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On two open problems on Ehrhart polynomials of fences

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Abstract.

We show how the coefficients of the order polynomial of a poset can be computed as a weighted sum over all decompositions into convex subposets. Using this result, we provide an explicit combinatorial interpretation of the coefficients of the order polynomial of a fence poset and conjecture a generalization to skew shape posets. Moreover, we use the decomposition into convex subposets to provide a combinatorial interpretation of the coefficients of the Ehrhart polynomial evaluated at $t-3/2$ associated with the zigzag posets. This allows us to express its generating series as a Hadamard product of two explicit series. Using this last result, we give an alternative proof of the second part of the Watanabe–Yoshida conjecture from 2005 by establishing an explicit lower bound on the evaluation of these shifted Ehrhart polynomials.

Keywords: Ehrhart polynomial, Order polynomial, Zigzag posets, Fence posets, skew shape posets, Hilbert-Kunz multiplicity

1 Introduction

We are interested in analyzing the coefficients of the order polynomial $n! \cdot \Omega_P(t) = \sum_{k=1}^{d} a_k(P) t^k$ of a poset P of size n. This polynomial encodes useful information about P: the leading coefficient of $n! \cdot \Omega_P(t)$ is the number of linear extensions in P denoted by $e(P)$ (see [16, Ch3]) which is one of the most important invariants of a poset. From a geometric point of view, following Stanley [15], we consider the *order polytope* : $\mathcal{O}(P) := \{f \in [0,1]^P : f(u) \leq f(v) \text{ for all } u \preceq v \text{ in } P\}$. In particular, the Ehrhart polynomial of $\mathcal{O}(P)$ is equal to $\Omega_P(t+1)$. It is a well-known result that the coefficients $a_k := n!\, c_k(P)$ are (not necessarily nonnegative) integers. Determining when these polynomials have positive coefficients is still an open problem [8]. Ferroni, Morales and Panova [5] recently reduced establishing non-negativity of the coefficients to the positivity of the linear coefficients of all the *convex subposets* of P (see Definition 8). They used this to prove the positivity of the coefficients $a_k(P)$ for fence and more generally skew shape posets. To expand their result, we define decompositions into convex subposets, a concept first observed in [14, Prop. 2.9].

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This formula is particularly interesting for families of posets in which convex subposets remain in the family. This property is equivalent to requiring that each ideal and coideal remains in the same family.

Skew shape posets, which are cell posets of a skew partition, form such a family. Moreover, in this case, Ferroni, Morales, Panova [5], gave an explicit formula of the linear term $a_1(P)$. They left as an open question to find a combinatorial interpretation for these nonnegative coefficients a_k, since in particular $\sum_k a_k = n!$. Our first result, which appears as Theorem 18, is answering this question for an important family of skew shape posets, namely for fences (the posets whose Hasse diagram is a path), via a new block statistic $bl(\cdot)$ (see Definition 15).

Theorem 1 (Theorem 18)**.** *Let P be a fence poset of size n. Then*

$$n! \cdot \Omega(P;t) = \sum_{\sigma} t^{bl(\sigma)},$$

where the sum is over all $n!$ labelings of P.

We also conjecture a generalization $\widetilde{bl}(\cdot)$ for skew shape posets in Conjecture 21.

Zigzag posets Z_n are the fence posets where every element is either maximal or minimal. These posets appear in various parts of combinatorics [2, 11], and other areas like algebra, probability, and algebraic geometry [3, 4, 1]. In the latter context, recently Pak, Shapiro, Smirnov and Yoshida [12] used this poset to study the *Hilbert-Kunz multiplicity* of simple (A_1)-singularities of dimension n and character p, denoted by $e_{HK}(\mathcal{A}_{p,n})$, an important invariant that gives information on the singularities of a local ring. They showed that this multiplicity respects the following inequality that can be deduced from [12, Corollary 3.10] : $e_{HK}(\mathcal{A}_{p,n}) \geq 1 + \Omega(Z_n; q - 1/2)/q^n$, where $q = p/2$.

Because of this inequality, [12] raised the question of whether Ehrhart theory of the order polytope of the zigzag poset could be used to prove the second part of the Yoshida–Watanabe conjecture from 2005 which states that $e_{HK}(\mathcal{A}_{p,n}) \geq 1 + E_n/n!$. Here E_n denotes the n-th Euler number, which counts the number of linear extension of the Zigzag poset Z_n. Recently, Meng [9] proved this conjecture by different methods.

