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Algebra and geometry of ASM weak order

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Abstract. Much of modern Schubert calculus is centered on Schubert varieties in the complete flag variety $\mathcal{F}\ell(n)$ and on their classes in the integral cohomology ring $\mathcal{H}^*(\mathcal{F}\ell(n))$. Under the Borel isomorphism, these classes are represented by distinguished polynomials called Schubert polynomials, introduced by Lascoux and Schützenberger. Schubert polynomials form an additive basis of $\mathcal{H}^*(\mathcal{F}\ell(n))$.

Knutson and Miller showed that Schubert polynomials are multidegrees of matrix Schubert varieties, affine varieties introduced by Fulton, which are closely related to Schubert varieties. Many roads to studying Schubert polynomials pass through unions and intersections of matrix Schubert varieties. The third author showed that the natural indexing objects of arbitrary intersections of matrix Schubert varieties are alternating sign matrices (ASMs), which had previously enjoyed rich study in enumerative combinatorics and, as the six-vertex ice model, in statistical mechanics. Intersections of matrix Schubert varieties are now called ASM varieties. Every ASM variety is expressible as a union of matrix Schubert varieties.

Many fundamental algebro-geometric invariants (e.g., Castelnuovo–Mumford regularity, codimension, degree) are well understood combinatorially for matrix Schubert varieties, substantially via the combinatorics of strong Bruhat order on S_n. The extension of strong order to $\mathsf{ASM}(n)$, the set of $n \times n$ ASMs, has so far not borne as much algebro-geometric fruit for ASM varieties.

Hamaker and Reiner proposed an extension of weak Bruhat order from S_n to $\mathsf{ASM}(n)$, which they studied from a combinatorial perspective. In the present paper, we place this work on algebro-geometric footing. We use weak order on ASMs to give a characterization of codimension of ASM varieties. We also show that weak order operators commute with K-theoretic divided difference operators.

Keywords: alternating sign matrices, Schubert calculus, weak order

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1 Introduction

Schubert varieties in the complete flag variety are central objects of study within Schubert calculus. The complete flag variety $\mathcal{F}\ell(n)$ is the left quotient of $\mathrm{GL}(n)$ by the Borel subgroup B_- of lower triangular matrices. The Borel subgroup of upper triangular matrices B_+ acts on $\mathcal{F}\ell(n)$ on the right by multiplication. The closures of the orbits under this action are called Schubert varieties, which are indexed by permutations $w \in S_n$. The classes of Schubert varieties form an additive basis for the integral cohomology ring of the complete flag variety. Under the Borel isomorphism [5], representatives of these classes are given by Schubert polynomials $\mathfrak{S}_w$, which were introduced by Lascoux and Schützenberger [20] and defined directly from the combinatorics of the permutation w. Similarly, double Schubert polynomials, which can also be defined combinatorially, are representatives of Schubert classes in the torus-equivariant cohomology ring of $\mathcal{F}\ell(n)$.

Knutson and Miller [17] showed that Schubert polynomials and double Schubert polynomials also arise as multidegrees of Fulton's matrix Schubert varieties, which are affine varieties closely related to Schubert varieties [10]. Because multidegrees are preserved under Gröbner degeneration, Schubert polynomials can also be read from Gröbner degenerations of matrix Schubert varieties, as Knutson and Miller did by using Bergeron and Billey's [1] pipe dreams to index antidiagonal initial varieties of matrix Schubert varieties. Recently, Klein and Weigandt [14] gave a similar connection between diagonal initial schemes of matrix Schubert varieties (which need not be reduced) and Lam, Lee, and Shimozono's [19] bumpless pipe dreams.

A primary motivation in modern Schubert calculus is to understand the Schubert structure constants, i.e. the coefficients $c_{u,v}^{w}$ arising in the expansion $[\mathcal{X}_u][\mathcal{X}_v] = \sum c_{u,v}^{w}[\mathcal{X}_w]$ in the integral cohomology ring of $\mathcal{F}\ell(n)$. These coefficients coincide with those in the expansion $\mathfrak{S}_u\mathfrak{S}_v = \sum c_{u,v}^{w}\mathfrak{S}_w$ of a product of Schubert polynomials as a sum of other Schubert polynomials. Several authors have understood special cases of this problem in terms of pipe dream combinatorics (see, e.g., [18, 1, 2, 13, 12]).

In order to further our understanding of this framework, we are motivated to study unions and intersections of matrix Schubert varieties, both because these unions and intersections allow for inclusion-exclusion counting of multidegrees and also because unions of matrix Schubert varieties arise naturally from Gröbner degenerations of matrix Schubert varieties that are compatible with moves on bumpless pipe dreams [14].

Our broad goal within this program is to understand multidegrees of arbitrary unions and intersections of matrix Schubert varieties in a manner that allows us to glean information about Schubert structure constants. One step in this program is to understand basic invariants of ASM varieties, such as codimension. Ultimately, one would want to be able to read algebro-geometric invariants of the ASM variety X_A from the ASM A with the same ease that these invariants can be read for a matrix Schubert variety X_w from the permutation w, as is done in [10, 28, 27, 24], for example.

Traditionally, the combinatorial tool underpinning algebro-geometric inquiry into matrix Schubert varieties has been strong (Bruhat) order, under which the lattice of ASMs emerges as the MacNeille completion of S_n [21]. In assessing invariants of ASM varieties, this path has proved difficult. In the present work, we lean instead on the combinatorics of weak (Bruhat) order, a tack also taken in [25] in their study of matrix Schubert varieties. Hamaker and Reiner [11] proposed an extension of weak order from S_n to $\mathsf{ASM}(n)$, the set of $n \times n$ ASMs, arguing for its naturality for combinatorial reasons. Terwilliger [29] introduced a poset whose maximal chains are in bijection with ASMs. Hamaker and Reiner showed that linear extensions of the weak order on $\mathsf{ASM}(n)$ give shelling orders on Terwilliger's poset.

In this paper, we present an algebro-geometric case that agrees with [11]'s conclusion. Using their notion of ASM weak order, we show the following:

Theorem 1 (Theorem 3.3)**.** *The codimension of the ASM variety X_A is equal to the minimum length of a saturated chain from the identity permutation to A in weak order.*

Grothendieck polynomials correspond to representatives of the K-theory classes of Schubert varieties. They are key ingredients in the known combinatorial formulas for Castelnuovo–Mumford regularity of matrix Schubert varieties [27, 28, 25, 7, 24, 26]. Grothendieck polynomials are known to satisfy K-theoretic divided difference operators. We define ASM Grothendieck polynomials, extending the definition from Grothendieck polynomials indexed by permutations, and show the following:

Theorem 2 (Theorem 4.1)**.** *ASM Grothendieck polynomials satisfy K-theoretic divided difference recurrences that are compatible with weak order on $\mathsf{ASM}(n)$.*

We also show that weak order operators respect the components of an ASM variety (Proposition 3.2) and that saturated chains in weak order detect the components of ASM varieties (Theorem 3.3) via reduced word combinatorics.

2 Preliminaries

In this section, we will define the set of $n \times n$ alternating sign matrices $\mathsf{ASM}(n)$ and both strong (Bruhat) order and weak (Bruhat) order on $\mathsf{ASM}(n)$.

Given $m \leq n \in \mathbb{Z}$, let $[n] = \{1, \ldots, n\}$ and let $[m, n] = \{m, m+1, \ldots, n-1, n\}$. Fix throughout this paper an arbitrary field κ.

An $n \times n$ ***alternating sign matrix (ASM)*** $A = (A_{i,j})$ is an $n \times n$ matrix with entries in $\{-1, 0, 1\}$ so that (1) for all $i, m \in [n]$, $\sum_{j=1}^{m} A_{i,j} \in \{0, 1\}$, (2) for all $j, m \in [n]$, $\sum_{i=1}^{m} A_{i,j} \in \{0, 1\}$, and (3) $\sum_{(i,j) \in [n] \times [n]} A_{i,j} = n$.

These three conditions are equivalent to the conditions that the nonzero entries in each row and column alternate in sign and sum to one. We write $\mathsf{ASM}(n)$ for the set of $n \times n$ alternating sign matrices.

We may view the symmetric group S_n as the subset of $\mathsf{ASM}(n)$ consisting of the permutation matrices. Specifically, if $w \in S_n$, we will associate to w the permutation matrix M_w with 1's in positions $(i, w(i))$ for all $i \in [n]$ and 0's elsewhere. Identifying an automorphism of $[n]$ with its permutation matrix, we will refer to either as an element of S_n. Given $w \in S_n$ we write $\ell(w) = |\{(i,j) : i < j \text{ and } w(i) > w(j)\}|$ for the ***Coxeter length*** of w.

To $A \in \mathsf{ASM}(n)$, we associate its ***corner sum function*** rk_A defined by $\mathrm{rk}_A(i,j) = \sum_{a \in [i], b \in [j]} A_{a,b}$ for all $i, j \in [n]$. To each $A \in \mathsf{ASM}(n)$, there is an associated subvariety of affine n^2-space constructed using rk_A. Let $\mathrm{Mat}(n)$ denote the set of $n \times n$ matrices with coefficients in the field κ. Given $M \in \mathrm{Mat}(n)$ we write $M_{[i],[j]}$ for the restriction of M to the first i rows and j columns. The ***ASM variety*** of $A \in \mathsf{ASM}(n)$ is $X_A = \{M \in \mathrm{Mat}(n) : \mathrm{rk}(M_{[i],[j]}) \leq \mathrm{rk}_A(i,j) \text{ for all } i, j \in [n]\}$.

We let $z_{i,j}$ with $i, j \in [n]$ denote the coordinates of $\mathrm{Mat}(n)$, $S = \kappa[z_{i,j} : i, j \in [n]]$, and $Z = (z_{i,j})_{i,j \in [n]}$. Let $I_k(Z_{[i],[j]}) \subset S$ be the ideal generated by the k-minors of $Z_{[i],[j]}$. We define $I_A = \sum_{i,j \in [n]} I_{\mathrm{rk}_A(i,j)+1}(Z_{[i],[j]})$, which we call the ***ASM ideal*** of A. Then $X_A = \mathbb{V}(I_A)$. By [15, Section 7.2] and [30, Lemma 5.9], or [14, Lemma 2.6], I_A is radical.

When $A \in S_n$, X_A is called a ***matrix Schubert variety***, and I_A is called a ***Schubert determinantal ideal***. Matrix Schubert varieties were introduced by Fulton [10], who proved, for $A \in S_n$, that X_A is irreducible and Cohen–Macaulay, that $\mathrm{codim}(X_A) = \ell(A)$, and that I_A is a prime ideal defining X_A [10, Theorem 3.3]. For further information on matrix Schubert varieties, we refer the reader to [10, 17].

Proposition/Definition 2.1. *Let $A, B \in \mathsf{ASM}(n)$. The following are equivalent: (1) $X_A \supseteq X_B$. (2) $I_A \subseteq I_B$. (3) $\mathrm{rk}_A(i,j) \geq \mathrm{rk}_B(i,j)$ for all $i, j \in [n]$. When these equivalent conditions are satisfied, we say $A \leq B$ in* **strong (Bruhat) order***.*

If $A, B \in S_n$, the strong order relation $A \leq B$ is equivalent to the usual notion in terms of reduced words. From this combinatorial perspective, we could also characterize $\mathsf{ASM}(n)$ under strong order as arising from (specifically, as the MacNeille completion of) S_n under strong order [21]. As usual, let $\wedge$ denote meet and $\vee$ denote join.

Proposition 2.2 ([30, Proposition 5.4])**.** *If $A, B_1, \ldots, B_k \in \mathsf{ASM}(n)$ and $A = B_1 \vee \cdots \vee B_k$, then $I_A = I_{B_1} + \cdots + I_{B_k}$ and $X_A = X_{B_1} \cap \cdots \cap X_{B_k}$..*

Although Proposition 2.2 shows intersections of arbitrary ASM varieties are again ASM varieties, arbitrary unions of ASM varieties need not be. However, there is an analogous statement of Proposition 2.2 for unions of ASM varieties that do happen to be another ASM variety.

Proposition 2.3. *If $A, B_1, \ldots, B_k \in \mathsf{ASM}(n)$, $A = B_1 \wedge \cdots \wedge B_k$, and $X_{B_1} \cup \cdots \cup X_{B_k}$ is an ASM variety, then $I_A = I_{B_1} \cap \cdots \cap I_{B_k}$ and $X_A = X_{B_1} \cup \cdots \cup X_{B_k}$.*

For $A \in \mathsf{ASM}(n)$, let $\mathrm{Perm}(A) = \{w \in S_n : w \geq A$, and, if $w \geq v \geq A$ for some $v \in S_n$, then $w = v\}$, which we call the ***permutation set of A***.

