in the sense specified in Section 6.1) in type A. This formula extends the pipe dream model for double Schubert and Grothendieck polynomials. We use the *generic pipe dreams* of Knutson and Zinn-Justin [7], which they use to compute the SSM and SMC classes of Schubert cells of full flag varieties in type A. Thus, our result also extends theirs to the elliptic case.

A generic pipe dream is a tiling of the following $n \times n$ square grid by the following tiles:



For $w \in S_n$, we denote by $\mathsf{GPD}(w)$ the set of generic pipe dreams with $w(a_i) = i$, i.e., the i-th pipe from the left boundary goes to the $w(i)$-th position of the upper boundary. To get a polynomial representative, we need to associate with each tile an *elliptic weight*. This is determined by: the position of the tile, the indices of pipes inside, and another statistic called level (Definition 15). In particular, this definition depends on associating the dynamical variable λ_i with the i-th pipe (from the left boundary).

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2 Preliminaries

We recall the classical *Jacobi theta function*

$$\theta(u) = (x^{1/2} - x^{-1/2}) \prod_{n>0} (1 - q^n x)(1 - q^n/x) \text{ where } x = e^{2\pi i u}. \tag{2.1}$$

In the present paper, we treat θ formally. More precisely, given a lattice Λ and an element u, we could view

$$\theta(u) \in \mathbb{Q}[\Lambda][[q]], \text{ where } \mathbb{Q}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Q} \cdot e^{\pi i \lambda} \text{ is the group algebra.} \tag{2.2}$$

We fix the following two functions:

$$\mathsf{P}(x,y) = \frac{\theta(x-y)\theta(\hbar)}{\theta(y+\hbar)\theta(x)}, \qquad \mathsf{Q}(x,y) = \frac{\theta(x+\hbar)\theta(y)}{\theta(y+\hbar)\theta(x)}. \tag{2.3}$$

We record two identities.

Proposition 1. *We have the following identities:*

1. $\mathsf{Q}(x,y)\mathsf{Q}(y,x) = 1$ *and* $\mathsf{P}(x,y) + \mathsf{Q}(x,y)\mathsf{P}(y,x) = 0$*;*
2. $\mathsf{P}(-\hbar, y) = 1$ *and* $\mathsf{Q}(-\hbar, y) = 0$*;*
3. $\mathsf{P}(x, 0) = 1$ *and* $\mathsf{Q}(x, 0) = 0$*.*

Let G be a reductive group with Borel subgroup B. Let T be the maximal torus and $X^*(T), X_*(T)$ the set of characters and cocharacters, respectively. Let Σ be the set of simple roots, Φ the set of roots, and Φ^+ the set of positive roots. Let W be the Weyl group. Let P be a parabolic subgroup of G, which corresponds to a subset Σ_P of Σ. Let $\Phi_P \subset \Phi$ be the subset of roots corresponding to P. Let W_P be the subgroup generated by $s_\alpha, \alpha \in \Sigma_P$, and let W^P be the set of minimal length representatives of W/W_P.

3 Elliptic Schubert Classes

In this section, we review the elliptic Schubert classes via the elliptic Demazure-Lusztig operators and compare them with the elliptic Schubert classes in [13].

3.1 Twisted group algebras

Let T be the maximal torus of G, and let $\hbar$ be a formal parameter. We consider

$$\mathcal{Q} = \operatorname{Frac} \mathbb{Q}[\Lambda][[q]] \text{ where } \Lambda = X^*(T) \oplus X_*(T) \oplus \mathbb{Z}\hbar. \tag{3.1}$$

For a character $\alpha \in X^*(T)$ (resp., a cocharacter $\beta^\vee \in X_*(T)$), we denote by z_α (resp. $\lambda_{\beta^\vee}$) the corresponding element in Λ. We think of z-variables as "equivariant parameters" and λ-variables as "dynamical parameters." The algebra $\mathbb{Q}[\Lambda]$ admits two commutative W-actions (and so does $\mathcal{Q}$): by acting on the z-variables and λ-variables, respectively; the latter is called the dynamical action. These actions are given by ${}^{wv^{\mathsf{d}}}z_\alpha = z_{w\alpha}$ and ${}^{wv^{\mathsf{d}}}\lambda_{\beta^\vee} = \lambda_{v\beta^\vee}$. Define the *twisted group algebra* $\mathcal{Q}_{W^2} = \mathcal{Q} \rtimes \mathbb{Q}[W \times W^{\mathsf{d}}]$. It is a free left $\mathcal{Q}$-module with basis $\{\delta_w\delta_v^{\mathsf{d}} : w, v \in W\}$, with multiplication given by

$$a\,\delta_w\delta_v^{\mathsf{d}} \cdot a'\,\delta_{w'}\delta_{v'}^{\mathsf{d}} = a \cdot {}^{wv^{\mathsf{d}}}a'\,\delta_{ww'}\delta_{vv'}^{\mathsf{d}}, \quad a, a' \in \mathcal{Q}.$$