In the literature, one can find recursive formulas for the ordinary generating function of $\Omega(Z_n;t)$ [13, Thm. 3.15] and [16, Ex. 3.66]. Here we provide a formula for the exponential generating function of the order polynomial of zigzag posets as a Hadamard product $\odot$ of the generating function of Euler numbers and an explicit algebraic generating function.

Theorem 2 (Theorem 32)**.** *For $n \geq 1$ and for all t :*

$$\sum_{n\geq 0} \Omega(Z_n;t)\frac{z^n}{n!} = (\tan(t+1/2) + \sec(t+1/2)) \odot_t G(t+1/2;z), \tag{1.1}$$

where $G(t;z) = \sqrt{1 - z^2/4} \cdot (1 + 2t \arcsin(z/2)) / (1 - 4t^2 \arcsin(z/2))$ *and* $\sum_{m\geq 0} A_m(z)t^m \odot_t \sum_{m\geq 0} B_m(z)t^m := \sum_{m\geq 0} A_m(z)B_m(z)t^m$

To the best of our knowledge, this is the first known non-recursive formula for this generating function. It has an immediate application: using this formula for explicit computation, we show that the second part of the Watanabe-Yoshida conjecture holds, fullfiling the Ehrhart theory approach to this problem suggested in [12].

Theorem 3 (Theorem 34)**.** *For all* $p \geq 3$ *and* $n \geq 2$*, we have the inequality*

$$e_{HK}(\mathcal{A}_{p,n}) \geq 1 + \frac{E_n}{n!}.$$

This abstract is organized as follows: Sections 2 and 3 include background on order polynomials and details on the poset decomposition. Section 4 includes the results on the statistic of order polynomials, based on [7]. Section 5 includes the details behind Theorem 32 and how it leads to the proof of the Watanabe-Yoshida conjecture, based on [6].

2 Order polynomials

Definition 4. Let $P = (X, \preceq)$ be a partially ordered set (poset) and let $|X| = n$. The *order polynomial* of P is the unique polynomial $\Omega(P;t) \in \mathbb{Q}[t]$ such that

$$\Omega(P;t) := \#\{f : X \to [1,\ldots,t] \quad \text{s.t.} \quad f(x) \leq f(y) \text{ if } x \preceq y\} \tag{2.1}$$

for $t \in \mathbb{N}$

It is a classical result that this function is a polynomial and that $n! \cdot \Omega(P;t)$ has integer coefficients. Therefore, we write $\sum_{k=0}^{n} a_k(P)t^k := n!\,\Omega(P;t)$ and $\sum_{k=0}^{n} c_k(P)t^k := \Omega(P;t)$.

Definition 5. The *order polytope* of P, introduced by Stanley [15], is defined as :

$$\mathcal{O}(P) := \{f \in [0,1]^P \text{ s.t. } f(u) \leq f(v) \text{ for all } u \preceq v \text{ in } P\}. \tag{2.2}$$

The polytope $\mathcal{O}(P)$ is integral. Stanley showed in [15, Thm. 4.1] that the *Ehrhart polynomial* of $\mathcal{O}(P)$, which counts the number of lattice points in the dilation $t \cdot \mathcal{O}(P)$, is given by a shift of the order polynomial $\Omega(P;t)$.

Proposition 6 ([15, Stanley])**.** *For any poset* P*, the Ehrhart polynomial of* $\mathcal{O}(P)$ *satisfies*

$$\#(t \cdot \mathcal{O}(P) \cap \mathbb{Z}^n) = \Omega(P;t+1).$$

Recently, Ferroni, Morales and Panova [5, Prop. 3.3] showed that the coefficients c_k satisfy the following recurrence :

Lemma 7 ([5])**.** *For any poset P, and every $k \geq 2$:*

$$c_k(P) = \frac{1}{2^k - 2} \left(\sum_{I \subsetneq P, I \neq \emptyset} \sum_{i=1}^{k-1} c_i(I) c_{k-i}(P \setminus I) \right).$$

where I is a nonempty proper ideal of P.

They used this to prove that the nonnegativity of the coefficients of the order polynomial of a poset P is implied by the positivity of the linear coefficient of the order polynomial of every ideal and coideal of P.

3 Decomposition into convex subposets

We can extend the result of Lemma 7 using decomposition into convex subposets. The main result of this section (Theorem 12) first appeared in a preprint by Shareshian, Wright, and Zhao [14].

Definition 8. Let $P = (X_P, \preceq_P)$ be a poset. A **convex subposet** R of P is a nonempty poset $R = (X_R, \preceq_R)$ satisfying the following properties:
- R is connected,
- $X_R \subset X_P$,
- (Subposet) For all $a, b \in X_R$, $a \preceq_R b$ if and only if $a \preceq_P b$,
- (Convexity) If $a \preceq_P b \preceq_P c$ and $a, c \in X_R$, then $b \in X_R$.

