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On an extension problem on the moment curve

Seunghun Lee*[1] and Eran Nevo†[2]

[1] *Department of Mathematics, Keimyung University, Daegu, South Korea.*
[2] *Mathematics Research Institute, Universidad de Valladolid, Spain and Einstein Institute of Mathematics, Hebrew University of Jerusalem, Israel, and CMSA, Harvard University, Cambridge, USA.*

Abstract. We show that for $2 \leq d \leq 4$, every finite geometric simplicial complex Δ in $\mathbb{R}^d$ with vertices on the moment curve can be extended to a triangulation T of the cyclic polytope C where Δ, T and C all have the same vertex set. Further, for $d \geq 5$ we construct complexes Δ for which no such triangulations T exist.

Keywords: moment curve, higher Stasheff-Tamari poset, triangulation, cyclic polytope, extension.

1 Introduction

We consider the following natural extension problem on the *moment curve*

$$\gamma_d = \{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\}.$$

Question 1.1. *Does* every *geometric simplicial complex in* $\mathbb{R}^d$ *on a finite point set* $A \subseteq \gamma_d$ *extend to a triangulation of* $\mathrm{conv}(A)$ *without adding new vertices?*

Let us remark that this problem is in fact a combinatorial one, as intersections correspond to interlacing patterns (see Proposition 2.1) which are forbidden, and boundary faces are identified by the Gale evenness condition (e.g. [9]) and one checks that exactly these $(d-1)$-faces are contained in a unique d-simplex.

As we shall see, the answer to Question 1.1 depends on the dimension d. Considering a more general situation first, note that for $d = 2$ every simplicial complex on a finite point set $A \subseteq \mathbb{R}^2$ extends to a triangulation of $\mathrm{conv}(A)$, while for $d = 3$ there are finite point sets $A \subseteq \mathbb{R}^3$ in convex general position, and simplicial complexes on them, which do not extend to a triangulation of $\mathrm{conv}(A)$ without adding new vertices. As a simple

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†nevo@math.huji.ac.il. Eran Nevo was partially supported by Israel Science Foundation grants ISF-2480/20 and ISF-687/24.

6-vertex example consider the Schönhardt's polyhedron [17], see [6, Lemma 3.6.2, Figure 3.43].

Perhaps surprisingly, we prove that the answer to Question 1.1 is Yes iff $d \leq 4$: For an affinely independent set $\sigma \subset \mathbb{R}^d$ of size at most $d+1$, we will frequently refer to the geometric simplex $\text{conv}(\sigma)$ as σ when it is appropriate. We say that two simplices σ_1 and σ_2 in $\mathbb{R}^d$ *overlap* if the intersection of the simplices is not a face of each of them, namely

$$\text{conv}(\sigma_1) \cap \text{conv}(\sigma_2) \supsetneq \text{conv}(\sigma_1 \cap \sigma_2).$$

In this definition, non-overlapping simplices may intersect, but only in a common face.

For convenience in the proof we use a different notation for dimension in our main result:

Theorem 1.2. *(i) For every $D \leq 4$ and every finite collection $\mathcal{F}$ of pairwise non-overlapping simplices in $\mathbb{R}^D$ on $A \subseteq \gamma_D$, $\mathcal{F}$ can be extended into a triangulation T of the cyclic polytope* $\text{conv}(A)$ *such that A is exactly the vertex set of T.*

(ii) For every $D \geq 5$ and $n \geq D+3$, there exists a collection of pairwise non-overlapping D-simplices in $\mathbb{R}^D$ exactly on an n-vertex set $A \subseteq \gamma_D$ which cannot be extended into a triangulation of $\text{conv}(A)$ *without adding new vertices.*

In order to prove Theorem 1.2(i) we will use the known facts that for $d = 2, 3$ the higher Stasheff-Tamari poset $\text{HST}(n, d)$ is a lattice; see [10] and [8, Thm.3.6] respectively and Theorem 2.8. In order to prove Theorem 1.2(ii) we construct an example with $D = 5$ and $n = 8$, and then lift it to all $D \geq 5$ and $n \geq D+3$.