Proposition 2.4. *[30, Proposition 5.4] Let $A \in \mathsf{ASM}(n)$. Then $I_A = \bigcap_{w \in \mathrm{Perm}(A)} I_w$ is the minimal prime decomposition of the radical ideal I_A. Consequently, $\mathrm{codim}(X_A) = \min\{\ell(w) : w \in \mathrm{Perm}(A)\}$.*

We now recall facts about weak order on $\mathsf{ASM}(n)$, which was introduced by Hamaker and Reiner in [11]. Hamaker and Reiner gave their definition in terms of objects called monotone triangles, which are combinatorially equivalent to ASMs. Specifically,

Lemma 2.5. *Given $A, B \in \mathsf{ASM}(n)$, let m_A and m_B be the monotone triangles corresponding to A and B, respectively, under the bijection of Mills, Robbins, and Rumsey [23]. Then $m_A \leq m_B$ if and only if $A \leq B$. Hence $\mathsf{ASM}(n) \cong \mathsf{MT}(n)$ as posets.*

In order to define weak order, we consider certain sublattices of $\mathsf{ASM}(n)$.

Definition 2.6. *Given $A \in \mathsf{ASM}(n)$ and $i \in [n-1]$, let*

$$\pi_i(A) = \min\{B \in \mathsf{ASM}(n) : \mathrm{rk}_A(a,b) = \mathrm{rk}_B(a,b) \text{ for all } a, b \in [n] \text{ with } a \neq i\}.$$

When $w \in S_n$, then $\pi_i(w) = \begin{cases} w & \text{if } w(i) < w(i+1) \\ ws_i & \text{if } w(i) > w(i+1) \end{cases}$ (see [11, Remark 3.3]).

We say that $A \in \mathsf{ASM}(n)$ has a ***descent*** at $i \in [n-1]$ (or that i is a descent of A) if $\pi_i(A) \neq A$. Note that in this case, $\pi_i(A) < A$. If $\pi_i(A) = A$ or if $i = n$, then we say that A has an ***ascent*** at i (or that i is an ascent of A). We define ***weak order*** $\preceq$ on $\mathsf{ASM}(n)$ to be the transitive closure of the covering relations $\pi_i(A) \prec A$ with i a descent of A.

Note that weak order refines strong order ([11, Remark 3.5]), from which it follows that the transitive closure is well-defined. The weak order operators π_i satisfy commutation and braid relations [11, Proposition 3.2]. That is $\pi_i \circ \pi_j = \pi_j \circ \pi_i$ when $|i-j| > 1$ and $\pi_i \circ \pi_{i+1} \circ \pi_i = \pi_{i+1} \circ \pi_i \circ \pi_{i+1}$. Furthermore, they are idempotent, i.e. $\pi_i^2 = \pi_i$.

As with strong order, the restriction of the given definition of weak order restricted to S_n recovers the usual notion in terms of reduced words. S_n under weak order already forms a lattice [3]. For that reason, we cannot view weak order on $\mathsf{ASM}(n)$ as canonically induced from weak order on S_n in the same way that we saw strong order on $\mathsf{ASM}(n)$ as induced from strong order on S_n.

Let $A \in \mathsf{ASM}(n)$. We say that (i,j) is an ***inversion*** of A if $\sum_{k=1}^{j} A_{i,k} = \sum_{l=1}^{i} A_{l,j} = 0$. We write $\mathrm{inv}(A)$ for the set of inversions of A. When $A \in S_n$, then $|\mathrm{inv}(A)|$ is the Coxeter length (also known as the inversion number) of A. Using the term "inversion" for ASMs in this way was first done in [23] and is now standard in the literature.

We introduce notation for the set of positions of -1's of A, namely $N(A) = \{(i,j) : A_{i,j} = -1\}$. Notice that $N(A) \subseteq \mathrm{inv}(A)$. The ***Rothe diagram*** of $A \in \mathsf{ASM}(n)$ is $D(A) = \{(i,j) : (i,j)$ is an inversion of A and $A_{i,j} = 0\}$. In particular, $\mathrm{inv}(A) = N(A) \sqcup D(A)$.

We remark that when $A \in S_n$, $|D(A)| = \ell(A) = \operatorname{codim}(X_A)$. However, for $A \in \mathsf{ASM}(n) - S_n$, necessarily $D(A) \subsetneq \operatorname{inv}(A)$, and $\operatorname{codim}(X_A)$ may be less than $|D(A)|$, greater than $|\operatorname{inv}(A)|$, or between the two numbers (see, e.g., the examples at the top of Page 21 of [11], interpreted through the lens of Theorem 3.3). Given that these natural candidates fail to characterize $\operatorname{codim}(X_A)$ combinatorially, one must look elsewhere for such a characterization.

3 Codimension of X_A via weak order

We note first that weak order operators on ASMs respect strong order relations and, consequently, containments of ASM varieties.

Theorem 3.1. *Let $A, B \in \mathsf{ASM}(n)$ so that $A \leq B$. Let $i \in [n-1]$. Then $\pi_i(A) \leq \pi_i(B)$.*

In the case that $A, B \in S_n$, Theorem 3.1 follows by a routine application of the Lifting Property for permutations ([4, Proposition 2.2.7]).

Proposition 3.2. *Let $A \in \mathsf{ASM}(n)$. Write $I_A = I_{w_1} \cap \cdots \cap I_{w_r}$ with $w_j \in S_n$. Then $I_{\pi_i(A)} = I_{\pi_i(w_1)} \cap \cdots \cap I_{\pi_i(w_r)}$. In particular, $\operatorname{Perm}(\pi_i(A))$ consists of the minimal elements of $\{\pi_i(w) : w \in \operatorname{Perm}(A)\}$.*

We next characterize $\operatorname{codim}(X_A)$ in terms of saturated chains in the weak order poset. Let $A \in \mathsf{ASM}(n)$, and let e denote the identity element of S_n. Given a saturated weak order chain $e = A_0 \prec A_1 \prec \cdots \prec A_k = A$, we may encode the chain by the tuple $(a_1, \ldots, a_k) \in [n-1]^k$ where $A_{i-1} = \pi_{a_i}(A_i)$ for all $i \in [k]$. Let ***chains***(A) denote the set of all such tuples representing saturated chains from the identity to A in weak order.

Theorem 3.3. *(1) Let $A \in \mathsf{ASM}(n)$. Then $(a_1, \ldots, a_k) \in \operatorname{chains}(A)$ if and only if a substring of $(a_1, \ldots, a_k)$ forms a reduced word for some $w \in \operatorname{Perm}(A)$, $\pi_{a_k}(A) \neq A$, and no π_{a_j} acts trivially on $\pi_{a_{j+1}} \circ \cdots \circ \pi_{a_k}(A)$ for $j \in [k-1]$.*

(2) If $(a_1, \ldots, a_k)$ forms a reduced word for some $w \in \operatorname{Perm}(A)$, then $(a_1, \ldots, a_k) \in \operatorname{chains}(A)$. In particular,

$$\operatorname{Perm}(A) \subseteq \{s_{a_1} s_{a_2} \cdots s_{a_k} : (a_1, \ldots, a_k) \in \operatorname{chains}(A)\}.$$

(3) $\operatorname{codim}(X_A)$ is equal to the minimum length of a saturated chain in the weak order poset from the identity permutation e to A.

Example 3.4. *Let* $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. *We have that* $\operatorname{Perm}(A) = \{4123, 3412\}$. *Furthermore, $\ell(4123) = 3$ and $\ell(3412) = 4$. Thus, $\operatorname{codim}(X_A) = 3$ by Proposition 2.4. We can also see $\operatorname{codim}(X_A) = 3$ using weak order chains. Pictured below is the order ideal of elements below A in weak order.*

$$\begin{tikzcd}
 & \begin{pmatrix}0&0&1&0\\1&0&-1&1\\0&1&0&0\\0&0&1&0\end{pmatrix} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix} \arrow[dd, "\pi_2"'] & & \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&-1&1\\0&0&1&0\end{pmatrix} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
 & \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-1&1\\0&0&1&0\end{pmatrix} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_3"] & & \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \arrow[dl, "\pi_1"'] \\
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \arrow[dr, "\pi_3"] & & \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \arrow[dl, "\pi_2"'] & \\
 & \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} & &
\end{tikzcd}$$

The shortest element of $\mathrm{chains}(A)$ *is* $(3,2,1)$, *which is of length* 3. *We also have* $3 = \mathrm{codim}(X_A)$, *as predicted by Theorem 3.3.*

We see in this example that there may be elements of $\mathrm{chains}(A)$ *that are not reduced words for any elements of* $\mathrm{Perm}(A)$. *For example,* $(3,2,1,2) \in \mathrm{chains}(A)$, *but* $s_3s_2s_1s_2 = 4213 \notin \mathrm{Perm}(A)$. *However,* $(3,2,1)$ *is also an element of* $\mathrm{chains}(A)$, *and* $s_3s_2s_1 = 4123 \in \mathrm{Perm}(A)$. *The issue is that* $\pi_2 \circ \pi_1 \circ \pi_2(A) = \pi_2 \circ \pi_1(A)$, *a possibly unintuitive relation. We use the next proposition to give some information about the chains in the weak order poset that do not come from elements of* $\mathrm{Perm}(A)$.

Given a reduced word $(a_1, \ldots, a_k)$ for w, define $\pi_w = \pi_{a_1} \circ \cdots \circ \pi_{a_k}$. By [11, Proposition 3.2], π_w is independent of the choice of reduced word.

Proposition 3.5. *Let* $A \in \mathsf{ASM}(n)$.

1. *If* $(a_1, \ldots, a_k) \in \mathrm{chains}(A)$, *then it is a reduced word.*

2. *If* $w = s_{a_1}s_{a_2}\cdots s_{a_k}$ *for some* $(a_1, \ldots, a_k) \in \mathrm{chains}(A)$ *then* $w \geq v$ *for some* $v \in \mathrm{Perm}(A)$.

Proposition 3.6. *Let* $A \in \mathsf{ASM}(n)$. *The weak order saturated chains from the identity to* A *are all of the same length if and only if* X_B *is equidimensional for all* $B \preceq A$.

The authors consider Proposition 3.6 to be somewhat surprising for the following reason: It is easy to see that X_A is equidimensional if and only if X_{A^T} is. Indeed, the assignments $z_{i,j} \mapsto z_{j,i}$ induce an isomorphism on the varieties' coordinate rings. Meanwhile, the weak order intervals between the identity and A and A^T, respectively, do not enjoy any such strong, naive connection. See Figure 1 and Figure 2 for an example where the weak order intervals for A and A^T are not anti-isomorphic. We thank Vic

Reiner for pointing out that, moreover, these intervals have different homotopy types and different Möbius functions. (For $w \in S_n$, it is true that the order ideals for w and w^T are anti-isomorphic posets.)

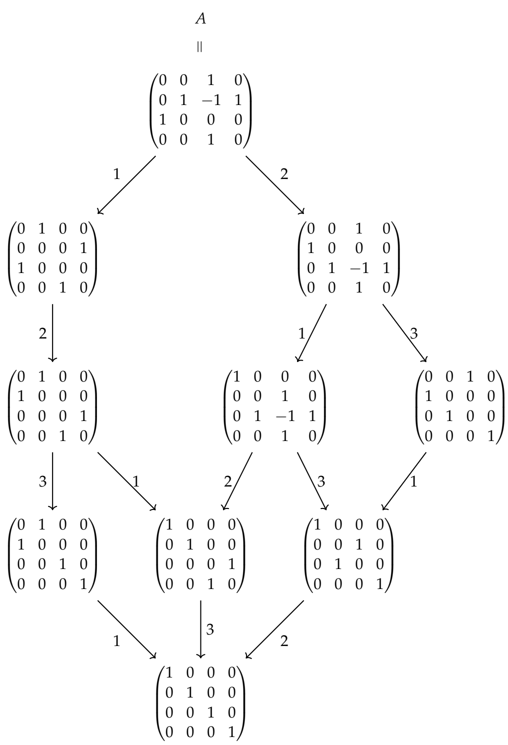

Figure 1: Weak order interval from an ASM A to the identity. An edge label $i \in [3]$ indicates a covering relation determined by π_i. This interval also appears in [11, Figure 1].

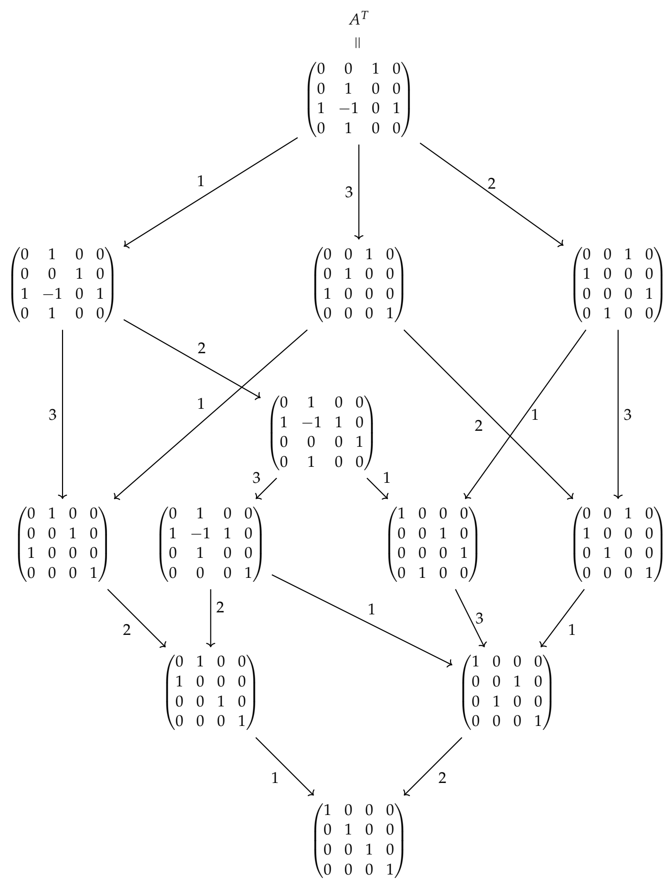

Figure 2: Weak order interval from the ASM A^T to the identity where A is the ASM from Figure 1. An edge label $i \in [3]$ indicates a covering relation determined by π_i.

4 Schubert and Grothendieck polynomials of ASM varieties

Our next goal is to study Grothendieck polynomials - otherwise known as twisted K-polynomials - of ASM varieties. The main goal of this section is to show that weak order operators on ASMs are compatible with K-theoretic divided difference operators on ASM Grothendieck polynomials (Theorem 4.1). In order to do so, we will first introduce the relevant concepts and notation, following the reference [22, Chapter 8].

Given $R = \kappa[z_{i,j} : i, j \in [n]]$ we consider the $\mathbb{N}^{2n}$-grading given by $\deg(z_{i,j}) = e_i + e_{n+j}$. Let J be an ideal that is homogeneous with respect to this grading. The ***multigraded Hilbert series*** of R/J is the formal power series in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ given by $\mathrm{Hilb}(R/J; \mathbf{x}, \mathbf{y}) = \sum_{\mathbf{a} \in \mathbb{N}^{2n}} \dim_\kappa((R/J)_\mathbf{a})(\mathbf{x}, \mathbf{y})^\mathbf{a}$, where $(\mathbf{x}, \mathbf{y})^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n} y_1^{a_{n+1}} \cdots y_n^{a_{2n}}$. There exists a unique polynomial $\mathcal{K}(R/J; \mathbf{x}, \mathbf{y}) \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, called the ***K-polynomial*** of R/J, such that $\mathrm{Hilb}(R/J; \mathbf{x}, \mathbf{y}) = \frac{\mathcal{K}(R/J;\mathbf{x},\mathbf{y})}{\prod_{i,j \in [n]}(1-(\mathbf{x},\mathbf{y})^{\deg(z_{i,j})})}$. Note $(\mathbf{x}, \mathbf{y})^{\deg(z_{i,j})} = x_i y_j$. We define $\widetilde{\mathcal{K}}(R/J; \mathbf{x}, \mathbf{y}) = \mathcal{K}(R/J; \mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y})$ to be the ***twisted K-polynomial*** of R/J. Twisted K-polynomials of matrix Schubert varieties have a cancellation-free combinatorial formula in terms of pipe dreams ([9, 1, 16, 17]) which aid in the analysis of these polynomials. The ***multidegree*** of R/J, denoted $\mathcal{C}(R/J; \mathbf{x}, \mathbf{y})$, is the sum of the lowest degree terms of $\widetilde{\mathcal{K}}(R/J; \mathbf{x}, \mathbf{y})$.

Given $A \in \mathsf{ASM}(n)$ we define $\mathfrak{G}_A(\mathbf{x}, \mathbf{y}) = \tilde{\mathcal{K}}(R/I_A; \mathbf{x}, \mathbf{y})$ and call it the ***double ASM Grothendieck polynomial***. We write $\mathfrak{G}_A(\mathbf{x}) = \mathfrak{G}_A(\mathbf{x}, \mathbf{0})$ for the ***(single) ASM Grothendieck polynomial*** i.e. the result of specializing the y_i's to 0. Likewise, we define $\mathfrak{S}_A(\mathbf{x}, \mathbf{y}) = \mathcal{C}(R/I_A; \mathbf{x}, -\mathbf{y})$ to be the ***double ASM Schubert polynomial*** and $\mathfrak{S}_A(\mathbf{x}) = \mathfrak{S}_A(\mathbf{x}, \mathbf{0})$ to be the ***single ASM Schubert polynomial***. When $A \in S_n$, these definitions agree with the usual definitions of Schubert polynomial and Grothendieck polynomial of a permutation by [17, Theorem A]. See also [6, 8].

We discussed the role of Schubert polynomials in intersection theory in Section 1, where we recalled that expanding products of Schubert polynomials as sums of other Schubert polynomials records complete information about the cohomology of the complete flag variety. From this perspective, it is clear that distinct permutations give rise to distinct Schubert polynomials. Correspondingly, distinct matrix Schubert varieties give rise to distinct Schubert polynomials as their multidegrees (see [17, Section 3.2]).

By contrast, ASM Schubert polynomials cannot reliably see enough of an ASM variety to be as complete a tool. Specifically, because multidegrees are additive over top-dimensional components of a variety [17, Theorem 1.7.1], they necessarily fail to record any information about components of other dimensions. It is to accommodate this type of failure of sensitivity of cohomology that we are motivated to study the K-theory classes of ASM varieties, specifically their ASM Grothendieck polynomials. One goal of this section is to show that distinct ASMs have distinct ASM Grothendieck polynomials (Proposition 4.2). More generally, we aim to build the case that ASM Grothendieck polynomials are often a more suitable tool for studying ASM varieties than ASM Schubert polynomials are, even in settings when Schubert polynomials are an excellent tool for studying matrix Schubert varieties. In order to do this, we will now study divided difference operators.

For permutations, Grothendieck polynomials are usually defined via divided difference operators. In this subsection, we will show that these divided difference relations also hold for ASM Grothendieck polynomials.

Let $\mathbb{Z}[\mathbf{x},\mathbf{y}] = \mathbb{Z}[x_1,\dots,x_n,y_1,y_2,\dots,y_n]$. There is an action of S_n on $\mathbb{Z}[\mathbf{x},\mathbf{y}]$ which permutes the indices of the variables x_i in the usual way and fixes each y_i. Precisely, given $f \in \mathbb{Z}[\mathbf{x},\mathbf{y}]$ and $w \in S_n$ we define $w \cdot f = f(x_{w(1)},\dots,x_{w(n)},y_1,\dots,y_n)$.

The ***divided difference operator*** δ_i acts on $\mathbb{Z}[\mathbf{x},\mathbf{y}]$ as follows: Given $f \in \mathbb{Z}[\mathbf{x},\mathbf{y}]$, $\delta_i(f) = (f - s_i \cdot f)/(x_i - x_{i+1})$. Note that $\delta_i(f) \in \mathbb{Z}[\mathbf{x},\mathbf{y}]$. We also define the ***K-theoretic divided difference operator*** π_i by $\pi_i(f) = \delta_i(1 - x_{i+1}f)$. Divided difference operators satisfy the same braid and commutation relations as do the simple reflections in the symmetric group. Additionally, $\delta_i^2 = 0$ and $\pi_i^2 = \pi_i$.

We are now ready to state the main theorem of this section:

Theorem 4.1. *Let $A \in \mathsf{ASM}(n)$. Then $\pi_i(\mathfrak{G}_A(\mathbf{x},\mathbf{y})) = \mathfrak{G}_{\pi_i(A)}(\mathbf{x},\mathbf{y})$, and $\pi_i(\mathfrak{G}_A(\mathbf{x})) = \mathfrak{G}_{\pi_i(A)}(\mathbf{x})$.*

One may use Theorem 4.1 to show that Grothendieck polynomials distinguish ASM varieties. Specifically:

Proposition 4.2. *Let $A, B \in \mathsf{ASM}(n)$. The following are equivalent (1) $A = B$. (2) $\mathfrak{G}_A(\mathbf{x},\mathbf{y}) = \mathfrak{G}_B(\mathbf{x},\mathbf{y})$. (3) $\mathfrak{G}_A(\mathbf{x}) = \mathfrak{G}_B(\mathbf{x})$.*

Finally, we will see that a result analogous to Theorem 4.1 holds for ASM Schubert polynomials exactly when $\operatorname{codim}(X_{\pi_i(A)}) < \operatorname{codim}(X_A)$. The contrast between Theorem 4.1 and Theorem 4.3 gives an example of the K-theory behaving more predictably with respect to ASMs than cohomology, a consequence of the potential of ASM varieties to fail to be equidimensional.

Theorem 4.3. *Let $A \in \mathsf{ASM}(n)$. If $\operatorname{codim}(X_A) = \operatorname{codim}(X_{\pi_i(A)})$, then $\delta_i(\mathfrak{S}_A(\mathbf{x},\mathbf{y})) = 0 = \delta_i(\mathfrak{S}_A(\mathbf{x}))$. Otherwise, $\delta_i(\mathfrak{S}_A(\mathbf{x},\mathbf{y})) = \mathfrak{S}_{\pi_i(A)}(\mathbf{x},\mathbf{y})$, and $\delta_i(\mathfrak{S}_A(\mathbf{x})) = \mathfrak{S}_{\pi_i(A)}(\mathbf{x})$.*

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