Definition 9 (Decomposition into convex subposets)**.** Let P be a poset. A *decomposition μ of P into convex subposets* is a partition of the ground set of P into k convex subposets $P_1, \ldots, P_k$. For two subposets P_i, P_j of P from the decomposition μ, we say that $P_i \tilde{\leq}_\mu P_j$ if and only if there exist $a_i \in P_i$ and $a_j \in P_j$ such that $a_i \leq_P a_j$.

The relation $\tilde{\leq}_\mu$ is not necessarily a partial order. To obtain one, we restrict to a smaller class of decompositions into convex subposets.

Definition 10 (Standard decomposition)**.** A decomposition $\mu = \{P_1, \ldots, P_k\}$ into convex subposets is a *standard* decomposition if there do not exist integers $2 \leq r \leq k$ and distinct indices $i_1, i_2, \ldots, i_r$ such that $P_{i_1} \tilde{\leq}_\mu P_{i_2} \tilde{\leq}_\mu \ldots \tilde{\leq}_\mu P_{i_r} \tilde{\leq}_\mu P_{i_1}$. Figure 1 shows an example of a standard decomposition.

Proposition 11. *Let P a poset and μ a decomposition of P into subposets. Let $\leq_\mu$ be the transitive closure of $\tilde{\leq}_\mu$. Then,*

$$\leq_\mu \text{ is a partial order if and only if } \mu \text{ is a standard decomposition.}$$

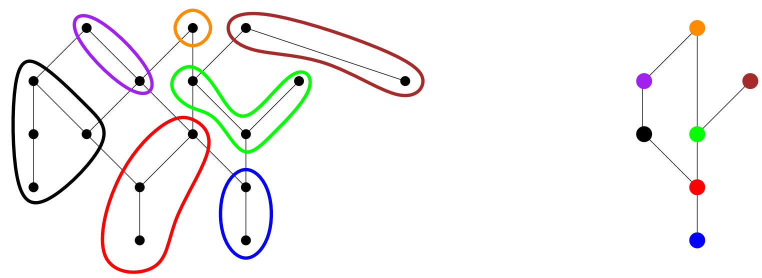

Figure 1: A standard decomposition μ into convex subposets and its induced poset.

Figure 1 shows an example of a standard decomposition μ and the induced poset $(\mu, \leq_\mu)$.We denote by $\mathsf{SDecompo}_k(P)$ the set of standard decompositions of P into k convex subposets. In this case, we define $e(\mu)$ to be the number of linear extensions of $(\mu, \leq_\mu)$.

Theorem 12 ([14]). *Let $\Omega(P;t) = \sum_{k=0}^{n} c_k(P)t^k$ be the order polynomial of a poset P.Then*

$$c_k(P) = \frac{1}{k!} \sum_{\substack{\mu \in \mathsf{SDecompo}_k(P) \\ \mu = \{P_1, P_2, \ldots, P_k\}}} e(\mu) \cdot \prod_{i=1}^{k} c_1(P_i).$$

for every $k \in \mathbb{N}$.

Sketch of proof. The result follows by applying Lemma 7 together with an inductive argument. The quantity $e(\mu)$ appears as the number of ways of splitting the poset whenever variables are split in its order polynomial. The full proof appears in [6]. □

4 Statistic for order polynomial of fences and skew shapes

Since $n! \cdot \Omega(P;1) = n!$, which is the cardinality of all labelings of the elements of P, and since, when P is a skew shape poset, all coefficients of $n!\,\Omega(P;t)$ are nonnegative, the authors of [5] asked whether there exists a statistic on the set of bijective labelings of the elements of the poset with $1, \ldots, n$ describing the coefficients of $n! \cdot \Omega(P;t)$. This section provides a partial answer to that question.

4.1 Block statistic for fences

A *fence poset* P is a poset whose Hasse diagram is a path. It can be described by a collection of positions $2 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$ on the ground set $\{z_1, \ldots, z_n\}$ such that

$$z_1 \prec z_2 \prec \cdots \prec z_{\alpha_1 - 1} \succ z_{\alpha_1} \prec \cdots \prec z_{\alpha_2 - 1} \succ z_{\alpha_2} \prec \ldots \tag{4.1}$$

The elements z_{α_i} are called the *descent elements*, and the others are called the *ascent elements*. The set of descent (respectively ascent) elements is denoted $\mathrm{Des}(P)$ (respectively $\mathrm{asc}(P)$).