We can view it as the algebra generated by two W-actions on $\mathcal{Q}$ and the operators of multiplication by elements of $\mathcal{Q}$. Next, consider the algebra $\mathcal{Q}_W^* = \operatorname{Map}(W, \mathcal{Q})$ given by componentwise addition and multiplication. We denote by $f_w \in \mathcal{Q}_W^*$ the element defined by $f_w(v) = \delta_{w,v}^{\mathsf{Kr}}$. In other words, for any $f \in \mathcal{Q}_W^*$, we can write $f = \sum_{w\in W} f(w) f_w \in \mathcal{Q}_W^*$.

3.2 Elliptic Schubert classes

Recall the definition of P and Q in (2.3) and the convention (2.2).

Definition 2. *For a simple root* $\alpha \in X^*(T)$, *the* elliptic Demazure-Lusztig operator *is*

$$T_\alpha = \delta_\alpha^{\mathsf{d}}\big(\mathsf{P}(z_\alpha, \lambda_{\alpha^\vee}) + \mathsf{Q}(z_\alpha, \lambda_{\alpha^\vee})\delta_\alpha\big) \in \mathcal{Q}_{W^2}, \quad \textit{where } \delta_\alpha = \delta_{s_\alpha},\ \delta_\alpha^{\mathsf{d}} = \delta_{s_\alpha}^{\mathsf{d}}. \tag{3.2}$$

We have $T_\alpha = -T_\alpha^{RW}$, where the latter operator was defined in [13, Theorem 1.3]; see [9, Remark 3.2]. So T_α satisfies the braid relations, and therefore T_w is well defined. Note that a different normalization of elliptic classes was used in [14].

For $u \in W$, it is a standard fact about $\mathcal{Q}_{W^2}$ that one can expand

$$\delta_{u^{-1}}^{\mathsf{d}} T_u = \sum_{w\le u} a_{u,w} \cdot \delta_w, \qquad \delta_u = \sum_{w \le u} b_{u,w} \cdot \delta_{w^{-1}}^{\mathsf{d}} T_w, \qquad \text{where } a_{u,w},\, b_{u,w} \in \mathcal{Q}. \tag{3.3}$$

In this setup, *3D mirror symmetry* can be stated as follows, cf. [11, 14]:

$$a_{u^{-1},w^{-1}} = b_{u,w}^{\mathsf{d}}, \qquad b_{u^{-1},w^{-1}} = a_{u,w}^{\mathsf{d}}. \tag{3.4}$$

Definition 3. *For* $w \in W$, *we define the elliptic Schubert class* $\mathbf{E}_w \in \mathcal{Q}_W^*$ *by* $\mathbf{E}_w(u) = b_{u,w}$.

3.3 Elliptic Schubert classes for G/P

In order to generalize the above setting to a parabolic subgroup, we need to assume that G is simply connected, i.e., $X_*(T)$ is the coroot lattice $\check{Q}$. We denote

$$\check{Q}^P = \bigoplus_{\alpha \in \Sigma \setminus \Sigma_P} \mathbb{Z}\alpha^\vee \subset \bigoplus_{\alpha \in \Sigma} \mathbb{Z}\alpha^\vee = \check{Q} = X_*(T). \tag{3.5}$$

We denote

$$\mathcal{Q}^P = \operatorname{Frac} \mathbb{Q}[\Lambda^P][[q]] \text{ where } \Lambda^P = X^*(T) \oplus \check{Q}^P \oplus \mathbb{Z}\hbar, \tag{3.6}$$

and

$$\mathcal{Q}^*_{W/W_P} = \operatorname{Map}(W/W_P, \mathcal{Q}^P) = \{f \in \operatorname{Map}(W, \mathcal{Q}^P) : \forall v \in W_P,\, f(uv) = f(u)\}. \tag{3.7}$$

We now define the parabolic version of elliptic Schubert classes. We have the following linear map of lattices

$$\check{Q} \oplus \mathbb{Z}\hbar \twoheadrightarrow \check{Q}^P \oplus \mathbb{Z}\hbar, \qquad \alpha^\vee \longmapsto \begin{cases} \alpha^\vee, & \alpha \notin \Sigma_P, \\ -\hbar, & \alpha \in \Sigma_P. \end{cases} \tag{3.8}$$

Denote by $\mathcal{Q}' \subset \mathcal{Q}$ the subset of elements having no poles along $\lambda_{\alpha^\vee} = -\hbar$ for all roots $\alpha \in \Phi$. Then the map (3.8) induces a map

$$\mathcal{Q}' \longrightarrow \mathcal{Q}^P, \qquad c \longmapsto [c]_P, \tag{3.9}$$

by specializing $\lambda_{\alpha^\vee}$ to $-\hbar$ for all $\alpha \in \Sigma_P$.

Definition 4. *For $w \in W^P$, we define the* parabolic elliptic Schubert class $\mathbf{E}^P_w \in \mathcal{Q}^*_{W/W_P}$ *by*

$$\mathbf{E}^P_w(u) = [b_{u,w}]_P \in \mathcal{Q}^P.$$

4 Elliptic Billey Formula

In this section, we derive a Billey-type formula for the elliptic classes $\mathbf{E}_w$. In type A, it is shown to have a diagrammatic version.

4.1 The dynamical R-matrix

We consider $\mathcal{Q}_{W^{\mathsf{d}}} = \mathcal{Q} \rtimes \mathbb{Q}[W^{\mathsf{d}}]$. It is a free left $\mathcal{Q}$-module with basis δ^{d}_v, and there is a multiplication given by $a\,\delta^{\mathsf{d}}_v \cdot a'\,\delta^{\mathsf{d}}_{v'} = a \cdot {}^{v^{\mathsf{d}}}a'\,\delta^{\mathsf{d}}_{vv'}$. For a simple root α_i and a weight $\beta \in X^*(T)$, we consider the *dynamical R-matrix*

$$h_i(\beta) = \mathsf{P}(\lambda_{\alpha_i^\vee}, z_\beta) + \delta^{\mathsf{d}}_i \cdot \mathsf{Q}(\lambda_{\alpha_i^\vee}, z_\beta) \in \mathcal{Q}_{W^{\mathsf{d}}}, \tag{4.1}$$

where $\delta_i^{\mathsf{d}} = \delta_{s_i}^{\mathsf{d}}$. We are going to construct a generating function of $b_{u,w} = \mathbf{E}_w(u)$ via dynamical R-matrices.

We first fix a reduced word of $u \in W$ as $u = s_{i_1} \cdots s_{i_\ell}$, which determines a sequence of roots $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$, $j = 1, \dots, \ell$. Our first main result is the following theorem.

Theorem 5. *With the notation introduced above, we have*

$$h_{i_1}(\beta_1) \cdots h_{i_\ell}(\beta_\ell) = \sum_{w \in W} \delta_w^{\mathsf{d}} \cdot \mathbf{E}_w(u). \tag{4.2}$$

In particular, the left-hand side does not depend on the choice of decomposition of u.

Corollary 6. *The dynamical R-matrices* (4.1) *satisfy the following properties.*

- *Unitary property*
$$h_i(x) h_i(-x) = 1.$$
- *Yang–Baxter equations*
$$\begin{aligned} h_i(x) h_j(y) &= h_j(y) h_i(x), && \text{if } (s_i s_j)^2 = \mathsf{id}; \\ h_i(x) h_j(x+y) h_i(y) &= h_j(y) h_i(x+y) h_j(x), && \text{if } (s_i s_j)^3 = \mathsf{id}. \end{aligned}$$

There are similar expressions for $(s_i s_j)^4 = \mathsf{id}$ and $(s_i s_j)^6 = \mathsf{id}$, see also [4, Proposition 3.1].

