Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #64, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Interpolation in Species and a Lift of the Hopf Algebra of r-Quasisymmetric Functions

Aaron Lauve[1] and Anthony Lazzeroni[*2]

[1]*Department of Mathematics and Statistics, Loyola University Chicago – Chicago, IL, USA*
[2]*Department of Mathematics, North Park University – Chicago, IL, USA*

Abstract. In this extended abstract we lift the theory of r-quasisymmetric functions to the theory of Hopf monoids. We provide a general method of interpolating between two Hopf monoids, one being the free monoid on a positive comonoid and the other being the free commutative monoid on a positive comonoid.

Keywords: Hopf Monoids, Hopf Algebras, Species

1 Introduction

The Hopf algebras $\{{}^r\mathrm{QSym}\}_{r\in\mathbb{Z}_+}$ form a one-parameter family that interpolates between the Hopf algebra QSym of quasisymmetric functions (at $r = 1$) and the Hopf algebra Sym of symmetric functions (at $r = \infty$). That is, there are injections of Hopf algebras

$$\mathrm{Sym} \hookrightarrow \cdots {}^{r+1}\mathrm{QSym} \hookrightarrow {}^{r}\mathrm{QSym} \hookrightarrow \cdots \mathrm{QSym} \quad (\forall r = 2, 3, \ldots).$$

for this interpolating family. For fixed r and n, Hivert [9] defines ${}^r\mathrm{QSym}_n$ as the invariant space of an "r-local" action of the symmetric group $\mathfrak{S}_n$ on the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ in commuting variables $\mathbf{x}_n = (x_1, x_2, \ldots, x_n)$. Specifically, for adjacent transposition $\sigma_i = (i, i{+}1)$, he defines

$$\sigma_i *_r (x_i^a x_{i+1}^b) := \begin{cases} x_{i+1}^a x_i^b & \text{if } a < r \text{ or } b < r \\ x_i^a x_{i+1}^b & \text{else} \end{cases}.$$

This action is not multiplicative. Nevertheless, the space is shown to be closed under ordinary polynomial product. A coproduct is defined via a standard alphabet-doubling trick; and it is shown to be compatible with polynomial product as n tends to infinity. So ${}^r\mathrm{QSym} = \lim_{n\to\infty} {}^r\mathrm{QSym}_n$ is afforded the structure of graded connected Hopf algebra. A variant ${}^r\mathrm{NCQSym}$ built from noncommuting variables $\mathbf{x}$ was also introduced, interpolating between NCQSym and NCSym.

*aalazzeroni@northpark.edu

In [9] one also finds a conjecture, later proven by Garsia and Wallach in [7], that ${}^r\mathrm{QSym}_n$ is a free module over Sym_n with dimension $n!$. These interpolating families have recently regained attention: [13] introduces a powersum basis that interpolates between natural bases for Sym and QSym; and [2] poses ${}^r\mathrm{NCQSym}$ as a natural setting for generating functions for certain digraph colorings.

The present work was motivated by a desire to unify the construction of these two families of Hopf algebras. The formalism of Hopf monoids in the monoidal category $(\mathsf{Sp}, \cdot)$ of species seemed the right place to start our investigation for a few reasons: (i) the study of combinatorial structures under break and join operations (as promoted by Rota and collaborators [11]) easily carries over to this category; (ii) after the work of Joyal [12], Bergeron–Labelle–Leroux [4] and others, there are easy ways ($+$, $\times$, $\cdot$, and $\circ$) to build new species of structures from old; and (iii) in [1] one finds a systematic study of species and their functors to the category $(\mathsf{gVec}, \otimes)$, where the Hopf algebras Sym, NCSym and the like make their home. As a bonus, (iv) it also happens that (co-, bi-)monoid axioms are easier to verify for species than for graded vector spaces. (See Proposition 8.35 and Remarks 8.8 and 8.36 in [1] for one perspective on this phenomenon.)

In Section 2 we give further background on $(\mathsf{Sp}, \cdot)$, and assorted Hopf monoids therein. Of particular importance are the species $\mathbf{L}$ of linear orders and the exponential species $\mathbf{E}$. We recount some universal constructions defined using these Hopf monoids. In Section 3, we make precise what we mean by an interpolating family $\{{}^r\mathbf{c}\}_{r\in\mathbb{Z}_+}$ of Hopf monoids between two given ones, $\mathbf{b}$ and $\mathbf{d}$. Theorem 2 gives an interpolating family of Hopf monoids between $\mathbf{L} \circ \mathbf{p}$ and $\mathbf{E} \circ \mathbf{q}$ starting from any surjective morphism $\theta : \mathbf{p} \twoheadrightarrow \mathbf{q}$ of positive comonoids in $(\mathsf{Sp}, \cdot)$. We employ the machinery of Fock functors from $(\mathsf{Sp}, \cdot)$ to $(\mathsf{Vec}, \otimes)$ in Section 4, to capture the motivating examples ${}^r\mathrm{QSym}$ and ${}^r\mathrm{NCQSym}$ among many others. We close with a number of related questions in Section 5.

2 Vector Species, and Hopf Monoids Therein

We briefly catalog what is needed from the theory of species and Hopf monoids. We follow closely the notation in [1], where further proofs and all details may be found.

Let Set (respectively, Vec) be the category of sets (vector spaces), with morphisms being arbitrary set maps (vector space maps). Let $\mathsf{Set}^\times$ be the category of finite sets with bijections as morphisms. A *set species* is a functor $\mathrm{p} : \mathsf{Set}^\times \to \mathsf{Set}$ (assigning a set $\mathrm{p}[I]$ for each $|I| < \infty$, and a bijection $\mathrm{p}[\sigma] : \mathrm{p}[I] \to \mathrm{p}[J]$ for each bijection $\sigma : I \to J$). A *(vector) species* $\mathbf{p} : \mathsf{Set}^\times \to \mathsf{Vec}$ is defined similarly. We consider here only *finite* species: $\dim \mathbf{p}[I] < \infty$ for all $|I| < \infty$. We also assume $\operatorname{char} \Bbbk = 0$. Call $\mathbf{p}$ *connected* if $\mathbf{p}[\emptyset] \cong \Bbbk$ and *positive* if $\mathbf{p}[\emptyset] = \{0\}$.

Example 1. Three elementary species, before we continue our discussion. The *exponential species* $\mathbf{E}$, where $\mathbf{E}[I]$ is the 1-dimensional space with basis $\{*_I\}$ for all finite sets I. The *trivial species* $\mathbf{1}$, concentrated in degree 0: $\mathbf{1}[I] = \Bbbk$ if $I = \emptyset$; and $\{0\}$ otherwise. The *linear orders* $\mathbf{L}$, where $\mathbf{L}[I]$ is the space $\Bbbk\{l \mid l$ a linear (total) ordering of $I\}$.

Below and throughout: $S \sqcup T = I$ indicates an ordered decomposition of a set I into (possibly empty) subsets; while $X \vdash I$ indicates an ordinary, unordered partition of I.

Definition 1. The *sum*, *Cauchy product*, and *substitution product* of two species $\mathbf{p}$ and $\mathbf{q}$ are defined, respectively, by

$$(\mathbf{p} + \mathbf{q})[I] := \mathbf{p}[I] \oplus \mathbf{q}[I], \quad (\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{S \sqcup T = I} \mathbf{p}[S] \otimes \mathbf{q}[T],$$
$$\text{and} \quad (\mathbf{p} \circ \mathbf{q})[I] := \bigoplus_{X \vdash I} \mathbf{p}[X] \otimes \mathbf{q}(X),$$

where in the third construction, we take: $\mathbf{q}[\emptyset] = \{0\}$; $(\mathbf{p} \circ \mathbf{q}_+)[\emptyset] = \mathbf{p}[\emptyset]$; and $\mathbf{q}(X)$ as shorthand for the unordered tensor product $\bigotimes_{A \in X} \mathbf{q}[A]$. The Cauchy product of two species $\mathbf{p}, \mathbf{q}$ is viewed as the space of pairs, *i.e.*, $(p, q) \in \mathbf{p}[S] \otimes \mathbf{q}[T] \subseteq (\mathbf{p} \cdot \mathbf{q})[I]$.

For $n \geq 0$, let $\mathbf{p}_n$ be the new species concentrated in cardinality n,

$$\mathbf{p}_n[I] = \begin{cases} \mathbf{p}[I] & \text{if } |I| = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

Example 2. The preceding constructions allow us to recast $\mathbf{1}$ and $\mathbf{L}$ as follows:

$$\mathbf{1} = \mathbf{E}_0 \quad \text{and} \quad \mathbf{L} = \mathbf{1} + \mathbf{E}_1 + \mathbf{E}_1^{\cdot 2} + \mathbf{E}_1^{\cdot 3} + \cdots .$$

Here are some additional species that will be important in what follows.

- Any species is built from its restrictions via infinite sum. Or, viewed the other way around, every species $\mathbf{p}$ has a *canonical decomposition* $\mathbf{p} = \sum_{n \geq 0} \mathbf{p}_n$ [4, Ex. 1.3.6]. Write $\mathbf{p}_+$ for $\sum_{n > 0} \mathbf{p}_n$. Also, put

$$\mathbf{p}_{<r} := \sum_{0 \leq n < r} \mathbf{p}_n \quad \text{and} \quad \mathbf{p}_{\geq r} := \sum_{r \leq n} \mathbf{p}_n.$$

- The species of *set partitions* $\mathbf{\Pi}$ is $\mathbf{E} \circ \mathbf{E}_+$, while the species of *ordered set partitions* (or, set compositions) $\mathbf{\Sigma}$ is $\mathbf{L} \circ \mathbf{E}_+$.

2.1 Monoids, comonoids, and Hopf monoids

The collection $(\mathsf{Sp}, \cdot)$ of species under Cauchy product is a symmetric monoidal category. A *monoid* $\mathbf{b}$ (respectively, *comonoid* $\mathbf{p}$) in $(\mathsf{Sp}, \cdot)$ comes with a bundle of species morphisms $\mu_{S,T} : \mathbf{b}[S] \otimes \mathbf{b}[T] \to \mathbf{b}[I]$ (resp., morphisms $\Delta_{S,T} : \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T]$), one for each decomposition $S \sqcup T = I$, which are (co)associative and (co)unital. The (co)unit and (co)unital axioms are trivial in what follows, so we suppress further mention.

A *Hopf monoid* $\mathbf{b}$ in $(\mathsf{Sp}, \cdot)$ is a monoid and comonoid whose two structures are compatible in a certain sense (making $\mathbf{b}$ a *bimonoid*) and which carries an antipode map in $\mathrm{End}(\mathbf{b})$. After [1, Prop. 8.10], every connected bimonoid in $(\mathsf{Sp}, \cdot)$ is Hopf. We'll be working exclusively with connected bimonoids and won't discuss the antipode further. We also need the following for species $\mathbf{b}$ and $\mathbf{d}$:

- If $\mathbf{b}$ and $\mathbf{d}$ are Hopf, then so is $\mathbf{b} \cdot \mathbf{d}$.
- If $\mathcal{I} \subseteq \mathbf{b}$ is a bi-ideal of a connected bimonoid $\mathbf{b}$, then $\mathbf{b}/\mathcal{I}$ is a Hopf monoid.
- if $\mathbf{b} \overset{\pi}{\hookrightarrow} \mathbf{d}$ is a morphism of Hopf monoids, then so is $\mathbf{d}^* \overset{\pi^*}{\twoheadrightarrow} \mathbf{b}^*$. (In particular, the dual species $\mathbf{b}^* := \bigoplus_I \mathbf{b}[I]^*$ is Hopf whenever $\mathbf{b}$ is.)

2.2 The functors $\mathcal{T}(\text{-})$ and $\mathcal{S}(\text{-})$

We consider the free Hopf monoid $\mathcal{T}(\mathbf{p}) = \mathbf{L} \circ \mathbf{p}$, and free commutative Hopf monoid $\mathcal{S}(\mathbf{p}) = \mathbf{E} \circ \mathbf{p}$, built on a positive comonoid $\mathbf{p}$. As free objects, they satisfy certain universal properties. We give the first here.

Theorem 1 ([1, Thm. 11.10])**.** *Let $\mathbf{h}$ be a Hopf monoid, $\mathbf{p}$ a positive comonoid, and $\zeta : \mathbf{p} \to \mathbf{h}_+$ a morphism of positive comonoids. Let $\eta(\mathbf{p})$ be the natural inclusion of $\mathbf{p}$ into $\mathcal{T}(\mathbf{p})_+$. There exists a unique morphism of Hopf monoids $\hat{\zeta} : \mathcal{T}(\mathbf{p}) \to \mathbf{h}$ such that*

$$\begin{tikzcd} \mathcal{T}(\mathbf{p})_+ \arrow[rr, "\hat{\zeta}_+"] & & \mathbf{h}_+ \\ & \mathbf{p} \arrow[ul, "\eta(\mathbf{p})"] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

is a commutative diagram of positive comonoids.

To describe the Hopf monoid $\mathcal{T}(\mathbf{p})$, we need some notation. From Definition 1, we might write an element of $\mathcal{T}(\mathbf{p})$ as $l \otimes \bigotimes_{A \in X} p_A$, for a given set partition $X \vdash I$ and given elements $l \in \mathbf{L}[X]$ and $\{p_A \in \mathbf{p}[A] \mid A \in X\}$. But if l is a linear order on X, we use the simpler notation "$(p_A)_{A \in F}$" where $F \vDash I$ is a *set composition* of I. We write simple tensors in $\mathcal{S}(\mathbf{p})$ as $\{p_A\}_{A \in X}$.

Example 3. (*The free Hopf monoid $\mathcal{T}(\mathbf{p})$ on $\mathbf{p}$.*) Let $S \sqcup T = I$ be a decomposition, with $F \vDash S$ and $G \vDash T$. The product maps are simple enough:

$$\mu_{S,T} : \mathcal{T}(\mathbf{p})[S] \otimes \mathcal{T}(\mathbf{p})[T] \to \mathcal{T}(\mathbf{p})[I] \qquad (p_A)_{A \in F} \otimes (p_B)_{B \in G} \mapsto (p_C)_{C \in F \cdot G},$$

where $F \cdot G$ is the concatenation of two compositions. For details on the coproduct, see "dequasi-shuffling" in [1, Sec. 11.2.4]. Briefly, we first remark that any composition $H \vDash I$ induces a pair of set compositions $F \vDash S$ and $G \vDash T$, with any given block $C \in H$ decomposing as $A \sqcup B$ for $A \in F$, $B \in G$. We use a modified Sweedler notation for $\Delta^{\mathbf{p}}$: given $p_C \in \mathbf{p}[C]$ with $A \sqcup B = C$, we write $\Delta^{\mathbf{p}}_{A,B}(p_C) = \sum_{(p_C)} p_A \otimes p_B$. Extending this to the multi-tensor product $(p_c)_{C \in H}$, we define the coproduct in $\mathcal{T}(\mathbf{p})$ as follows.

$$\Delta_{S,T} : \mathcal{T}(\mathbf{p})[I] \to \mathcal{T}(\mathbf{p})[S] \otimes \mathcal{T}(\mathbf{p})[T] \qquad (p_C)_{C\in H} \mapsto \sum_{((p_C)_{C\in H})} (p_A)_{A\in F} \otimes (p_B)_{B\in G}.$$

Edge cases (either $A = S \cap C$ or $B = T \cap C$ is empty) are handled in the obvious way.[1]

We write $()$ and $\{\}$ for the units in $\mathcal{T}(\mathbf{p})[\varnothing]$ and $\mathcal{S}(\mathbf{p})[\varnothing]$, respectively.

3 Interpolation in $(\mathsf{Sp}, \cdot)$

Fix two species $\mathbf{b}$ and $\mathbf{d}$ and a totally-ordered set $\mathcal{N}$. We say the (one-parameter) family of species $\{{}^r\mathbf{c}\}_{r\in\mathcal{N}}$ *interpolates between* $\mathbf{b}$ *and* $\mathbf{d}$ if there exist surjective morphisms making the following diagram commute

$$\begin{tikzcd} \mathbf{b} \arrow[r, "\mathsf{port}_r"'] \arrow[rr, bend left, "\mathsf{port}_s"] & {}^r\mathbf{c} \arrow[r, "\mathsf{port}^r_s"'] \arrow[rr, bend left, "\mathsf{port}^r"] & {}^s\mathbf{c} \arrow[r, "\mathsf{port}^s"'] & \mathbf{d}\,. \end{tikzcd} \tag{3.1}$$

And additionally, $\mathsf{port}^r_t = \mathsf{port}^s_t \circ \mathsf{port}^r_s$ $(\forall\, r \le s < t)$.

Remark 1. We focus our discussion, and all proofs that follow, on the case where $\mathcal{N} = \mathbb{Z}_{\ge 1}$, with $\mathbf{b} = {}^1\mathbf{c}$ and $\mathbf{d} = {}^\infty\mathbf{c} = \lim_{r\to\infty} {}^r\mathbf{c}$. (The latter notion is defined carefully below.) But in examples, it may be convenient to allow our indexing to proceed in the opposite direction (say, $\mathbf{b} = {}^\infty\mathbf{c}$ and ${}^1\mathbf{c} = \mathbf{d}$).

More importantly, we extend the definition of interpolating family (as in the introduction) to allow for a chain of injections in place of surjections. One can pass between the two notions by taking duals of the associated (finite) species, *cf.* Section 2.1.

Recall from [4, Def. 1.21] that a sequence $({}^r\mathbf{p})_{r\ge1}$ of species *converges to* $\mathbf{q}$ up to isomorphism (written $\lim_{r\to\infty} {}^r\mathbf{p} \cong \mathbf{q}$) if for any n, there exists an r_0 such that ${}^r\mathbf{p}_{\le n} \cong \mathbf{q}_{\le n}$ for all $r > r_0$. A *decreasing filtration* of a species $\mathbf{b}$, denoted $\mathbf{b} = \bigcup_r \mathbf{b}_{\llcorner r \lrcorner}$, is a sequence of species satisfying $\mathbf{b} = \mathbf{b}_{\llcorner 1 \lrcorner} \supseteq \mathbf{b}_{\llcorner 2 \lrcorner} \supseteq \cdots \supseteq \mathbf{1}$. An *increasing filtration* of a species $\mathbf{d}$, denoted $\mathbf{d} = \bigcup_r \mathbf{d}_{\widehat{r}}$, has $\mathbf{1} \subseteq \mathbf{d}_{\widehat{r}}$, $\mathbf{d}_{\widehat{r-1}} \subseteq \mathbf{d}_{\widehat{r}}$ and $\lim_{r\to\infty} \mathbf{d}_{\widehat{r}} \cong \mathbf{d}$.

Definition 2. Suppose $\pi : \mathbf{b} \twoheadrightarrow \mathbf{d}$ is a morphism of Hopf monoids and there are filtrations $\bigcup_r \mathbf{b}_{\llcorner r \lrcorner}$ and $\bigcup_r \mathbf{d}_{\widehat{r}}$. We say ${}^r\mathbf{c} := \mathbf{b}_{\llcorner r \lrcorner} \cdot \mathbf{d}_{\widehat{r}}$ $(r \ge 1)$ is an *interpolating family of Hopf monoids* if each ${}^r\mathbf{c}$ is Hopf and the diagram (3.1) comprises Hopf maps that factor π.

[1] Since $\mathbf{p}$ is positive, we have no definition for the expression, *e.g.*, $\Delta^{\mathbf{p}}_{\varnothing,C}$, appearing implicitly in the definition of $\Delta_{S,T}$. In such cases, omit the corresponding $\Delta^{\mathbf{p}}$ and give p_C, unaltered, to F or G as appropriate.

3.1 Generalizing and interpolating the abelianization map

Recall the universal property of the $\mathcal{T}(\mathbf{p})$ functor from Section 2.2. In case $\mathbf{h} = \mathcal{S}(\mathbf{p})$, $\hat{\zeta}$ is the *abelianization map* of [1, §11.6.2]. In the further case that $\mathbf{p} = \mathbf{E}_+$, the Hopf monoids $\mathcal{T}(\mathbf{p})$ and $\mathcal{S}(\mathbf{p})$ are $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$, respectively. These will produce in the coming section all the Hopf algebras mentioned in the introduction.

Here, starting from any map $\theta : \mathbf{p} \twoheadrightarrow \mathbf{q}$ of positive comonoids, we use increasing and decreasing filtrations on $\mathbf{p}$ and $\mathbf{q}$ to provide a Hopf monoid structure on $\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})$. From θ we also get a (positive) comonoid map $\mathbf{p}_+ \to \big(\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})\big)_+$. Then Theorem 1 readily provides the requisite Hopf morphisms for an interpolating family (and factors the natural generalization π_θ of the abelianization map). Let us begin.

Put $\mathbf{b}_{\underline{r}} = \mathcal{T}(\mathbf{p}_{\underline{r}}) := \mathbf{L} \circ \mathbf{p}_{\geq r}$ as species; and put $\mathbf{d}_{\widehat{r}} = \mathcal{S}(\mathbf{q}_{\widehat{r}}) := \mathbf{E} \circ \mathbf{q}_{<r}$ as Hopf monoids. (Note that $\mathbf{q}_{<r}$ is a subcomonoid of $\mathbf{q}$, so that $\mathcal{S}(\mathbf{q}_{<r})$ is a Hopf monoid as in Section 2.1.) Then the Cauchy product $\mathcal{T}(\mathbf{p}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})$ is a Hopf monoid as well. We denote this object by ${}^r\mathbf{h}$ to avoid confusion with our target species $\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})$. Lemma 1 establishes the latter as a (Hopf monoid) quotient of the former.

Lemma 1. *Suppose $\theta : \mathbf{p} \to \mathbf{q}$ is a morphism of positive comonoids. Let ${}^r\mathcal{I}$ be the ideal of ${}^r\mathbf{h} = \mathcal{T}(\mathbf{p}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})$ generated by elements*

$$\Big\{ (p_C) \otimes \{\} - () \otimes \{\theta(p_C)\} \ \Big| \ p_C \in \mathbf{p}[C] \text{ with } |C| < r \Big\}. \tag{3.2}$$

Then the quotient ${}^r\mathbf{h}/{}^r\mathcal{I}$ is isomorphic to $\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})$ as species. Moreover, ${}^r\mathcal{I}$ is also a coideal.

Proof. The isomorphism as species is clear, as $\mathcal{T}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q}_{\widehat{r}})$ are free monoids. It suffices to show that any generator x from (3.2) satisfies

$$\Delta(x) \in {}^r\mathbf{h} \cdot {}^r\mathcal{I} \ + \ {}^r\mathcal{I} \cdot {}^r\mathbf{h}.$$

A judicious choice of adding zero will give the result. We have:

$$\begin{aligned}
\Delta_{S,T}(x) &= \sum_{(p_C)} (p_S) \otimes \{\} \mathbin{\dot\otimes} (p_T) \otimes \{\} - \sum_{(\theta(p_C))} () \otimes \{\theta(p_C)_S\} \mathbin{\dot\otimes} () \otimes \{\theta(p_C)_T\} \\
&= \sum_{(p_C)} (p_S) \otimes \{\} \mathbin{\dot\otimes} (p_T) \otimes \{\} - () \otimes \{\theta(p_S)\} \mathbin{\dot\otimes} () \otimes \{\theta(p_T)\},
\end{aligned}$$

since θ is a comonoid map. Dropping the "$\otimes$" and continuing,

$$\begin{aligned}
&= \sum_{(p_C)} (p_S)\{\} \mathbin{\dot\otimes} (p_T)\{\} \ \mp \ (p_S)\{\} \mathbin{\dot\otimes} ()\{\theta(p_T)\} \ - \ ()\{\theta(p_S)\} \mathbin{\dot\otimes} ()\{\theta(p_T)\} \\
&= \sum_{(p_C)} (p_S)\{\} \mathbin{\dot\otimes} \Big[(p_T)\{\} - ()\{\theta(p_T)\} \Big] + \sum_{(p_C)} \Big[(p_S)\{\} - ()\{\theta(p_S)\} \Big] \mathbin{\dot\otimes} ()\{\theta(p_T)\},
\end{aligned}$$

as needed. $\square$

Theorem 2. *Let $\theta : \mathbf{p} \twoheadrightarrow \mathbf{q}$ be a surjective map of positive comonoids. Then $\{\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})\}_{r\geq 1}$ is a one-parameter family of Hopf monoids interpolating between $\mathcal{T}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$.*

Proof. Let θ also denote the obvious composite $\mathbf{p} \to \mathbf{q} \hookrightarrow \mathcal{S}(\mathbf{q})$. The map π_θ alluded to above is the one provided by Theorem 1, $\hat{\theta}$, starting from this map. It remains to define Hopf monoid maps port_r, port^r_s, and port^s that factor π_θ.

The maps port_r. Let θ_r be the composite $\mathbf{p} \xhookrightarrow{\eta} {}^r\mathbf{h}_+ \xrightarrow{\vartheta_r} \big(\mathcal{T}(\mathbf{p}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})\big)_+$. (The first map from the embedding $\mathbf{p} \hookrightarrow \mathcal{T}(\mathbf{p})$; the second from the projection in the lemma.) Then Theorem 1 provides a unique bimonoid map $\mathsf{port}_r : \mathcal{T}(\mathbf{p}) \to {}^r\mathbf{h}$ for all $r > 1$.

The maps port^r_s. Let port^r_s be the composite

$$\mathcal{T}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}}) \xhookrightarrow{\varsigma^r} {}^r\mathbf{h} \xhookrightarrow{\hat{\zeta}} {}^s\mathbf{h} \xrightarrow{\vartheta_s} \mathcal{T}(\mathbf{p}_{\underline{s}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{s}}),$$

where ς^r is any lift of the projection ϑ_r and $\hat{\zeta}$ is induced by the natural embedding of $\mathbf{q}_{\widehat{r}}$ into $\mathbf{q}_{\widehat{s}}$. One checks that ${}^r\mathcal{I} \subseteq {}^r\mathbf{h}$ belongs to the kernel of $\vartheta_s \circ \hat{\zeta}$, so port^r_s is a bimonoid map. Theorem 1 and the fact that θ is surjective then combine to give $\mathsf{port}^r_t = \mathsf{port}^s_t \circ \mathsf{port}^r_s$.

The maps port^r. Follow the steps used to define port^r_s, only replace $\vartheta_s \circ \hat{\zeta}$ with the composition $\mathcal{T}(\mathbf{p}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}}) \xrightarrow{\pi_\theta \cdot \mathrm{id}} \mathcal{S}(\mathbf{q}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}}) \xrightarrow{\mu} \mathcal{S}(\mathbf{q})$. (Analyze the ideal ${}^\infty\mathcal{I}$ of ${}^r\mathbf{h}$ with generators as in (3.2), minus the cardinality condition on I.) This gives us two morphisms from $\mathcal{T}(\mathbf{p})$ to $\mathcal{S}(\mathbf{q})$: $\mathsf{port}^r \circ \mathsf{port}_r$ and $\mathsf{port}^s \circ \mathsf{port}^r_s \circ \mathsf{port}_r$. The uniqueness statement in Theorem 1 and the surjectivity of θ tells us $\mathsf{port}^r = \mathsf{port}^s \circ \mathsf{port}^r_s$, as needed. □

Since $\mathcal{S}(\mathbf{q})$ is commutative, all appeals made to the universal property of $\mathcal{T}(\mathbf{p})$ in the proof above can be made instead to $\mathcal{S}(\mathbf{p})$. We have the following.

Corollary 1. *Let $\theta : \mathbf{p} \twoheadrightarrow \mathbf{q}$ be a surjective map of positive comonoids. Then $\{\mathcal{S}(\mathbf{p}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{q}_{\widehat{r}})\}_{r\geq 1}$ is a one-parameter family of Hopf monoids interpolating between $\mathcal{S}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$.* □

4 From Hopf Monoids to Hopf Algebras

Let $(\mathsf{gVec}, \otimes)$ be the category of graded vector spaces. Consider the *Fock functors* $\mathcal{K}$, $\overline{\mathcal{K}}, \mathcal{K}^\vee : \mathsf{Sp} \to \mathsf{gVec}$ defined by

$$\mathcal{K}(\mathbf{b}) := \bigoplus_{n\geq 0} \mathbf{b}[n], \quad \overline{\mathcal{K}}(\mathbf{b}) := \bigoplus_{n\geq 0} \mathbf{b}[n]_{\mathfrak{S}_n}, \quad \text{and} \quad \mathcal{K}^\vee(\mathbf{b}) := \bigoplus_{n\geq 0} \mathbf{b}[n],$$

where $\mathbf{b}[n]_{\mathfrak{S}_n}$ is the vector space of $\mathfrak{S}_n$-coinvariants of $\mathbf{b}[n]$. The first and the last functors are the Full Fock functors and the middle is the bosonic Fock functor. The last Full Fock functor is the contragradient of the first. That is, $\mathcal{K}^\vee(\mathbf{b}^*) \cong \mathcal{K}(\mathbf{b})^*$.

An in-depth discussion of Fock functors can be found in [1, Part III]. In particular, they are *bilax* functors, meaning they take bimonoids to bimonoids; indeed, connected Hopf monoids in $(\mathsf{Sp}, \cdot)$ are mapped to Hopf algebras in $(\mathsf{gVec}, \otimes)$.

4.1 r-Compositions

Recall that $\mathbf{\Sigma} = \mathcal{T}(\mathbf{E}_+)$ and $\mathbf{\Pi} = \mathcal{S}(\mathbf{E}_+)$. The abelianization map $\pi : \mathbf{\Sigma} \twoheadrightarrow \mathbf{\Pi}$ gives us:

$$\begin{tikzcd} \mathcal{K}(\mathbf{\Sigma}) = \mathrm{M\Pi} \arrow[d, two heads] & \mathcal{K}^\vee(\mathbf{\Sigma}) = \mathrm{NCQSym}^* \arrow[d, two heads] & \overline{\mathcal{K}}(\mathbf{\Sigma}) = \mathrm{NSym} \arrow[d, two heads] \\ \mathcal{K}(\mathbf{\Pi}) = \mathrm{NCSym} & \mathcal{K}^\vee(\mathbf{\Pi}) = \mathrm{NCSym}^* & \overline{\mathcal{K}}(\mathbf{\Pi}) = \mathrm{Sym} \end{tikzcd}$$

and dually, $\mathrm{Sym} \hookrightarrow \mathrm{QSym}$, $\mathrm{NCSym} \hookrightarrow \mathrm{NCQSym}$, and $\mathrm{NCSym}^* \hookrightarrow \mathrm{M\Pi}^*$. So each of the six Hopf pairs has an interpolating family provided by Theorem 2. (Including two new families not previously appearing in the literature.)

Remark 2. The notation for $\mathcal{K}(\mathbf{\Sigma})$ is taken from [1, §17.3.1], this Hopf algebra was first introduced in the context of twisted bialgebras (essentially bimonoids in species) by Patras and Schocker [17]. The notation for $\mathcal{K}^\vee(\mathbf{\Sigma})$ in taken from [5]. After the work of Novelli–Thibon and Foissy [16, 6] it is known to be a self-dual Hopf algebra.

Give increasing and decreasing filtrations to $\mathbf{E}_+$ using the canonical decomposition, $\mathbf{E}_{\underline{r}} := \mathbf{E}_{\geq r}$ and $\mathbf{E}_{\widehat{r}} := \mathbf{E}_{<r}$. Let ${}^r\mathbf{\Sigma} = \mathcal{T}(\mathbf{E}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{E}_{\widehat{r}})$. For a set composition Φ, call a block $B \in \Phi$ large if $|B| \geq r$ and small otherwise. Let $\Phi^{\lfloor r \rfloor}$ and $\Phi^{\lceil r \rceil}$ denote the compositions (induced by Φ) of its large and small blocks, respectively. For $1 < r < s$, the map

$$\begin{aligned} \mathsf{sort}^r_s : {}^r\mathbf{\Sigma} &\twoheadrightarrow {}^s\mathbf{\Sigma} \\ (\Phi; \phi) &\mapsto \left(\Phi^{\lfloor s \rfloor}; \phi \cup \pi\left(\Phi^{\lceil s \rceil}\right)\right) \end{aligned}$$

satisfies the conditions of port. Theorem 2 says that ${}^r\mathbf{\Sigma}$ is a one-parameter family of Hopf monoids interpolating between $\mathbf{\Sigma}$ and $\mathbf{\Pi}$.

4.2 r-Bijections

The Grossman-Larson Hopf algebra of heap-ordered trees, defined in [8], is isomorphic to the Hopf algebra $\mathfrak{S}\mathrm{Sym}$ of bijective endofunctions introduced by Hivert, Novelli, and Thibon in [10]. We will now define a family of Hopf algebras that interpolate between $\mathfrak{S}\mathrm{Sym}$ and NCSym.

Consider the Hopf monoid of bijections $\mathbf{Bij}$. Every bijection is a collection of cycles, so $\mathbf{Bij} = \mathcal{S}(\mathbf{cyc})$, where $\mathbf{cyc}$ is the comonoid of cycles. Then $\mathcal{K}(\mathcal{S}(\mathbf{cyc})) = \mathfrak{S}\mathrm{Sym}$, *cf.* [14, Section 5.4]. At the level of Hopf monoids, we will define an interpolation of Hopf monoids between $\mathcal{S}(\mathbf{cyc})$ and $\mathcal{S}(\mathbf{E})$. Thus our approach will be to use Corollary 1.

Give $\mathbf{cyc}$ the comonoid structure mentioned in [14]. Consider the increasing filtration, through the canonical decomposition, on $\mathbf{cyc}$ by $\mathbf{cyc}_{\underline{r}} := \mathbf{cyc}_{\geq r}$. This leads to our interpolating species $\{{}^r\mathbf{Bij} := \mathcal{S}(\mathbf{cyc}_{\underline{r}}) \cdot \mathcal{S}(\mathbf{E}_{\widehat{r}})\}_{r \geq 1}$.

The map $\theta : \mathbf{cyc}_+[I] \to \mathbf{E}_+[I]$ given by $\theta(\gamma) \mapsto *_I$ is a comonoid morphism. Thereby the port map is defined as

$$\begin{aligned} \mathsf{port}^r_s : {}^r\mathbf{Bij} &\twoheadrightarrow {}^s\mathbf{Bij} \\ (\sigma;\phi) &\mapsto (\sigma^{\lfloor s \rfloor}; \phi \cup \pi_\theta(\sigma^{\lceil s \rceil})) \end{aligned}$$

where $\lfloor s \rfloor$ (resp. $\lceil s \rceil$) is in terms of cycles with size at least s (resp. less than s) and π_θ as in Corollary 1. Through this corollary, we have that $\mathcal{K}({}^r\mathbf{Bij})$ interpolates between $\mathfrak{S}$Sym and NCSym. (And similarly for the other Fock functors.)

4.3 r-Rooted Forests

The Hopf algebra of rooted forests $\mathcal{H}_R$ was first defined in [8]. This is also known as the Connes-Kreimer Hopf algebra, as they define this Hopf algebra in the context of renormalization in quantum field theory. The noncommutative analog $\mathcal{H}_{PR}$ on planar rooted forests was introduced by Foissy. A more complete history of development can be found in [1, §17.5.4]. To define a family of Hopf algebras that interpolates between $\mathcal{H}_R$ and $\mathcal{H}_{PR}$, we consider the Hopf monoids of planar rooted forests $\vec{\mathbf{F}}$ and rooted forests $\mathbf{F}$, which satisfy $\overline{\mathcal{K}}(\vec{\mathbf{F}}) = \mathcal{H}_{PR}$ and $\overline{\mathcal{K}}(\mathbf{F}) = \mathcal{H}_R$.

Writing $\vec{\mathbf{a}}$ (respectively, $\mathbf{a}$) for the species of planar rooted trees (rooted trees), we have $\vec{\mathbf{F}} = \mathbf{L} \circ \vec{\mathbf{a}}$ and $\mathbf{F} = \mathbf{E} \circ \mathbf{a}$. These are isomorphic to $\mathcal{T}(\vec{\mathbf{a}})$ and $\mathcal{S}(\mathbf{a})$ as monoids, but their comonoid structures are different. (While we cannot directly apply Theorem 2, we can mimic its proof to build an interpolating family in the present context.)

We describe the coproduct on $\vec{\mathbf{F}}$ briefly, by its action on a planar rooted forest $f \in \vec{\mathbf{F}}[I]$. A subset $S \subseteq I$ is *f-admissible* if whenever $a \in S$, every ancestor of a is also in S. Given $I = S \sqcup T$, we have

$$\Delta^{\vec{\mathbf{F}}}_{S,T}(f) = \begin{cases} f|_S \otimes f|_T & \text{if } S \text{ is } f\text{-admissible} \\ 0 & \text{else,} \end{cases}$$

where $f|_U$ indicates the forest on $U \subseteq I$ induced by f. The coproduct for $\mathbf{F}$ is similar. Forgetting planar embeddings of trees gives the Hopf monoid morphism $\pi : \vec{\mathbf{F}} \twoheadrightarrow \mathbf{F}$.

Let ${}^r\vec{\mathbf{F}} = \vec{\mathbf{F}}_{\underline{r}} \cdot \mathbf{F}_{\widehat{r}} = (\mathbf{L} \circ \vec{\mathfrak{a}}_{\geq r}) \cdot (\mathbf{E} \circ \mathfrak{a}_{<r})$. (So a basis of ${}^r\vec{\mathbf{F}}$ is pairs $(\vec{f}_1, f_2)$ where every tree in $\vec{f}_1$ has at least r nodes, and every tree in f_2 has fewer than r nodes.) As a family of species, these clearly interpolate between $\vec{\mathbf{F}}$ and $\mathbf{F}$. To give ${}^r\vec{\mathbf{F}}$ a Hopf structure, and verify the conditions in Definition 2, we first consider the larger Hopf monoid ${}^r\mathbf{h} = \vec{\mathbf{F}} \cdot \mathbf{F}_{\widehat{r}}$. Note ${}^r\mathbf{h}$ projects onto ${}^r\vec{\mathbf{F}}$ via the linear map ${}^r\pi : \vec{\mathbf{F}} \cdot \mathbf{F}_{\widehat{r}} \to \vec{\mathbf{F}}_{\underline{r}} \cdot \mathbf{F}_{\widehat{r}}$ that slides small trees from $\vec{\mathbf{F}}$ over to $\mathbf{F}$ and forgets their planar embedding. If $\ker {}^r\pi$ is a Hopf ideal, we're done (see Section 2.1). Concretely, the product and coproduct on ${}^r\vec{\mathbf{F}}$ takes the form

$$\mu^{{}^r\vec{\mathbf{F}}} = \pi \circ \mu^{\vec{\mathbf{F}}} \circ \varsigma^{\otimes 2} \quad \text{and} \quad \Delta^{{}^r\vec{\mathbf{F}}} = \pi^{\otimes 2} \circ \Delta^{\vec{\mathbf{F}}} \circ \varsigma,$$

where ς is a section map of ${}^r\pi$. It is easy to see that the kernel is the ideal

$$\left\langle \vec{t} \otimes 1 - 1 \otimes \pi(\vec{t}) \;\middle|\; \vec{t} \in \vec{\mathbf{a}}_{<r}[I] \right\rangle .$$

It is sufficient to check the coideal condition on generators:

$$\begin{aligned}
\Delta_{S,T}(\vec{t} \otimes 1 - 1 \otimes \pi(\vec{t})) &= \left(\vec{t}|_S \otimes 1\right) \otimes \left(\vec{t}|_T \otimes 1\right) - \left(1 \otimes \pi(\vec{t})|_S\right) \otimes \left(1 \otimes \pi(\vec{t})|_T\right) \\
&= \left(\vec{t}|_S \otimes 1\right) \otimes \left(\vec{t}|_T \otimes 1 - 1 \otimes \pi(\vec{t}|_T)\right) \\
&\quad + \left(\vec{t}|_S \otimes 1 - 1 \otimes \pi(\vec{t}|_S)\right) \otimes \left(1 \otimes \pi(\vec{t}|_T)\right), \qquad (4.1)
\end{aligned}$$

since π is a comonoid map. An easy induction argument shows for any small trees $\vec{t}_1, \ldots, \vec{t}_k$ that $(\vec{t}_1 | \cdots \vec{t}_k) \otimes 1 - 1 \otimes \{\pi(\vec{t}_1), \ldots, \pi(\vec{t}'_k)\}$ belongs to $\ker {}^r\pi$. (Notation as in Section 2.2.) So (4.1) is an element of ${}^r\mathbf{h} \cdot (\ker {}^r\pi) + (\ker {}^r\pi) \cdot {}^r\mathbf{h}$, as needed.

Given a forest $\vec{f} \in \vec{\mathbf{F}}$, define $\vec{f}^{\lfloor r \rfloor}$ and $\vec{f}^{\lceil r \rceil}$ as for set compositions. Let $r < s$ and define the map on the family of species of r-planar rooted forests

$$\begin{aligned}
\pi^r_s : {}^r\vec{\mathbf{F}}[I] &\twoheadrightarrow {}^s\vec{\mathbf{F}}[I] \\
\pi^r(\vec{f}; f) &\mapsto (\vec{f}^{\lfloor s \rfloor}; f \cup \pi(\vec{f}^{\lceil s \rceil})).
\end{aligned}$$

Under the view of π^s_r, $\{{}^r\vec{\mathbf{F}}\}_{r \geq \bullet}$ is a family of Hopf monoids that interpolates between $\vec{\mathbf{F}}$ and $\mathbf{F}$. Thus $\overline{\mathcal{K}}({}^r\vec{\mathbf{F}})$ interpolates between $\mathcal{H}_{PR}$ and $\mathcal{H}_R$.

This example suggests it may be possible to give a more general result than what is presented in Theorem 2.

5 Further Remarks on Interpolation of Species

We highlight some avenues for future research.

Noncommutative examples

In our three examples of interpolating families $\mathbf{b} \twoheadrightarrow {}^r\mathbf{c} \twoheadrightarrow \mathbf{d}$, it transpired that $\mathbf{d}$ was commutative. Is this necessary when building a Hopf monoid structure on $\mathbf{b}_{\underline{r}} \cdot \mathbf{d}_{\widehat{r}}$? (The answer is, "No." But we do expect commutativity is forced by the additional requirement of port^r and port_r factoring a given map $\pi : \mathbf{b} \to \mathbf{d}$ for all r.)

Alternate interpolations

In our framework, we stitch together two filtrations via the Cauchy product. What can be said for the other natural operations on species, *e.g.*, $\mathbf{b}_{\underline{r}} + \mathbf{d}_{\widehat{r}}$, $\mathbf{b}_{\underline{r}} \times \mathbf{d}_{\widehat{r}}$, and $\mathbf{b}_{\underline{r}} \circ \mathbf{d}_{\widehat{r}}$?

The recoil classes of [15] fit into this story (with species operation "$+$"), but a general framework for each of these alternate schemes remains elusive.

In a different direction, we have not considered here the cases of colored and decorated species [1, Chs. 14, 19].

Nor have we spoken about interpolating between Fock functors. Is there a sensible notion of ${}^{s}\mathcal{K}$ interpolating between $\mathcal{K}$ and $\overline{\mathcal{K}}$ (equivalently, between $\mathcal{K}^{\vee}$ and $\overline{\mathcal{K}}^{\vee}$)?

Species extensions

We have focused our attention on the case $\mathbf{b} \twoheadrightarrow {}^{r}\mathbf{c} \twoheadrightarrow \mathbf{d}$ or dually, $\mathbf{b} \hookrightarrow {}^{r}\mathbf{c} \hookrightarrow {}^{r}\mathbf{d}$. Depending on the context, one may allow for more general factorizations. The short exact sequence $\mathbf{b} \hookrightarrow {}^{r}\mathbf{c} \twoheadrightarrow \mathbf{d}$ springs to mind and suggests the question: *given two Hopf monoids in species* $\mathbf{b}, \mathbf{d}$, *can we classify the extensions of* $\mathbf{d}$ *by* $\mathbf{b}$? In the theory of finite dimensional Hopf algebras, all such extensions are *cleft*[2] and there is a suitable cohomology theory that can be worked out in specific examples [3]. The Hopf algebras arising in combinatorics (while not finite) hew closely to the finite case, so we ask for a cohomology theory in $\mathsf{Bimon}(\mathsf{Sp}, \cdot)$.

Acknowledgements

The authors thank Marcelo Aguiar and Rafael González D'León for several helpful conversations throughout the writing of this manuscript.

References

[1] M. Aguiar and S. Mahajan. *Monoidal functors, species and Hopf algebras*. Vol. 29. CRM Monograph Series. With forewords by Kenneth Brown and Stephen Chase and André Joyal. American Mathematical Society, Providence, RI, 2010, pp. lii+784. DOI.

[2] F. Aliniaeifard, S. X. Li, and S. van Willigenburg. "Generalized chromatic functions". *Int. Math. Res. Not. IMRN* 5 (2024), pp. 4456–4500. DOI.

[3] N. Andruskiewitsch and J. Devoto. "Extensions of Hopf algebras". *Algebra i Analiz* **7**.1 (1995), pp. 22–61.

[4] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*. Vol. 67. Encyclopedia of Mathematics and its Applications. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota. Cambridge University Press, Cambridge, 1998, pp. xx+457.

[2] Translating to our context, there are certain (co)actions λ, ρ between $\mathbf{b}$ and $\mathbf{d}$ that twist the componentwise structure on the Cauchy product of species $\mathbf{b} \cdot \mathbf{d}$.

[5] N. Bergeron and M. Zabrocki. "The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free". *J. Algebra Appl.* **8**.4 (2009), pp. 581–600. DOI.

[6] L. Foissy. "Bidendriform bialgebras, trees, and free quasi-symmetric functions". *J. Pure Appl. Algebra* **209**.2 (2007), pp. 439–459. DOI.

[7] A. M. Garsia and N. Wallach. "r-Qsym is free over Sym". *J. Combin. Theory Ser. A* **114**.4 (2007), pp. 704–732. DOI.

[8] R. Grossman and R. G. Larson. "Hopf-algebraic structure of families of trees". *J. Algebra* **126**.1 (1989), pp. 184–210.

[9] F. Hivert. "Local action of the symmetric group and generalizations of quasi-symmetric functions". preprint: `http://igm.univ-mlv.fr/~hivert/paper.html`. 2005.

[10] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. "Commutative combinatorial Hopf algebras". *J. Algebraic Combin.* **28**.1 (2008), pp. 65–95. DOI.

[11] S.-n. A. Joni and G.-C. Rota. "Coalgebras and bialgebras in combinatorics". *Stud. Appl. Math.* **61**.2 (1979), pp. 93–139. DOI.

[12] A. Joyal. "Une théorie combinatoire des séries formelles". *Adv. in Math.* **42**.1 (1981), pp. 1–82. DOI.

[13] A. Lazzeroni. "Powersum bases in quasisymmetric functions and quasisymmetric functions in non-commuting variables". *Electron. J. Combin.* **30**.4 (2023), Paper No. 4.43, 36 pp. DOI.

[14] E. Marberg. "Strong forms of linearization for Hopf monoids in species". *J. Algebraic Combin.* **42**.2 (2015), pp. 391–428. DOI.

[15] J.-C. Novelli, C. Reutenauer, and J.-Y. Thibon. "Generalized descent patterns in permutations and associated Hopf algebras". *European J. Combin.* **32**.4 (2011), pp. 618–627. DOI.

[16] J.-C. Novelli and J.-Y. Thibon. "Polynomial realizations of some trialgebras". *18th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2006)*. 2006, pp. 243–254.

[17] F. Patras and M. Schocker. "Trees, set compositions and the twisted descent algebra". *J. Algebraic Combin.* **28**.1 (2008), pp. 3–23. DOI.