For $1 \leq i < j \leq n$, we define $P_{[i,j]}$ as the subposet of P restricted to the elements z_k with $i \leq k \leq j$. For $2 \leq i \leq n$, we define $P_{/i}$ as the poset obtained by removing the element z_i and the inequality immediately preceding it in (4.1). For an ascent element i, we also define $f(i)$ as the first descent element after i (if there is no such element, set $f(i) := n+1$).

The interpretation of $[t^n]\, n!\, \Omega(P;t) = e(P)$ as counting linear extensions of P, together with the formula for $[t^1]\, \Omega(P;t)$ from [5] and the decomposition into convex subposets in Theorem 12, inspired the following statistic describing the coefficient a_k. See Figure 2 as an example.

Definition 13 (Block fences)**.** Let P be a fence poset and let σ be a labeling of the elements of P. A *decomposition into blocks* of σ is a partition of P into convex subposets (called **blocks**) such that for each subposet P_i:

- The label of the leftmost element of the block is called the **root** of the block,
- For each other element of the block, if the element is an ascent element, then its label is smaller than that of the root,
- For each other element of the block, if the element is a descent element, then its label is greater than that of the root.

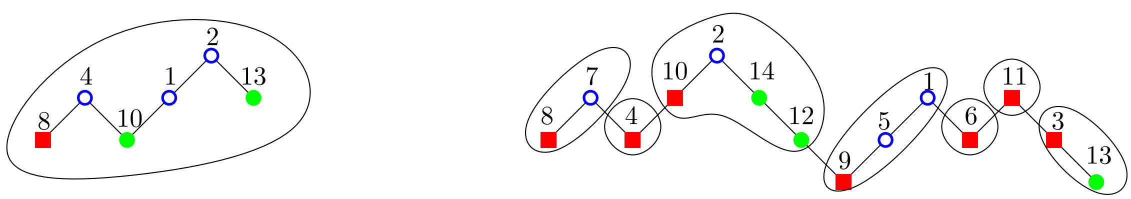

Figure 2: A block and the block decomposition of a labeling σ. Here, $bl(\sigma) = 7$. Roots are represented as red squares, ascents as blue circles and descents as green disks.

Definition 14. Given a fence poset P with a labeling σ, the *block decomposition* of σ is obtained by scanning from left to right and greedily constructing the largest possible leftmost block at each step.

We can now define the polynomial $A(P;t)$, which will turn out to equal $n!\,\Omega(P;t)$.

Definition 15. Let $bl(\sigma)$ denote the number of blocks of σ. We define $A(P;t) := \sum_\sigma t^{bl(\sigma)}$, where the sum ranges over all $n!$ labelings σ of P.

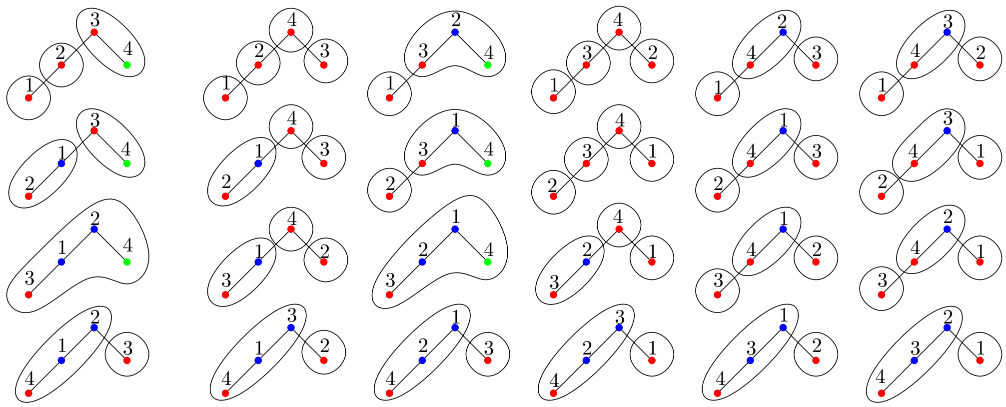

Figure 3: The 4! possible labelings for a fence poset of size 4. Counting the number of blocks for each labeling gives $A(P;t) = 3t^4 + 10t^3 + 9t^2 + 2t$.

By considering the position of the label n in P, note that either n is in $\mathrm{Des}(P)$—in which case it can be erased without changing the number of blocks—or n is in $\mathrm{asc}(P)$, in which case it creates a new block ending at the first encountered element of $\mathrm{Des}(P)$. This gives the following recurrence for $A(P;t)$.

Theorem 16. *Let P be a fence poset. Then,*

$$n \cdot A(P_{[0,n-1]};t) = \sum_{i \in \mathrm{Des}(P)} A(P_{[0,n-1]/i};t) + \sum_{i \in \mathrm{asc}(P)} A(P_{[0,i-1]};t)\, t\, A(P_{[f(i),n-1]};t).$$

Using Corollary **??**, and considering the position of the subposet with the highest label, we can derive a recurrence formula for $\Omega(P;t)$ when P is a fence poset.

Theorem 17. *Let P be a fence poset. Then,*

$$\frac{d}{dt}\Omega(P_{[0,n-1]};t) = \sum_{\substack{0 \le k < \ell \le n-1 \\ k \in \mathrm{asc}(P),\, \ell \in \mathrm{Des}(P)}} \Omega(P_{[0,k-1]};t)\, a_1(P_{[k,\ell-1]};t)\, \Omega(P_{[\ell,n-1]};t).$$

Combining these two theorems and performing careful algebraic manipulations, we obtain our first main result.

Theorem 18. *Let P be a fence poset of size n. Then*

$$n!\,\Omega(P;t) = A(P;t).$$

Remark 19. *If the fence poset has only ascent elements, we recover exactly the Foata correspondence, in which case $[t^k]\, A(P;t)$ counts the number of permutations with k cycles.*

4.2 Conjectured block statistic for skew shapes

We also conjecture a generalization of the statistic bl when P is a skew shape. However, in this case, there is no simple recurrence formula arising from this statistic.

Consider any skew shape λ/μ, represented as a skew Young diagram. The poset $P_{\lambda/\mu}$ is the poset whose elements are the boxes $X(\lambda/\mu) = \{\, (i,j) \mid j \in [\mu_i + 1, \lambda_i] \,\}$, with covering relations given by $(i,j) \succeq (i+1,j) \quad \text{and} \quad (i,j) \succeq (i,j+1)$.

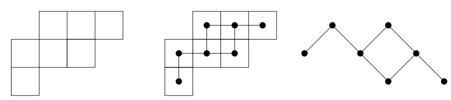

Figure 4: The skew shape 431/1 and its associated skew shape poset.

Definition 20 (Blocks in skew shapes)**.** Let P be a skew shape poset, and let σ be a labeling of this poset. A *block* B is a convex subposet of P satisfying the following conditions (see Fig. 5):

- The **root** x of B is the label of the leftmost element of B,
- Each element in the upper-left part of B has a label smaller than x,
- Each element in the lower-left part of B has a label greater than x.

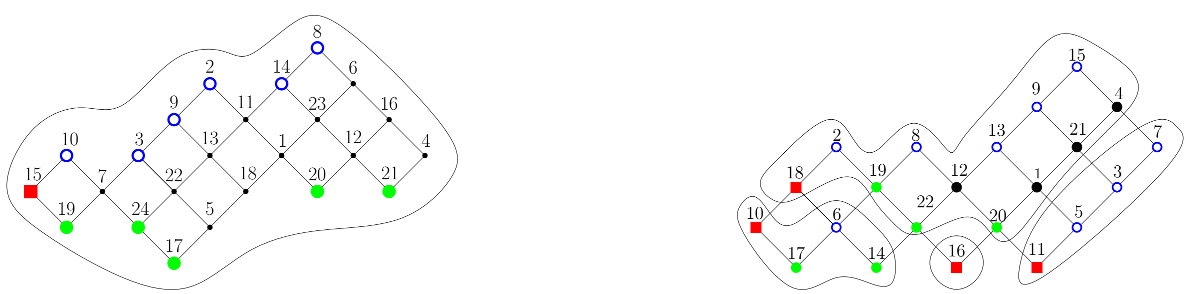

Figure 5: A block and the block decomposition of a labeling σ.In this example, $\tilde{bl}(\sigma) =$ 4. Roots are represented as red circles, upper-left elements as blue circles and lower-left elements as green disks.

As in the fence case, we define the *block decomposition* as the result of iteratively taking the largest possible leftmost block. Although fences have a natural left-to-right order, in skew shapes we can still greedily obtain maximal blocks by starting from the leftmost element of the skew shape poset P. We denote by $\tilde{bl}(\sigma)$ the number of blocks in the block decomposition of σ.

Conjecture 21. *Let P be the cell poset of a skew shape of size n. Then*

$$n!\,\Omega(P;t) = \sum_{\sigma} t^{\tilde{bl}(\sigma)},$$

where the sum is over all $n!$ labelings σ of P.

Remark 22. *The leading term corresponds to the case where each block has size* 1*. In this situation, the labeling must be a linear extension of the poset. