Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #6, 11 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Complementary vectors of doubly Cohen–Macaulay complexes

Matt Larson*[1] and Alan Stapledon†[2]

[1]*Princeton University, Princeton, NJ, USA*
[2]*Sydney Mathematical Research Institute, Sydney, NSW, Australia*

Abstract. We classify the complementary vectors of doubly Cohen–Macaulay complexes. This proves a conjecture of Swartz, negatively answers a question of Athanasiadis and Tzanaki, and gives new bounds on the number of independent sets in a matroid.

Keywords: simplicial complex, doubly Cohen–Macaulay, M-vector

1 Doubly Cohen–Macaulay complexes

Let Δ be a simplicial complex of dimension $(d-1)$ with vertex set $\{1,\dots,n\}$. We say that Δ is *doubly Cohen–Macaulay* (over a field k) if it is Cohen–Macaulay (over k), and $\Delta \setminus v$ is Cohen–Macaulay (over k) of dimension $(d-1)$ for every vertex $v \in \{1,\dots,n\}$. This can be described both algebraically and topologically:

1. Δ is doubly Cohen–Macaulay (over k) if and only if $\tilde{H}_i(|\Delta|;k) = H_i(|\Delta|,|\Delta| - p;k) = 0$ for all $p \in |\Delta|$ and all $i < d-1$, and $\tilde{H}_{d-2}(|\Delta| - p;k) = 0$ for all $p \in |\Delta|$.

2. Δ is doubly Cohen–Macaulay (over k) if and only if the Stanley–Reisner ring $k[\Delta]$ is Cohen–Macaulay, and, after possibly replacing k by an extension, there is a linear system of parameters $\theta_1,\dots,\theta_d$ for $k[\Delta]$ such that the socle of $k[\Delta]/(\theta_1,\dots,\theta_d)$ is concentrated in degree d.

Here $|\Delta|$ is the geometric realization of the simplicial complex Δ. A collection of elements $\theta_1,\dots,\theta_d$ in the degree 1 part of $k[\Delta]$ is a *linear system of parameters* if $k[\Delta]/(\theta_1,\dots,\theta_d)$ is a finite-dimensional vector space over k; a linear system of parameters exists as long as k is infinite. The socle of $k[\Delta]/(\theta_1,\dots,\theta_d)$ is the set of elements x such that $xy = 0$ for every $y \in k[\Delta]/(\theta_1,\dots,\theta_d)$ of positive degree. The socle is concentrated in degree d if and only if, for each nonzero $x \in k[\Delta]/(\theta_1,\dots,\theta_d)$, there is some y such that xy has

*mattlarson@princeton.edu Matt Larson is supported by the Charles Simonyi endowment and the Oswald Veblen fund at the Institute for Advanced Study.

†astapldn@gmail.com Alan Stapledon was partially supported by the Charles Simonyi endowment at the Institute for Advanced Study.

degree d and is nonzero, so $k[\Delta]/(\theta_1, \dots, \theta_d)$ is a *level* ring. If (2) holds for one linear system of parameters, then it holds for any linear system of parameters. See [24, Section III.3] and [5].

Whether a simplicial complex Δ is doubly Cohen–Macaulay over a field k depends only on the characteristic of k. For reasons that will become apparent later, in this article we say that Δ is doubly Cohen–Macaulay if it is doubly Cohen–Macaulay over a field of characteristic 2. Many complexes which arise naturally in combinatorics are doubly Cohen–Macaulay.

Example 1. *If Δ has dimension 1 and has at least three vertices, then Δ is doubly Cohen–Macaulay if and only if it is 2-connected, i.e., it is connected, and it remains connected after any vertex is deleted.*

Example 2. *If Δ is a triangulation of a sphere (i.e., $|\Delta|$ is homeomorphic to a sphere), then Δ is doubly Cohen–Macaulay.*

There are many examples of doubly Cohen–Macaulay complexes coming from complexes with a *convex ear decomposition* in the sense of Chari [9, Section 3.2]. Explicitly, Δ admits a convex ear decomposition if it contains a sequence of subcomplexes $\Sigma_1, \dots, \Sigma_m$, where each Σ_p is the boundary of a d-dimensional simplicial polytope, $\Delta = \cup_p \Sigma_p$, and $\Sigma_q \cap (\cup_{p<q} \Sigma_p)$ is a $(d-1)$-dimensional ball for all $1 < q \leq m$ (see [2, Remark 2.3]). Swartz proved that if Δ admits a convex ear decomposition, then it is doubly Cohen–Macaulay [25, Theorem 4.1]. Examples of simplicial complexes admitting convex ear decompositions (and hence of doubly Cohen–Macaulay complexes) include the independence complex of a coloop-free matroid [9, Theorems 2-3], the Bergman complex of a matroid [20, Section 4], and the augmented Bergman complex of a matroid [2, Theorem 1.2]. See [8, 11, 22, 23, 25, 27] for further examples of complexes with convex ear decompositions.

In some cases, it is easier to prove that a complex is doubly Cohen–Macaulay than to prove that it admits a convex ear decomposition [5, 26]. There are families of combinatorially-defined complexes which are known to be doubly Cohen–Macaulay, but are not known to admit a convex ear decomposition [3, 13].

Because of their ubiquity, it is interesting to obtain bounds on the h-vector of doubly Cohen–Macaulay complexes. Strengthening results of Nevo and Swartz [19, 25], a powerful inequality was proved by Adiprasito, Papadakis, and Petrotou. For a simplicial complex Δ of dimension $(d-1)$, its *g-vector* is $(h_0, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$. The g-vector determines the first half of the h-vector of Δ.

Theorem 1. *[1, Corollary 3.2] Let Δ be a doubly Cohen–Macaulay complex of dimension $(d-1)$. Then the h-vector $(h_0, h_1, \dots, h_d)$ satisfies $h_q \leq h_{d-q}$ for each $q \leq d/2$, and the g-vector of Δ is an M-vector.*

Recall that a sequence $(1, a_1, a_2, \ldots, a_q)$ is an M-vector if there is a standard graded algebra whose Hilbert function is $(1, a_1, \ldots, a_q)$. In Section 3, we recall the explicit inequalities describing M-vectors.

For example, if Δ is a triangulation of a sphere, then the Dehn–Sommerville equations state that $h_q = h_{d-q}$ for each q. In [6], Billera and Lee showed that every M-vector $(a_0, a_1, \ldots, a_{\lfloor d/2 \rfloor})$ occurs as the g-vector of the boundary of a d-dimensional simplicial convex polytope, so Theorem 1 exactly classifies the possible first halves of the h-vectors of doubly Cohen–Macaulay complexes. In light of this result, it is natural to study the *complementary vector* $c(\Delta) = (c_0, c_1, \ldots, c_{\lfloor (d-1)/2 \rfloor})$, where $c_q = h_{d-q} - h_q$, which is nonnegative by Theorem 1.

The following result completely classifies the complementary vectors of doubly Cohen–Macaulay complexes and answers a question of Swartz [25, Problem 4.8].

Theorem 2. *Let Δ be a doubly Cohen–Macaulay simplicial complex of dimension $(d-1)$. Then the complementary vector $c(\Delta)$ is a sum of M-vectors. Moreover, suppose that a sequence $a = (a_0, \ldots, a_{\lfloor (d-1)/2 \rfloor})$ is a sum of M-vectors. Then there is a $(d-1)$-dimensional simplicial complex Δ that admits a convex ear decomposition such that $a = c(\Delta)$.*

In Section 3, we give explicit inequalities describing sums of M-vectors. In the case when Δ admits a convex ear decomposition, that $c(\Delta)$ is a sum of M-vectors was previously shown in [25, Proposition 4.4].

2 Algebraic results

We now describe the algebraic results that underlie the proof of Theorem 2. Let k be a field of characteristic 2, and let $K = k(a_{i,j})_{1 \le i \le d, 1 \le j \le n}$. Let $K[\Delta]$ be the Stanley–Reisner ring of Δ over K, and set

$$H(\Delta) = K[\Delta]/(\theta_1, \ldots, \theta_d), \text{ where } \theta_i = \sum_j a_{i,j} x_j \text{ for each } i.$$

Then $H(\Delta) = H^0(\Delta) \oplus \cdots \oplus H^d(\Delta)$ is graded. Because Δ is Cohen–Macaulay, $\dim H^q(\Delta)$ is the qth entry of the h-vector of Δ. Because Δ is doubly Cohen–Macaulay, $H(\Delta)$ is level: for any nonzero $x \in H^q(\Delta)$, there is some $y \in H^{d-q}(\Delta)$ such that xy is nonzero. In order to prove Theorem 1, Adiprasito, Papadakis, and Petrotou proved the following result.

Theorem 3. *[1, Corollary 3.2] Let Δ be a doubly Cohen–Macaulay complex of dimension $(d-1)$. Let $\ell = x_1 + \cdots + x_n \in H^1(\Delta)$. For each $q \le d/2$, multiplication by ℓ^{d-2q} induces an injection from $H^q(\Delta)$ to $H^{d-q}(\Delta)$.*

This immediately implies Theorem 1, because the Hilbert function of $H(\Delta)/(\ell)$ begins $(h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1}, \ldots)$ and $H(\Delta)/(\ell)$ is a standard graded algebra. We will prove a strengthening of Theorem 3 in Theorem 4 below.

There is a natural identification of the dual vector space $\mathrm{Hom}(H^d(\Delta), K)$ with the reduced homology group $\tilde{H}_{d-1}(|\Delta|; K)$ [17, Theorem 10]. Given any $\mu \in \mathrm{Hom}(H^d(\Delta), K)$, let $H(\Delta, \mu)$ be the induced Gorenstein quotient of $H(\Delta)$. That is, $H(\Delta, \mu)$ is the quotient of $H(\Delta)$ by the ideal $(x \in H^q(\Delta) : \mu(xy) = 0 \text{ for all } y \in H^{d-q}(\Delta))$. By construction, the pairing $H^q(\Delta, \mu) \times H^{d-q}(\Delta, \mu)$ is nondegenerate for each q, so $\dim H^q(\Delta, \mu) = \dim H^{d-q}(\Delta, \mu)$.

Theorem 4. *Let Δ be a doubly Cohen–Macaulay complex of dimension $(d-1)$. There is a map $\mu \colon H^d(\Delta) \to K$ such that, for each $q \leq d/2$, $\dim H^q(\Delta, \mu) = h_q$ and multiplication by ℓ^{d-2q} induces an isomorphism from $H^q(\Delta, \mu)$ to $H^{d-q}(\Delta, \mu)$.*

This is indeed a strengthening of Theorem 3, as the isomorphism from $H^q(\Delta) = H^q(\Delta, \mu)$ to $H^{d-q}(\Delta, \mu)$ given by multiplication by ℓ^{d-2q} factors through $H^{d-q}(\Delta)$.

A sequence of nonnegative integers $(a_0, a_1, \dots, a_m)$ is a sum of M-vectors if and only if there is a graded module $M = M^0 \oplus M^1 \oplus \cdots$ over a polynomial ring in finitely many variables which is generated as a module in degree 0 and has $\dim M^q = a_q$ for all q. We show that $(a_0, a_1, \dots, a_m)$ is a sum of M-vectors if and only if it is identically 0, or if $a_0 > 0$ and $(1, a_1, \dots, a_m)$ is an M-vector. See Proposition 4. In particular, Macaulay's inequalities classifying M-vectors immediately imply a numerical classification of all sums of M-vectors.

Proof of Theorem 2. Theorem 4 states that there is a linear map $\mu \colon H^d(\Delta) \to K$ such that $\dim H^q(\Delta, \mu) = h_q$ for each $q \leq d/2$. Let M be the kernel of the quotient map $H(\Delta) \to H(\Delta, \mu)$. Let M^* be the graded dual of M, i.e., the degree q component of M^* is the dual of the vector space M^{d-q}. This is a module over $H(\Delta)$, with $y \in H(\Delta)$ acting on $f \in M^*$ via $(y \cdot f)(z) = f(y \cdot z)$ for each $z \in M$. See, for example, [10, Section 21.1]. The Hilbert function of M^* is $c(\Delta)$.

It suffices to show that M^* is generated in degree 0 as a module over $H(\Delta)$, as $H(\Delta)$ is generated in degree 1 and so is a quotient of a polynomial ring in finitely many variables. The perfect pairing between M and M^* induces a perfect pairing between $M^*/H^{>0} \cdot M^*$ and $\mathrm{Soc}(M) := \{y \in M : y \cdot z = 0 \text{ for all } z \in H^{>0}(\Delta)\}$. As M is an ideal in the level algebra $H(\Delta)$, $\mathrm{Soc}(M)$ is concentrated in degree d, so $M^*/H^{>0} \cdot M^*$ is concentrated in degree 0. Then M^* is generated in degree 0 by the graded version of Nakayama's lemma.