Previous studies. The combinatorics of triangulations of the cyclic polytope $C(n, d)$ has been extensively studied. Edelman and Reiner [8] defined two partial orders on this set of triangulations; one is given by bistellar flips and the other by comparing the "heights of the liftings in $\mathbb{R}^{d+1}$" for those triangulations. These two orders define the so-called *(first and second) higher Stasheff-Tamari poset*. They showed that these two orders are equal for $d = 2, 3$. Since then there have been active research on this topic [16, 7, 18, 15, 19, 20]. Recently the two orders were proved to be the same for every dimension d, or equivalently, the intermediate space between any two sections can be triangulated in dimension $d+1$, by Williams [21]. Our result is a stronger version of this when $d = 2, 3$.

The combinatorial criterion for having overlapping simplices on the moment curve (see Proposition 2.1) is well-known and is one of the ways of defining the *alternating oriented matroids* [2, Section 9.4]. It is also closely related to recent works on geometric hypergraph colorings and their representations where a similar combinatorial property called the $(AB)^{l/2}$*-freeness* was introduced and has been studied [1, 11, 12, 5].

Outline. In Section 2 we review the higher Stasheff-Tamari poset $\text{HST}(n, d)$, which is a partial order on the triangulations of the cyclic polytope $C(n, d)$, and analogously define a partial order on simplices in $\mathbb{R}^d$ by lifting to the moment curve one dimension higher. In Section 3 we prove Theorem 1.2; to prove (i) in Subsection 3.1 we use the existence of certain triangulations in $\mathrm{HST}(n,2)$ and $\mathrm{HST}(n,3)$ – the proof sketch of the existence is shown right after that. Theorem 1.2(ii) is proved in Subsection 3.2 by means of constructions. In Section 4 we discuss complexity aspects of this work and end with related open problems.

2 Ordering simplices and triangulations by height

2.1 Ordering simplices by height

Since combinatorial properties of finite point sets $\gamma_d(t_1), \gamma_d(t_2), \dots, \gamma_d(t_n)$ with $t_1 < t_2 < \cdots < t_n$ depend solely on the order, we will frequently abuse notation and refer to the point $\gamma_d(t_i)$ by its index i; so the whole point set is identified with $[n] := \{1, 2, \dots, n\}$. *The* cyclic polytope is defined as $C(n,d) = \mathrm{conv}\{\gamma_d(i) : i \in [n]\}$ on the vertex set $[n]$. We similarly define $C(V,d)$ for a subset $V \subseteq [n]$.

For subsets σ and τ of $[n]$, we say that σ and τ are *k-interlacing* if there is a sequence $v_1 < v_2 < \cdots < v_k$ of elements of $\sigma \cup \tau$ which satisfies one of the following alternating conditions:

$$v_1 \in \sigma, v_2 \in \tau, v_3 \in \sigma \dots \text{ (beginning with an element of } \sigma\text{), or} \tag{σ}$$

$$v_1 \in \tau, v_2 \in \sigma, v_3 \in \tau \dots \text{ (beginning with an element of } \tau\text{).} \tag{τ}$$

The following fact gives a combinatorial criterion for the overlapping condition on the moment curve and is well-known (for a proof, see e.g. [13, Lemma 2.5]).

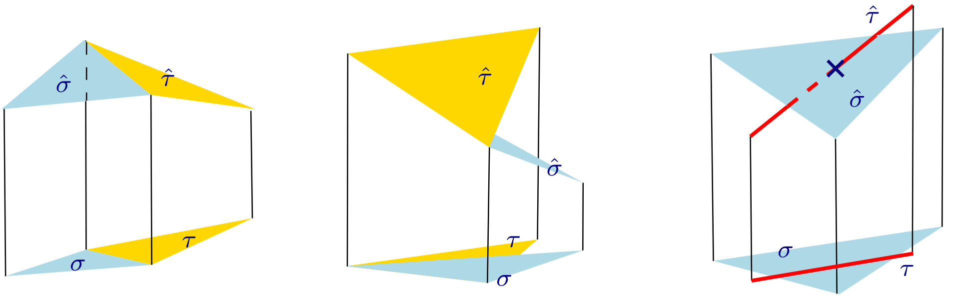
Proposition 2.1. *Simplices σ and τ on γ_d overlap in $\mathbb{R}^d$ if and only if they are $(d+2)$-interlacing.*

For $k \leq d+1$ and a simplex $\sigma = \{\gamma_d(t_1), \dots, \gamma_d(t_k)\}$ on γ_d, we define the *height function* of σ, $h_\sigma : \mathrm{conv}(\sigma) \to \mathbb{R}$, by setting $h_\sigma(p)$, for each $p \in \mathrm{conv}(\sigma)$, to be the last coordinate of the point in $\mathrm{conv}(\hat{\sigma})$ corresponding to p via projection, where $\hat{\sigma} = \{\gamma_{d+1}(t_1), \dots, \gamma_{d+1}(t_k)\}$. We call $\hat{\sigma}$ the *lifting of* σ. We define a relation $<_{d+1}$ on the set of simplices on γ_d. For two simplices $\sigma, \tau \subseteq \gamma_d$, we have $\sigma <_{d+1} \tau$ if and only if σ and τ overlap in $\mathbb{R}^d$ and $h_\sigma \leq h_\tau$ in the common domain.

Proposition 2.2. *For simplices σ and τ on γ_d, $\sigma <_{d+1} \tau$ if and only if σ and τ are $(d+2)$-interlacing by*

- *satisfying (σ) but not (τ) when d is even, and*
- *satisfying (τ) but not (σ) when d is odd.*

From Propositions 2.1 and 2.2, we have the following corollary, see Figure 1.



(A) σ and τ do not overlap in $\mathbb{R}^2$.

(B), (C) $\sigma <_3 \tau$

(D) $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^3$.

Figure 1: Illustration of the mutually exclusive cases (A)-(D).

Corollary 2.3. *Given a pair of simplices σ and τ on γ_d, one of the following four mutually exclusive cases occurs:*

(A) σ and τ do not overlap in $\mathbb{R}^d$, or equivalently, σ and τ are not $(d+2)$-interlacing.

(B) $\sigma <_{d+1} \tau$, or equivalently, σ and τ are $(d+2)$-interlacing by satisfying the combinatorial condition of Proposition 2.2.

(C) $\sigma >_{d+1} \tau$, or equivalently, σ and τ are $(d+2)$-interlacing by satisfying the combinatorial condition of Proposition 2.2 with the roles of σ and τ interchanged.

(D) the liftings $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^{d+1}$, or equivalently, σ and τ are $(d+3)$-interlacing.

For simplices $\sigma, \tau \subseteq \gamma_d$, we say $\sigma \preceq_{d+1} \tau$ if

- $\sigma = \tau$, or
- there is a positive integer k and a sequence of simplices $\sigma = \sigma_0 <_{d+1} \sigma_1 <_{d+1} \cdots <_{d+1} \sigma_k = \tau$ on γ_d.

Lemma 2.4. *The relation $\preceq_{d+1}$ is a partial order on the simplices on γ_d.*

Lemma 2.4 is used in the proof of Theorem 1.2(i).

Note that we never have $\tau <_{d+1} \sigma$ when $\sigma \preceq_{d+1} \tau$. It is easy to find examples of non-overlapping (in $\mathbb{R}^d$) $\sigma, \tau \subseteq \gamma_d$ with $\sigma \preceq_{d+1} \tau$. It is also not very difficult to find $\sigma, \tau \subseteq \gamma_d$ with $\sigma \preceq_{d+1} \tau$ and that the liftings $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^{d+1}$; these suggest that there is subtle difference between the "above-below" relation $<_{d+1}$ and the order $\preceq_{d+1}$:

Example 2.5. Consider the following simplices on γ_2: $\sigma = \{1,4,6\}$, $e = \{2,7\}$ and $\tau = \{3,5,8\}$. Then $\sigma <_3 e <_3 \tau$ (so in particular $\sigma \preceq_3 \tau$) but σ and τ are 6-interlacing.

2.2 Height of a triangulation and submersion sets

Let $S(n,d)$ be the set of all triangulations on vertex set $[n]$ of the cyclic polytope $C(n,d)$.

For a triangulation $T \in S(n,d)$, we can similarly define the height function $h_T : C(n,d) \to \mathbb{R}$ as the union of $h_\sigma : \operatorname{conv}(\sigma) \to \mathbb{R}$ for every $\sigma \in T$, that is, if a point $p \in \operatorname{conv}(\sigma)$, we let $h_T(p) = h_\sigma(p)$. For a triangulation $T \in S(n,d)$ and a simplex $\sigma \subseteq \gamma_d$, we say $T \leq_{d+1} \sigma$ ($\sigma \leq_{d+1} T$ resp.) if $h_T(p) \leq h_\sigma(p)$ ($h_\sigma(p) \leq h_T(p)$ resp.) for every point $p \in \operatorname{conv}(\sigma)$. By definition, for a triangulation $T \in S(n,d)$ and a simplex $\sigma \subseteq \gamma_d$, $\sigma \leq_{d+1} T$ if and only if for every $\tau \in T$ either σ and τ do not overlap in $\mathbb{R}^d$ or $\sigma <_{d+1} \tau$. A similar statment holds for $\sigma \geq_{d+1} T$.

The following generalizes [8, Propositions 3.2 and 4.1].

Lemma 2.6. *For every triangulation $T \in S(n,d)$ and every simplex $\sigma \subseteq \gamma_d$ there holds: $\sigma \leq_{d+1} T$ if and only if there are no $\sigma' \subseteq \sigma$ and $\tau \in T$ satisfying $\tau <_{d+1} \sigma'$ with $|\sigma'| = \lceil d/2 \rceil + 1$ and $|\tau| = \lfloor d/2 \rfloor + 1$.*

For two triangulations $T, T' \in S(n,d)$, say $T \leq_{d+1} T'$ if $h_T(p) \leq h_{T'}(p)$ for every point $p \in C(n,d)$. This relation $\leq_{d+1}$ is a partial order on $S(n,d)$, and was introduced by Edelman and Reiner [8] as the *second higher Stasheff-Tamari order*. This order (which coincides with their *first* higher Stasheff-Tamari order by [21]) defines the *higher Stasheff-Tamari poset* $\mathrm{HST}(n,d) = (S(n,d), \leq_{d+1})$. We similarly define $\mathrm{HST}(V,d)$ for a subset $V \subseteq [n]$.

The following result by Rambau [16, Theorem 1.1] will also be useful.

Theorem 2.7. *There is a surjective map Ψ from the set of maximal chains of $\mathrm{HST}(n,d)$ to $S(n,d+1)$ such that for a maximal chain $\mathcal{C} : T_1 \leq_{d+1} T_2 \leq_{d+1} \cdots \leq_{d+1} T_k$ in $\mathrm{HST}(n,d)$, the d-skeleton of $\Psi(\mathcal{C})$ is exactly the union of the liftings of all T_i (which in turn determines $\Psi(\mathcal{C})$).*

For a triangulation $T \in S(n,d)$, the *submersion set* $\operatorname{sub}(T)$ is the set

$$\operatorname{sub}(T) = \{\sigma \subseteq \gamma_d : \sigma \leq_{d+1} T, \dim(\sigma) = \lceil d/2 \rceil\}.$$

Submersion sets were used to provide the following encoding result which in particular implies that $\mathrm{HST}(n,d)$ is a lattice for $d = 2, 3$, see [8, Theorems 3.6, 4.9] and [10].

Theorem 2.8. *For $d = 2$ or 3, the poset map Φ from $\mathrm{HST}(n,d)$ to the poset $(\mathcal{P} = 2^{\binom{[n]}{\lceil d/2 \rceil + 1}}, \subseteq)$ given by $\Phi : T \mapsto \operatorname{sub}(T)$ is an injection, and the subposet of $\mathcal{P}$ induced by the image of Φ is a lattice where the meet $\operatorname{sub}(T_1) \wedge \operatorname{sub}(T_2)$ is given by the intersection $\operatorname{sub}(T_1) \cap \operatorname{sub}(T_2)$. In particular, $\mathrm{HST}(n,d)$ is a lattice.*

In order to prove that Φ is injective, one of the key ideas in [8] is to extract "maximal" elements from *diagonal-closed* or *triangle-closed* families, and (re)construct the triangulation from them. We reformulate this idea for our purpose.

Lemma 2.9. *For $T \in S(n,d)$ and a simplex $\tau \subseteq \gamma_d$ with $\tau \leq_{d+1} T$, τ is a face of T if and only if there are no $\sigma \in \operatorname{sub}(T)$ such that $\tau <_{d+1} \sigma$.*

3 Proof of Theorem 1.2

3.1 Proof of Theorem 1.2(i)

The following theorems are useful in the proof of Theorem 1.2(i); their proof are sketched later, after they are used.

Theorem 3.1. *Let* $\sigma \subseteq \gamma_2$ *be an edge or a triangle whose vertices are contained in* $[n]$*. Then there is* $T(\sigma) \in S(n,2)$ *such that* $\sigma \in T(\sigma)$ *and for every triangle or edge* τ *different from* σ *which does not overlap* σ *in* $\mathbb{R}^2$ *or does satisfy* $\tau <_3 \sigma$*, we have* $\tau \leq_3 T(\sigma)$*.*

Theorem 3.2. *Let* $\sigma, \tau_1, \tau_2, \ldots, \tau_m \subseteq \gamma_3$ *be distinct triangles whose liftings are pairwise non-overlapping in* $\mathbb{R}^4$ *and whose vertices are contained in* $[n]$*. We further assume that* $\min(\sigma) \leq \min(\tau_i)$ *for every* $i \in [m]$*, and* $\min(\sigma) = \min(\tau_i)$ *implies* $\max(\sigma) \geq \max(\tau_i)$ *for every* $i \in [m]$*. Then there is* $T \in S(n,3)$ *such that* $\sigma \in T$ *and* $\tau_i \leq_4 T$ *for every* $i \in [m]$*.*

Proof sketch of Theorem 1.2(i). We may assume $\mathcal{F}$ consists only of triangles and edges when $D = 3$, and may assume $\mathcal{F}$ consists only of triangles when $D = 4$; we omit the easy proof of this reduction. Let $d = D - 1$ and $m = |\mathcal{F}|$. We project the simplices in $\mathcal{F}$ into $\mathbb{R}^d$ by ignoring the last coordinate. Abusing notation, we also identify such projections with subsets of $[n]$.

We order these projections as $\sigma_1, \sigma_2, \ldots, \sigma_m$ to satisfy the following conditions: When $D = 3$, we require that for every $1 \leq i < j \leq m$ either $\sigma_i <_D \sigma_j$ or σ_i and σ_j do not overlap in $\mathbb{R}^d$; such an ordering can be given as a linear extension of the partial order $\preceq_D$ by Lemma 2.4. When $D = 4$, we require that whenever $i < j$ we have $\min(\sigma_j) \leq \min(\sigma_i)$, and if $\min(\sigma_i) = \min(\sigma_j)$ then $\max(\sigma_i) \leq \max(\sigma_j)$. By Theorems 3.1 and 3.2, we obtain $T_i \in S(n,d)$ for every $i \in [m]$ such that $\sigma_i \in T_i$ and $\sigma_j \leq_D T_i$ for $j < i$.

Now we use the lattice property of $\mathrm{HST}(n,d)$ in Theorem 2.8. For every $k \in [m]$, let $S_k = T_k \wedge T_{k+1} \wedge \cdots \wedge T_m$ where $\wedge$ denotes the meet operation. Clearly these S_k form a chain in $\mathrm{HST}(n,d)$; considering a maximal chain in $\mathrm{HST}(n,d)$ containing all S_k gives a triangulation $\widetilde{T} \in S(n,D)$ which contains the liftings of all S_k by Theorem 2.7.

It remains to show that S_k contains σ_k for every $k \in [m]$. It is enough to show that for every $\lceil d/2 \rceil$-dimensional face τ of σ_k, we have $\tau \in \mathrm{sub}(S_k)$ and τ satisfies the right-hand side condition of Lemma 2.9 with T replaced by S_k; once it is shown, then by Lemma 2.9 every $\lceil d/2 \rceil$-dimensional face of σ_k belongs to S_k, which implies $\sigma_k \in S_k$.

By Theorem 2.8, we have

$$\mathrm{sub}(S_k) = \mathrm{sub}\Big(\bigwedge_{i=k}^{m} T_i\Big) = \bigcap_{i=k}^{m} \mathrm{sub}(T_i). \tag{3.1}$$

Since $\sigma_k \in T_k$ and $\sigma_k \leq_D T_i$ for every $i \in [k,m]$, we have $\tau \in \mathrm{sub}(T_i)$ for every $i \in [k,m]$, which together with (3.1) implies that $\tau \in \mathrm{sub}(S_k)$. Now assume for a contradiction (by

Lemma 2.9), that there is another $\sigma' \in \operatorname{sub}(S_k)$ such that $\tau <_{d+1} \sigma'$. In particular, σ' and τ overlap in $\mathbb{R}^d$. Now by (3.1), we have $\sigma' \in \operatorname{sub}(T_k)$. By definition of $\operatorname{sub}(T_k)$ we have $\sigma' <_D \tau$, so we have a contradiction since $\sigma' <_D \tau$ and $\tau <_D \sigma'$ are exclusive events by Corollary 2.3. □

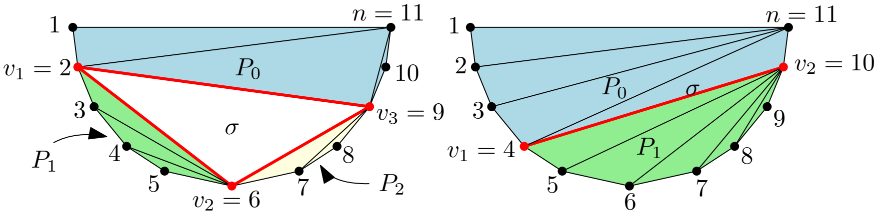


Figure 2: Extension of σ into a triangulation $T(\sigma) \in S(n,2)$.

Proof sketch of Theorem 3.1. For a given edge or a triangle $\sigma \subseteq \gamma_2$ from Theorem 3.1, we add in each polygon divided by σ all the diagonals which are incident with the "maximum" vertex in that polygon, see Figure 2. □

Note that the join of all the triangulations in the lattice $\mathrm{HST}(n,2)$ which contain σ again contains σ – indeed, this join equals $T(\sigma)$ in Theorem 3.1. However, this is not the case for $\mathrm{HST}(n,3)$, which leads us to take a more dynamic approach for proving Theorem 3.2.

Example 3.3. In $\mathrm{HST}(7,3)$, the following two triangulations T and T' are maximal among all triangulations in $S(7,3)$ which contain the simplex $\sigma = \{2,5,6,7\}$, see [8, Figure 4(b)]:

$$T = \{\{1,2,3,5\},\{1,2,5,6\},\{1,2,6,7\},\{1,3,4,5\},\{2,3,5,7\},\{2,5,6,7\},\{3,4,5,7\}\}$$
$$T' = \{\{1,2,3,4\},\{1,2,4,5\},\{1,2,5,6\},\{1,2,6,7\},\{2,3,4,7\},\{2,4,5,7\},\{2,5,6,7\}\}$$

We give a proof sketch of Theorem 3.2. Let $\sigma = \{v_1 < v_2 < v_3\}, \tau_1, \ldots, \tau_m \subseteq \gamma_3$ be the given triangles from Theorem 3.2. By Lemma 2.6, it is enough to find a triangulation T which contains σ and has no edges which are 5-interlacing with any τ_i.

For a simplicial polytope P and a point q outside P in $\mathbb{R}^d$ such that $\{q\} \cup V$ is in general position, we say that a face τ (as a vertex subset) of P *is visible from* q if $\operatorname{conv}(\tau \cup \{q\})$ does not intersect the interior of P. For a triangulation of T of P (without adding new vertices), the *cone of T over q*, denoted by $\operatorname{cone}(T,q)$, is the triangulation of $\operatorname{conv}(P \cup \{q\})$ given by the set of faces

$$T \cup \{\tau \cup \{q\} : \tau \text{ is a face of } P \text{ visible from } q\}.$$

We define four intervals as follows, see Figure 3.

$$J_L = \{i \in [n] : i < v_2\} \neq \emptyset, \qquad J_R = \{i \in [n] : i > v_3\},$$
$$I_M = \{i \in [n] : v_2 < i < v_3\}, \text{ and} \qquad J_M = I_M \cup \{v_2, v_3\} \neq \emptyset.$$

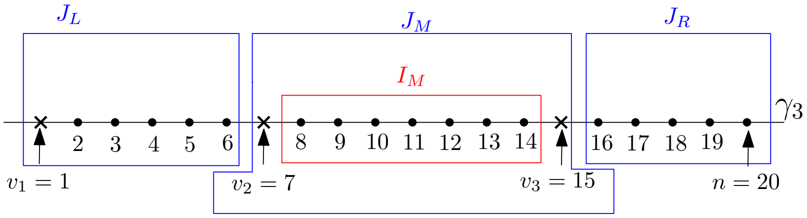


Figure 3: Illustrations of $\sigma = \{v_1, v_2, v_3\}$ and the intervals J_L, J_R and $I_M \subset J_M$.

Proof sketch of Theorem 3.2. We may assume that $v_1 = 1$; this reduction is via the coning operation described above. Let $V_0 = [n] \setminus I_M$ (we may assume that $|V_0| \geq 4$). Let $T_0 = T_M^{V_0}$ be the maximum triangulation of $\mathrm{HST}(V_0, 3)$ on vertex set V_0. By Gale's evenness criterion, σ is contained in T_0. Further the set of edges of T_0 is given as

$$\{\{i,n\}, \{1,j\}, \{k, s(k)\} : i \in V_0 \setminus \{n\}, j \in V_0 \setminus \{1\}, k \in V_0 \setminus \{1,n\}\}. \tag{3.2}$$

None of the edges in (3.2) is 5-interlacing with any triangle τ_i when I_M is empty, so we can take $T = T_0$ in the assertion of the theorem in this case. Thus, we may assume that $l := |I_M| > 0$. Still the edges other than $\{v_2, v_3\} \subseteq J_M$ cannot 5-interlace any of the given triangles τ_i.

Now, we find a certain total order $q_1, q_2, \ldots, q_l$ on I_M and use it to find a desirable triangulation by inductive coning: For $i \in [l]$, let $V_i = V_{i-1} \cup \{q_i\}$ and $T_i = \mathrm{cone}(T_{i-1}, q_i) \in S(V_i, 3)$. The edges newly added at the ith step to form T_i are exactly $\{1, q_i\}$, $\{p(q_i), q_i\}$, $\{q_i, s(q_i)\}$ and $\{q_i, n\}$ where $p(q_i)$ and $s(q_i)$ are the immediate predecessor and successor of q_i in V_i, respectively. Note that these edges are exactly the edges of the cyclic polytope $P_i = \mathrm{conv}(V_i)$ which are not in T_{i-1}. We can find a suitable order on I_M so that these edges never 5-interlace any of the given triangles τ_i; the details of constructing such order on I_M are delicate and are omitted for space reasons. □

3.2 Proof of Theorem 1.2(ii)

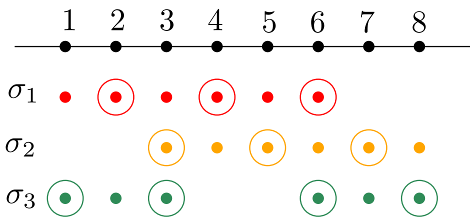


Figure 4: Illustration of $\mathcal{F}$. Circled elements in each σ_i describe the only possible choice of elements which can form a 7-interlacing pattern with other simplices.

Proof sketch of Theorem 1.2(ii). On the moment curve in $\mathbb{R}^5$, consider the family

$$\mathcal{F} = \{\sigma_1 = \{1,2,3,4,5,6\}, \sigma_2 = \{3,4,5,6,7,8\}, \sigma_3 = \{1,2,3,6,7,8\}\}$$

which is not extendable to a triangulation in $S(8,5)$, see Figure 4. Indeed, one cannot add a new 5-simplex in $[8]$ to $\mathcal{F}$ without creating a 7-interlacing with some element of $\mathcal{F}$. To increase the number of vertices n while maintaining dimension D, we add the convex hull of the new vertex $n+1$ and each facet of the cyclic polytope $C(n,D)$ which are visible from $n+1$. To increase both n and D, we take the join $\mathcal{F}' * \{\{n+1\}\}$ from a previous family on γ_D by adding a new vertex $n+1$. □

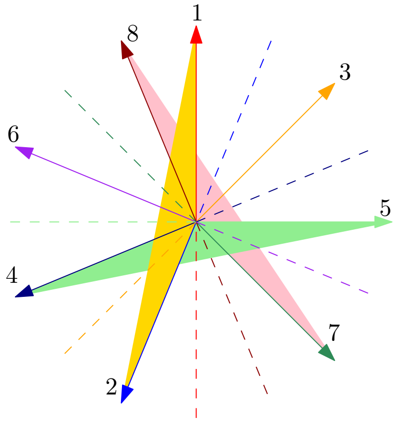


Figure 5: Description of the Gale transform of $\mathcal{F}$. The dual cones to σ_1, σ_2 and σ_3 are spanned by $\{7,8\}$, $\{1,2\}$ and $\{4,5\}$, respectively.

Remark 3.4 (Communicated with Francisco Santos)**.** An alternative way to see non-extendability when $n = D+3$ can be provided via Gale transform, see Figure 5. It is known that when $n = D+3$ every triangulation over n points in $\mathbb{R}^D$ is regular. A given list of D-simplices is extendable to a regular triangulation without adding new vertices if and only if the intersection of all dual cones (the dual cone of σ is the cone at the Gale dual over the complement $[n] \setminus \sigma$) has a full-dimensional intersection. Therefore, it is reduced to checking intersection of the dual cones in $\mathbb{R}^2$. For more backgrounds, consult [6].

4 Final remarks

Algorithmic aspects. Theorem 1.2(i) raises the question of how fast the extension can be computed:

Conditions	$n = d + 2$	$n = d + 3$	$n = d + 4$	$n = d + 5$
Condition (4.1) on $\mathrm{HST}(n, d)$	YES	NO [8]	-	-
Extendability in $\mathbb{R}^D = \mathbb{R}^{d+1}$	YES	YES	NO	NO
$\mathrm{HST}(n, d)$ being a lattice	YES	YES [21]	?	NO [7, 19]

Table 1: Comparison between the lattice conditions and extendability. Here $D = d + 1$.

Problem 4.1. *For $d = 3, 4$, given a geometric complex Δ on n-vertex set V on the moment curve γ_d, find the running time of computing a triangulation $T \in S(V, d)$ of the cyclic d-polytope* $\mathrm{conv}(V)$ *which is an extension of Δ, without adding new vertices; namely in the worst case, as n tends to infinity.*

For Problem 4.1, a greedy algorithm can be given which runs in $O(n^5)$ time.

Quantitative aspects. Theorem 1.2(ii) raises the question of how many new vertices suffice in order to extend:

Problem 4.2. *Let $d \geq 5$. Running over all geometric complexes Δ on an n-vertex set V on the moment curve γ_d, find the minimum number $m(d, n)$ of new vertices that suffices for extension of Δ to a triangulation of* $\mathrm{conv}(V)$*, in the worst case.*

By Theorem 1.2(ii), for every $d \geq 5$ and $n \geq d + 3$ we have $m(d, n) \geq 1$, but we were unable to find a better bound. There are studies on convex partitions of polyhedra [3, 4], but in this context one may subdivide simplices of Δ, which we forbid in our setting.

Connection to the lattice property. Recall that when $d = 2, 3$, Theorem 2.8 guarantees that $\mathrm{HST}(n, d)$ is a lattice where the meet is given by submersion sets to satisfy

$$\mathrm{sub}(T_1 \wedge T_2) = \mathrm{sub}(T_1) \cap \mathrm{sub}(T_2) \text{ for every } T_1, T_2 \in S(n, d). \tag{4.1}$$

Condition (4.1) was important in our proof of Theorem 1.2(i).

However, imposing (4.1) for $\mathrm{HST}(n, d)$ is not necessary for extendability at one higher dimension $D = d + 1$: While it is not difficult to see that given pairwise non-overlapping simplices on γ_D on $n \leq D + 2 = d + 3$ vertices are extendable to a triangulation in $S(n, D)$, it was shown that (4.1) is not satisfied when $d = D - 1$ is 4 or 5 and $n = d + 3 = D + 2$, see [8, Section V, Remark 1]. Still, $\mathrm{HST}(d + 3, d)$ is a lattice (communicated from Nicholas J. Williams) and this can be seen, for instance, by the explicit description of this poset [21, Lemma 3.1]. However for $d = 4, 5$, $\mathrm{HST}(d + 5, d)$ is not a lattice [7, 19]. See Table 1 for a summary of known results.

Question 4.3. *Is* $\mathrm{HST}(d + 4, d)$ *a lattice for every $d \geq 4$?*

Tightness for embedded d-simplices on γ_d. Our construction for Theorem 1.2(ii) gives a non-extendable example with respect to d-simplices that has three d-simplices on γ_d (and on $d + 3$ vertices), for every $d \geq 5$.

Problem 4.4. *For $d \geq 5$, is there a collection of two d-simplices on γ_d which cannot be extended to a triangulation of the convex hull of their union, without adding new vertices?*

Note that no such examples with one d-simplex exist, even if the vertex set includes extra vertices from the moment curve:

Lemma 4.5. *Let $d \geq 2$ and $\sigma \subseteq V \subseteq \gamma_d$ such that V is finite and $|\sigma| = d+1$. Then the simplex σ can be extended to a triangulation of $\operatorname{conv}(V)$ without adding vertices not in V.*

Proof. Order the vertices in $V \setminus \sigma$ in an arbitrary order $v_1, \dots, v_l$. Let $T_0 = \{\sigma' : \sigma' \subseteq \sigma\}$, and $T_i = \operatorname{cone}(T_{i-1}, v_i)$ for $i \in [l]$. Then T_i is a triangulation of $\operatorname{conv}(\sigma \cup \{v_1, \dots v_i\})$, hence T_l is a desired triangulation. $\square$

Remark 4.6. Lemma 4.5 answers [14, Question 5.4(1)] in the affirmative.

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