4.2 Billey-type formula

Now we expand the formula in Theorem 5. Recall that we fixed a reduced word of u. Consider a subword given by $J \subseteq \{1, \dots, \ell\}$, and define its evaluation in the usual way:

$$w(J) = s_{i_1}^{\epsilon_1} \cdots s_{i_\ell}^{\epsilon_\ell} = \overrightarrow{\prod_{j \in J}} s_{i_j} \le u\,, \qquad \text{where } \epsilon_j = \delta_{j \in J}^{\mathsf{Kr}} \in \{0, 1\} \text{ for } 1 \le j \le \ell\,.$$

For $j = 1, \dots, \ell$, we consider the coroots $\check{\gamma}_j^J = s_{i_\ell}^{\epsilon_\ell} \cdots s_{i_{j+1}}^{\epsilon_{j+1}} \alpha_{i_j}^\vee$.

Theorem 7. *With the notation introduced above, we have*

$$\mathbf{E}_w(u) = \sum_{w(J)=w} \prod_{j=1}^{\ell} \begin{cases} \mathsf{Q}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j}), & j \in J, \\ \mathsf{P}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j}), & j \notin J. \end{cases}$$

Remark 8. In [8, 14] there are similar formulas for computing restrictions of (different) elliptic classes, namely the coefficients $a_{u,w}$ in (3.3). Our Billey-type formula is for the coefficients $b_{u,w}$ instead (recall Definition 3). It can be translated to the formulas in [8, 14] via (3.4) − a phenomenon which only happens in the elliptic case. The latter formulas were derived geometrically, via the Bott-Samelson resolution. By contrast, our formula is derived in the original algebraic and combinatorial setup of [4], which is based on R-matrices. In the elliptic case, R-matrices also appeared in [18, Section 9] in a different context (for GL_n only). Finally, we note that, by contrast to [4], where the Yang-Baxter property is an input for the proof of the Billey formula, we derive it as a corollary.

We now state the parabolic generalization of the above Billey-type formula.

Theorem 9. *Using the notation introduced above, we have*

$$\mathbf{E}_w^P(u) = \sum_{w(J)=w} \prod_{j=1}^{\ell} \begin{cases} [\mathsf{Q}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j})]_P, & j \in J, \\ [\mathsf{P}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j})]_P, & j \notin J, \end{cases}$$

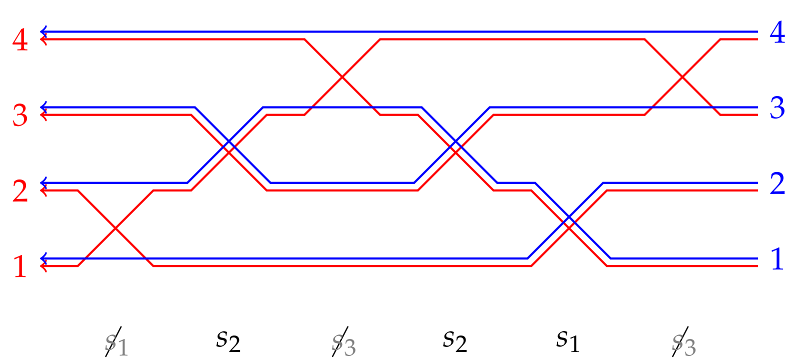
where the sum over $J \subset \{1, \ldots, \ell\}$ such that $\forall j = 1, \ldots, \ell$, we have $s_{i_j}^{\epsilon_j} \cdots s_{i_\ell}^{\epsilon_\ell} \in W^P$. In particular, $\mathbf{E}_w^P(u) = 0$ unless $w \in W^P$.

5 Diagrams for Type A

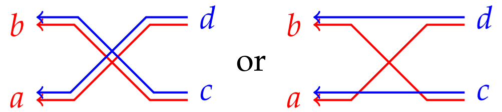
Now we switch to GL_n, for which the Weyl group W is the symmetric group S_n. We can identify $X^*(T) = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n = X_*(T)$ with simple roots $\epsilon_i - \epsilon_{i+1}$, for $i = 1, \ldots, n-1$. We denote $z_i = z_{\epsilon_i} \in \mathcal{Q}$ and $\lambda_i = \lambda_{\epsilon_i} \in \mathcal{Q}$.

5.1 Wiring diagrams

Let u be a permutation. A choice of a reduced decomposition of u can be illustrated by a *wiring diagram*, where each s_{i_j} is represented by a crossing of strings of heights i_j and $i_j + 1$ in the j-th interval. A choice of $J \subseteq \{1, \ldots, \ell\}$, corresponding to a choice of a subword, can be represented by a "sub-wiring diagram" obtained by resolving s_{i_j} for $j \notin J$. To distinguish them, we color the strings in the wiring diagram of u by red, and those in the sub-wiring diagram by blue. We label the red (resp., blue) strings by their targets (resp., sources). Here is an example.

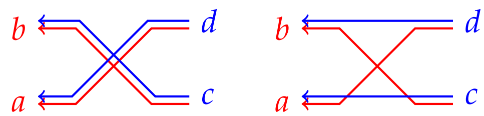


Then we can read β_j and $\check{\gamma}_j^J$ from the sub-wiring diagram. At the j-th crossing, we have the following possibilities.



$$\Longrightarrow \quad \begin{aligned} \beta_j &= \epsilon_a - \epsilon_b \\ \check{\gamma}_j^J &= \epsilon_c - \epsilon_d \end{aligned}$$

This leads us to define the weight of a sub-wiring diagram to be the product of local weights below.



(5.1)

weight $\mathsf{Q}(\lambda_c - \lambda_d, z_a - z_b)$ $\mathsf{P}(\lambda_c - \lambda_d, z_a - z_b)$

Theorem 7 can now be restated as follows.

Theorem 10. *Let $u, w \in S_n$. For a fixed wiring diagram D_0 of u, we have*

$$\mathbf{E}_w(u) = \sum_D \mathrm{weight}(D),$$

where the sum is over all sub-wiring diagrams D of D_0 whose permutation is w.

5.2 Parabolic version

We now rephrase Theorem 9 in the type A case. Theoretically, we need to switch to $G = \mathrm{SL}_n$, but everything below can be lifted to GL_n.

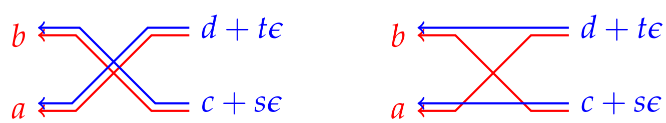
Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$ be a composition with $|\mathbf{a}| := a_1 + \cdots + a_m = n$. We have a parabolic subgroup $P \subset \mathrm{GL}_n$ whose Weyl group is the Young subgroup $W_P = S_{\mathbf{a}} := S_{a_1} \times \cdots \times S_{a_m} \subset S_n$. We define

$$A = \{a_1 + \cdots + a_i : 0 \le i \le n\} = \{1, a_1 + 1, a_1 + a_2 + 1, \dots, a_1 + \cdots + a_{m-1} + 1\}.$$

Then the specialization map (3.9) $f \mapsto [f]_P$ is given by $\lambda_i - \lambda_{i+1} \mapsto -\hbar$ for $i \notin A$. We denote $\lambda_a = [\lambda_a]_P$ for $a \in A$ by an abuse of notation. Then for any $i = 1, \dots, n$,

$$\lambda_i \mapsto \lambda_a + (i - a)\hbar \quad \text{where} \quad a = \min(a \in A : a \le i).$$

To compute $\mathbf{E}_w^P(u)$ using the sub-wiring diagrams introduced above, we relabel the blue strings by replacing the label i by $a + (i - a)\epsilon$ for $a \in A$ as above. We could view ϵ as the symbol for an infinitesimal number. The weights (5.1) specialize to



(5.2)

$[\mathrm{weight}]_P$ $\mathsf{Q}(\lambda_c - \lambda_d + (t-s)\hbar, z_a - z_b)$ $\mathsf{P}(\lambda_c - \lambda_d + (t-s)\hbar, z_a - z_b)$.

We can now translate Theorem 9 as follows.

Theorem 11. *Let $u, w \in W$. For a fixed wiring diagram D_0 of u, we have*

$$\mathbf{E}_w^P(u) = \sum_D [\text{weight}(D)]_P,$$

where the sum is over all sub-wiring diagrams D of D_0 whose underlying permutation is w. We can further assume that any pair of blue strings in D labeled by $c + \mathbb{N}\epsilon$ for the same $c \in A$ does not intersect.

6 Pipe Dream Model

In this section, we consider polynomial representatives of equivariant elliptic Schubert classes $\mathbf{E}_w$ in type A via the generic pipe dreams of Knutson and Zinn-Justin [7], which were recalled in Section 1.

6.1 Polynomial representatives

Recall the setup in Section 3.1. We take the *equivariant elliptic cohomology* to be $\mathbf{E}ll_T(G/B)$ equaling $K_T(G/B)[[q]]$, and consider the Borel model of $K_T(G/B)$. When $G = \mathrm{GL}_n$, we consider

$$\mathcal{P} = \mathbb{Q}[\Lambda][[q]], \qquad \Lambda = \left(\bigoplus_{i=1}^n \mathbb{Z}x_i\right) \oplus \left(\bigoplus_{i=1}^n \mathbb{Z}y_i\right) \oplus \left(\bigoplus_{i=1}^n \mathbb{Z}\lambda_i\right) \oplus \mathbb{Z}\hbar.$$

Denote by x_i, y_i, λ_i for $1 \le i \le n$ the coordinates of the three summands. Define the localization map, which is a ring homomorphism:

$$\mathrm{Loc} : \mathcal{P} \longrightarrow \mathcal{Q}_W^*, \qquad \mathrm{Loc}(f(x, y, \lambda))(u) = f(uz, z, \lambda),$$

where $uz = (z_{u(1)}, \ldots, z_{u(n)})$. A *polynomial representative* of $\mathbf{E}_w$ is an element $\mathcal{E}_w \in \mathcal{P}$ such that $\mathrm{Loc}(\mathcal{E}_w) = \mathbf{E}_w$. Note that the polynomial representative is not unique.

Remark 12. The above notation agrees up to sign with the traditional one for the variables in double Schubert and Grothendieck polynomials, in the sense that x_i are the Chern roots of the universal subquotient bundles on GL_n / B, and y_i are the equivariant parameters (characters of the torus); in addition, we now also have the dynamical variables λ_i. More precisely, the pipe dream formula for double Schubert polynomials (see, e.g., [6, Section 3]) yields, via specialization, a formula for the localizations of Schubert classes which is positive in $x_j - x_i$, $j > i$ (i.e., the negative roots), while our corresponding formula is positive in the positive roots.

We construct the polynomial representatives $\mathcal{E}_w$ for w in $W = S_n$. The construction is based on the Billey-type formula in a larger group $\widehat{G} = \mathrm{GL}_{2n}$, whose Weyl group is $\widehat{W} = S_{2n}$. Consider $\widehat{W}_{\widehat{P}} = S_1 \times \cdots \times S_1 \times S_n$, and let $u_0 \in S_{2n}$ be given by

$$1 \leq i \leq n, \qquad u_0(i) = n+i, \qquad u_0(n+i) = i.$$

For $w \in W = S_n$, we view $w = w \times \mathsf{id} \in S_n \times S_n \subset S_{2n}$. From the discussion in the previous sections, we have constructed

$$\mathbf{E}_w^{\hat{P}}(u_0) = [\mathbf{E}_w(u_0)]_{\widehat{P}} \in \mathbb{Q}[\Lambda][[q]], \qquad \Lambda = \left(\bigoplus_{i=1}^{2n} \mathbb{Z} z_i\right) \oplus \left(\bigoplus_{i=1}^{n+1} \mathbb{Z}\lambda_i\right) \oplus \mathbb{Z}\hbar.$$

In particular, we can write it as

$$\mathbf{E}_w^{\widehat{P}}(u_0) = \widehat{\mathcal{E}}_w(z_1, \ldots, z_{2n}; \lambda_1, \ldots, \lambda_n, \lambda_{n+1}).$$

We define

$$\mathcal{E}_w(x, y, \lambda) = \widehat{\mathcal{E}}_w(y_1, \ldots, y_n, x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n, 0) \in \mathcal{P}.$$

Let $w_\circ$ be the longest element of $W = S_n$.

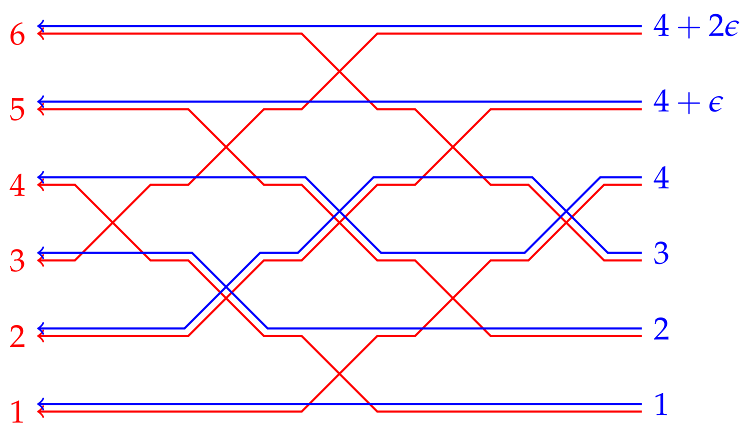
Lemma 13. *We have*

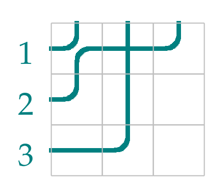
$$\mathcal{E}_{w_\circ}(x, y, \lambda) = \prod_{i+j \leq n} \mathsf{Q}(\lambda_i - \lambda_{n+1-j}, y_j - x_i) \prod_{i=1}^{n} \mathsf{P}(\lambda_i, y_{n+1-i} - x_i). \tag{6.1}$$

Theorem 14. *The element $\mathcal{E}_w \in \mathcal{P}$ is a polynomial representative of $\mathbf{E}_w$.*

6.2 Generic pipe dreams

From our definition and Theorem 11, there is a sub-wiring diagram model for $\mathcal{E}_w$. Note that the wiring diagrams for any reduced word of u_0 are isotopically equivalent. The set of such diagrams is in a natural bijection with the set of generic pipe dreams [7], as illustrated below for $n = 3$.





More precisely, we replace the red strings by the "Poincaré dual", i.e., an $n \times n$ grid, and the blue strings labeled by $1, \ldots, n$ by pipes in the grid. Since the blue strings labeled by $n+1+\mathbb{N}\epsilon$ are not allowed to intersects, by Theorem 11, we lose no information.

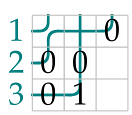
We now explain the translation of the weights to generic pipe dreams. We first need to define the *level* statistic.

Definition 15. *For a generic pipe dream, we define* level *for tiles containing only one pipe inside, i.e.* , , , , *as follows. For each pipe, imagine that we are walking along it with a number kept in mind. At the source of the pipe, the number is* 0*. Then once we meet* , , ,  *we define its* level *to be the number in mind, and then increase the number by* 1 *if we met* , *and by* -1 *if met* .

We can now translate the weights at the (i, j)-position as follows.

	$\mathsf{Q}(\lambda_a - \lambda_b, y_j - x_i)$		$\mathsf{Q}(\lambda_a - c\hbar, y_j - x_i)$		$\mathsf{Q}(c\hbar - \lambda_b, y_j - x_i)$		1
	$\mathsf{P}(\lambda_a - \lambda_b, y_j - x_i)$		$\mathsf{P}(\lambda_a - c\hbar, y_j - x_i)$		$\mathsf{P}(c\hbar - \lambda_b, y_j - x_i)$		

Here a (resp., b) is the index for the pipe from the lower (resp., left) boundary, and c is the level. See the following example, where the levels are displayed inside the respective tiles.

weight ()

$$= \prod \begin{Bmatrix} \mathsf{P}(\lambda_1 - \lambda_2, y_1 - x_1) & \mathsf{Q}(\lambda_2 - \lambda_3, y_2 - x_1) & \mathsf{P}(\lambda_2, y_3 - x_1) \\ \mathsf{P}(\lambda_2, y_1 - x_2) & \mathsf{Q}(-\lambda_3, y_2 - x_2) & 1 \\ \mathsf{Q}(\lambda_3, y_1 - x_3) & \mathsf{P}(\lambda_3 - \hbar, y_2 - x_3) & 1 \end{Bmatrix}$$

Theorem 16. *For any $w \in S_n$, we have*

$$\mathcal{E}_w = \sum_{\pi \in \mathsf{GPD}(w)} \text{weight}(\pi).$$

Remark 17. We can define polynomial representatives of equivariant elliptic classes for partial flag manifolds GL_n / P by applying the map $[\_]_P$ to the polynomial representatives for GL_n / B. In this way, we derive the analogue of Theorem 16 for GL_n / P; the respective sum is now restricted to those generalized pipe dreams for which the strings labeled by $c + \mathbb{N}\epsilon$ for the same $c \in A$ do not intersect (see the notation in Theorem 11), and the map $[\_]_P$ is applied to the corresponding weights (as in Theorem 11).

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