The construction of all sums of M-vectors as complementary vectors of simplicial complexes with convex ear decompositions is carried out in Proposition 5. □

For the proof of Theorem 4, we refer to [16]. We briefly indicate the main ideas. Recall that $\mathrm{Hom}(H^d(\Delta), K)$ is identified with $\tilde{H}_{d-1}(|\Delta|; K)$. We say that a map $\mu \colon H^d(\Delta) \to K$ is *defined over* $\mathbb{F}_2$ if, under this identification, it corresponds to a homology class which is in the image of the natural map $\tilde{H}_{d-1}(|\Delta|; \mathbb{F}_2) \to \tilde{H}_{d-1}(|\Delta|; K)$. A symmetric bilinear form $B \colon V \times V \to K$ is *anisotropic* if, for all nonzero $x \in V$, $B(x, x)$ is nonzero. One key input to Theorem 4 is the following result, which is similar to [1, Theorem 3] and [15, Theorem 4.4] and is based on an idea introduced in [21].

Proposition 1. *Let $\mu\colon H^d(\Delta) \to K$ be a nonzero map which is defined over $\mathbb{F}_2$, and let $H(\Delta, \mu)$ be the corresponding Gorenstein quotient. Let $\ell = x_1 + \cdots + x_n \in H^1(\Delta, \mu)$. For each $q \leq d/2$, the symmetric bilinear form $H^q(\Delta, \mu) \times H^q(\Delta, \mu) \to K$ given by $(x, y) \mapsto \mu(x \cdot y \cdot \ell^{d-2q})$ is anisotropic.*

Note that $\mathrm{Hom}(H^d(\Delta), K)$ has a basis $\mu_1, \dots, \mu_s$ where each μ_i is defined over $\mathbb{F}_2$. Because Δ is doubly Cohen–Macaulay, $H(\Delta)$ is level, and so we have an injection

$$H(\Delta) \hookrightarrow \bigoplus_{i=1}^{s} H(\Delta, \mu_i).$$

In the proof of Theorem 4, we construct the map μ as a linear combination of the chosen basis $\mu_1, \dots, \mu_s$. We inductively choose $\lambda_1, \dots, \lambda_t \in K$ such that, for each $q \leq d/2$, we have

$$\ker\left(H^q(\Delta) \to \bigoplus_{i=1}^{t} H^q(\Delta, \mu_i)\right) = \ker\left(H^q(\Delta) \to H^q(\Delta, \sum_{i=1}^{t} \lambda_i \mu_i)\right),$$

and $H(\Delta, \sum_{i=1}^{t} \lambda_i \mu_i)$ has the strong Lefschetz property with respect to ℓ. That this is possible for $t = 1$ follows from Proposition 1, with $\lambda_1 = 1$. We show that Proposition 1 guarantees that there is some $\lambda_{t+1} \in K$ such that $\sum_{i=1}^{t+1} \lambda_i \mu_i$ also has these properties, allowing us to continue the induction. Once we find $(\lambda_1, \dots, \lambda_s)$ with these properties, we have proven Theorem 4.

The proof of Proposition 1 requires the characteristic of k to be 2, and it is open whether an analogous statement holds in other characteristics. In [14], the authors established a weaker version of Proposition 1 in any positive characteristic p. This result can be used to deduce the following theorem, see [16, Theorem 5.1].

Theorem 5. *Let k be a field of characteristic $p > 0$, and let Δ be a doubly Cohen–Macaulay complex over k of dimension $(d-1)$. Then the vector composed of the first $\lfloor d/p \rfloor + 1$ entries of $c(\Delta)$ is a sum of M-vectors.*

The following example shows that the conclusion of Theorem 4 can fail if one quotients by a different linear system of parameters.

Example 3. *Let Δ be the 1-dimensional complex in Figure 1, which is doubly Cohen–Macaulay and has h-vector $(1, 3, 2)$. Set $\theta_1^N = a_{1,1}x_1 + a_{1,3}x_3 + a_{1,5}x_5$ and $\theta_2 = a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 + a_{2,5}x_5$. Let $H_N(\Delta) = K[\Delta]/(\theta_1^N, \theta_2)$, which is level and has Hilbert function $(1, 3, 2)$. A basis of $\tilde{H}_1(|\Delta|; K)$ is given by the classes of the induced subcomplexes with vertex sets $\{1, 2, 4, 5\}$ and $\{2, 3, 4, 5\}$, respectively. Call these classes μ_1 and μ_2. Let $\mu = \alpha\mu_1 + \beta\mu_2$ be a nonzero class. If α and β are both nonzero, then $a_{1,1}\alpha^{-1}x_1 + a_{1,3}\beta^{-1}x_3$ is in the kernel of $H_N(\Delta) \to H_N(\Delta, \mu)$. If $\alpha = 0$ or $\beta = 0$, then μ is supported on a subcomplex with h-vector $(1, 2, 1)$. Therefore there is no Gorenstein quotient of $H_N(\Delta)$ with Hilbert function $(1, 3, 1)$.*

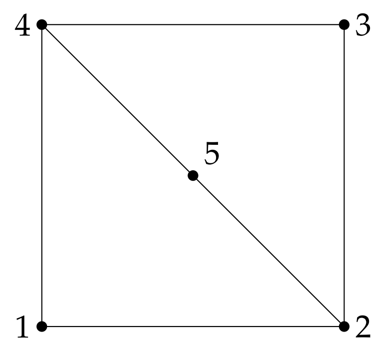

Figure 1: A counterexample to a possible extension of Theorem 4.

3 Constructions and M-vectors

In this section, we discuss M-vectors and Hilbert functions of modules. We give a strengthening of the bounds in Theorem 2. We also give a construction which shows that every sum of M-vectors occurs as the complementary vector of a complex with a convex ear decomposition, completing the proof of Theorem 2.

Recall that an M-vector is the Hilbert function of a finite-dimensional standard graded algebra. There are explicit inequalities which characterize M-vectors that were proved essentially by Macaulay (see [24, p. 56]), and which we now recall.

Given positive integers b and i, there is a unique expression

$$b = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \quad \text{with} \quad n_i > n_{i-1} > \cdots > n_j \geq j \geq 1.$$

We define $b^{\langle i \rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}$. We set $0^{\langle i \rangle} = 0$. Then we have the following result.

Proposition 2. *[18] A sequence of nonnegative integers $(a_0, \ldots, a_m)$ is an M-vector if and only if $a_0 = 1$ and $a_{i+1} \leq a_i^{\langle i \rangle}$ for $1 \leq i \leq m-1$.*

We will need a generalization of Proposition 2 which classifies the Hilbert function of modules which are generated in degree 0 over a polynomial ring in finitely many variables. This was proved by Hulett [12] in characteristic 0 and by Blancafort and Elias [7] in arbitrary characteristic.

Proposition 3. *[7, Theorem 3.2] A sequence of nonnegative integers $(a_0, \ldots, a_m)$ is the Hilbert function of a graded module which is generated in degree 0 over a polynomial ring with s generators if and only if $a_{i+1} \leq q\binom{s+i}{i+1} + r^{\langle i \rangle}$ for all $0 \leq i \leq m-1$, where q and r are the unique nonnegative integers with $a_i = q\binom{s+i-1}{i} + r$ and $r < \binom{s+i-1}{i}$.*

We use Proposition 3 to give a description of the inequalities satisfied by sums of M-vectors.

Proposition 4. *For a sequence $(a_0, \dots, a_m)$ of nonnegative integers, the following are equivalent:*

1. *$(a_0, \dots, a_m)$ is a sum of M-vectors.*
2. *Either the sequence is identically 0, or $a_0 > 0$ and $(1, a_1, \dots, a_m)$ is an M-vector.*
3. *$(a_0, \dots, a_m)$ is the Hilbert function of a graded module which is generated in degree 0 over a polynomial ring in finitely many variables.*

Proof. (1) implies (3): A standard graded algebra A is a module over the polynomial ring $\operatorname{Sym} A^1$ which is generated in degree 0, and so an M-vector is the Hilbert function of a module which is generated in degree 0. We realize a sum of M-vectors as the Hilbert function of a module which is generated in degree 0 by taking the direct sum of these modules.

(3) implies (2): If N is a module which is generated in degree 0 over a polynomial ring in s generators, then it is also a module which is generated in degree 0 over a polynomial ring in t generators for any $t \geq s$, with the extra generators acting by 0. For t sufficiently large, the inequalities in Proposition 3 become the inequalities in Proposition 2, which gives (2).

(2) implies (1): We can write $(a_0, \dots, a_m) = (1, a_1, \dots, a_m) + \sum_{i=1}^{a_0-1}(1, 0, \dots, 0)$ if $a_0 > 0$, and the sequence is the empty sum of M-vectors if it is identically zero. □

The proof of Theorem 2 shows that the complementary vector of Δ is the Hilbert function of a graded module which is generated in degree 0 over $H(\Delta)$. Because $H(\Delta)$ is a quotient of a polynomial ring with h_1 generators, Proposition 3 gives a sharper bound on $c(\Delta)$ in terms of h_1.

Corollary 1. *Let Δ be a doubly Cohen–Macaulay complex of dimension $(d-1)$. We have $c_{i+1} \leq q_i\binom{h_1+i}{i+1} + r_i^{\langle i \rangle}$ for all $0 \leq i \leq (d-3)/2$, where q_i and r_i are the unique nonnegative integers with $c_i = q_i\binom{h_1+i-1}{i} + r_i$ and $r_i < \binom{h_1+i-1}{i}$.*

Because the independence complex of a coloop-free matroid is doubly Cohen–Macaulay [24, p. 94], Corollary 1 gives new inequalities for the h-vectors of independence complexes of coloop-free matroids.

We next turn our attention to constructing examples. We will use the following lemma of Chari. If $h(\Delta) = (h_0, \dots, h_d)$ is the h-vector of a $(d-1)$-dimensional simplicial complex, then we write $h(\Delta; t) = \sum_{i=0}^{d} h_i t^i$.

Lemma 1. *[9, Lemma 3] Let Δ be a $(d-1)$-dimensional simplicial complex that admits a convex ear decomposition, i.e., it contains a sequence of subcomplexes $\Sigma_1, \dots, \Sigma_m$, where each Σ_p is the* *boundary of a d-dimensional simplicial polytope, $\Delta = \cup_p \Sigma_p$, and $\Sigma_q \cap (\cup_{p<q} \Sigma_p)$ is a $(d-1)$-dimensional ball for all $1 < q \leq m$. Then*

$$h(\Delta; t) = h(\Sigma_1; t) + \sum_{q=2}^{m} t^d h(B_q; t^{-1}),$$

where B_q is the closure of $\Sigma_q \setminus (\Sigma_q \cap (\cup_{p<q} \Sigma_p))$.

The following proposition completes the proof of Theorem 2.

Proposition 5. *Suppose that a sequence of nonnegative integers $a = (a_0, \ldots, a_{\lfloor (d-1)/2 \rfloor})$ is a sum of M-vectors. Then there is a $(d-1)$-dimensional simplicial complex Δ that admits a convex ear decomposition such that $a = c(\Delta)$.*

Proof. The zero vector is the complementary vector of any $(d-1)$-dimensional doubly Cohen–Macaulay complex Δ with $\dim \tilde{H}_{d-1}(|\Delta|; k) = 1$, e.g., the boundary of a d-dimensional simplicial polytope. Assume that $a = (a_0, \ldots, a_{\lfloor (d-1)/2 \rfloor})$ is nonzero. Let $\hat{a} = (1, a_1, \ldots, a_{\lfloor (d-1)/2 \rfloor})$. Then $a_0 > 0$ and $\hat{a}$ is an M-vector by Proposition 4. There is a $(d-2)$-dimensional simplicial complex Q that is the boundary of a $(d-1)$-dimensional simplicial polytope with $\hat{a} = g(Q)$ [6]. Recall also that $h(Q; t) = t^{d-1} h(Q; t^{-1})$. Let $S^0 = \{x, x'\}$ be the 0-dimensional sphere with two points. Then the join $Q * S^0$ is a $(d-1)$-dimensional simplicial sphere with $h(Q * S^0; t) = h(Q; t)(1 + t)$. The subcomplexes B and B' given by the joins $Q * \{x\}$ and $Q * \{x'\}$ respectively are $(d-1)$-dimensional balls with h-vectors $h(B; t) = h(B'; t) = h(Q; t)$. Let Δ' be obtained by gluing two copies of $Q * S^0$ along B. Equivalently, Δ' is obtained by gluing a copy of B' to $Q * S^0$ along Q. Then Δ' admits a convex ear decomposition by construction. By Lemma 1, $h(\Delta'; t) = h(Q * S^0; t) + t^d h(B'; t^{-1}) = h(Q; t)(1 + 2t)$. It follows that $t^d h(\Delta'; t^{-1}) - h(\Delta'; t) = h(Q; t)(1 - t)$, and hence $c(\Delta') = g(Q) = \hat{a}$.

Fix a facet F of Δ'. Let Δ be obtained by gluing a_0 copies of Δ' along the complement of the interior of F. Equivalently, Δ is obtained from Δ' by successively gluing $a_0 - 1$ copies of F along the boundary of F. Then Δ admits a convex ear decomposition by construction. Since $h(F; t) = 1$, Lemma 1 implies that $h(\Delta; t) = h(\Delta'; t) + (a_0 - 1)t^d$. Hence $c(\Delta) = c(\Delta') + (a_0 - 1, 0, \ldots, 0) = a$. □

In [4, Question 7.2], Athanasiadis and Tzanaki asked if the following inequalities hold: if Δ is a $(d-1)$-dimensional doubly Cohen–Macaulay complex and $h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta))$, then

$$\frac{h_0(\Delta)}{h_d(\Delta)} \leq \frac{h_1(\Delta)}{h_{d-1}(\Delta)} \leq \cdots \leq \frac{h_{d-1}(\Delta)}{h_1(\Delta)} \leq \frac{h_d(\Delta)}{h_0(\Delta)}. \tag{3.1}$$

As in the proof of Proposition 5, let Q be the boundary of a simplicial $(d-1)$-dimensional polytope with h-vector $h(Q) = (h_0(Q), \ldots, h_{d-1}(Q))$. Let $S^0 = \{x, x'\}$ be the 0-dimensional sphere. Let Δ be the $(d-1)$-dimensional doubly Cohen–Macaulay complex obtained by gluing two copies of the suspension $Q * S^0$ of Q along $Q * \{x\}$. Then we have seen that $h(\Delta; t) = h(Q; t)(1 + 2t)$. Then one computes that (3.1) is equivalent to the condition that $h_{i-1}(Q)h_{i+1}(Q) \leq h_i(Q)^2$ for all $i \geq 0$, i.e., $h(Q)$ is a log-concave sequence. The latter condition does not hold in general. We give a concrete example below.

Example 4. *With the notation above, let Q be the boundary of a simplicial 6-dimensional polytope with $h(Q) = (1, 10, 13, 17, 13, 10, 1)$, which exists by [6]. Note that $h(Q)$ is not log-concave since $10 * 17 = 170 > 13^2 = 169$. Recall that $S^0 = \{x, x'\}$ is the 0-dimensional sphere. Let Δ be the 6-dimensional doubly Cohen–Macaulay complex obtained by gluing two copies of the suspension $Q * S^0$ along $Q * \{x\}$. Then $h(\Delta) = (1, 12, 33, 43, 47, 36, 21, 2)$ and $c(\Delta) = g(Q) = (1, 9, 3, 4)$. We have $\frac{h_2(\Delta)}{h_5(\Delta)} = \frac{33}{36} > \frac{h_3(\Delta)}{h_4(\Delta)} = \frac{43}{47}$, and so Δ does not satisfy (3.1).*

Acknowledgements

We thank Christos Athanasiadis, Luis Ferroni, Anthony Iarrobino, Kalle Karu, Juan Migliore, and Fabrizio Zanello for useful conversations.

References

[1] K. Adiprasito, S. A. Papadakis, and V. Petrotou. "Anisotropy, biased pairings, and the Lefschetz property for pseudomanifolds and cycles" (2021). arXiv:2101.07245.

[2] C. A. Athanasiadis and L. Ferroni. "A convex ear decomposition of the augmented Bergman complex of a matroid". *Ark. Mat.* **64**.1 (2026), pp. 1–17. DOI.

[3] C. A. Athanasiadis and E. Tzanaki. "Shellability and higher Cohen-Macaulay connectivity of generalized cluster complexes". *Israel J. Math.* **167** (2008), pp. 177–191. DOI.

[4] C. A. Athanasiadis and E. Tzanaki. "Symmetric decompositions, triangulations and real-rootedness". *Mathematika* **67**.4 (2021), pp. 840–859. DOI.

[5] K. Baclawski. "Cohen-Macaulay connectivity and geometric lattices". *European J. Combin.* **3**.4 (1982), pp. 293–305. DOI.

[6] L. J. Billera and C. W. Lee. "A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes". *J. Combin. Theory Ser. A* **31**.3 (1981), pp. 237–255. DOI.

[7] C. Blancafort and J. Elias. "On the growth of the Hilbert function of a module". *Math. Z.* **234**.3 (2000), pp. 507–517. DOI.

[8] F. Breuer and A. Dall. "Bounds on the coefficients of tension and flow polynomials". *J. Algebraic Combin.* **33**.3 (2011), pp. 465–482. DOI.

[9] M. K. Chari. "Two decompositions in topological combinatorics with applications to matroid complexes". *Trans. Amer. Math. Soc.* **349**.10 (1997), pp. 3925–3943. DOI.

[10] D. Eisenbud. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785. DOI.

[11] P. Hersh and E. Swartz. "Coloring complexes and arrangements". *J. Algebraic Combin.* **27**.2 (2008), pp. 205–214. DOI.

[12] H. A. Hulett. "A generalization of Macaulay's theorem". *Comm. Algebra* **23**.4 (1995), pp. 1249–1263. DOI.

[13] M. Kallipoliti and M. Kubitzke. "A poset fiber theorem for doubly Cohen-Macaulay posets and its applications". *Ann. Comb.* **17**.4 (2013), pp. 711–731. DOI.

[14] K. Karu, M. Larson, and A. Stapledon. "Differential operators, anisotropy, and simplicial spheres" (2026). arXiv:2412.04561.

[15] K. Karu and E. Xiao. "On the anisotropy theorem of Papadakis and Petrotou". *Algebr. Comb.* **6**.5 (2023), pp. 1313–1330. DOI.

[16] M. Larson and A. Stapledon. "Complementary vectors of simplicial complexes" (2025). arXiv:2504.20264.

[17] C. W. Lee. "P.L.-spheres, convex polytopes, and stress". *Discrete Comput. Geom.* **15**.4 (1996), pp. 389–421. DOI.

[18] F. S. Macaulay. "Some Properties of Enumeration in the Theory of Modular Systems". *Proc. London Math. Soc. (2)* **26** (1927), pp. 531–555. DOI.

[19] E. Nevo. "Rigidity and the lower bound theorem for doubly Cohen-Macaulay complexes". *Discrete Comput. Geom.* **39**.1-3 (2008), pp. 411–418. DOI.

[20] K. Nyman and E. Swartz. "Inequalities for the h-vectors and flag h-vectors of geometric lattices". *Discrete Comput. Geom.* **32**.4 (2004), pp. 533–548. DOI.

[21] S. A. Papadakis and V. Petrotou. "The characteristic 2 anisotropicity of simplicial spheres" (2020). arXiv:2012.09815.

[22] J. Schweig. "A convex-ear decomposition for rank-selected subposets of supersolvable lattices". *SIAM J. Discrete Math.* **23**.2 (2009), pp. 1009–1022. DOI.

[23] J. Schweig. "Convex-ear decompositions and the flag h-vector". *Electron. J. Combin.* **18**.1 (2011), Paper 4, 14. DOI.

[24] R. P. Stanley. *Combinatorics and commutative algebra*. Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. x+164.

[25] E. Swartz. "g-elements, finite buildings and higher Cohen-Macaulay connectivity". *J. Combin. Theory Ser. A* **113**.7 (2006), pp. 1305–1320. DOI.

[26] V. Welker. "On the Cohen-Macaulay connectivity of supersolvable lattices and the homotopy type of posets". *European J. Combin.* **16**.4 (1995), pp. 415–426. DOI.

[27] R. Woodroofe. "Cubical convex ear decompositions". *Electron. J. Combin.* **16**.2 (2009), Research Paper 17, 33. DOI.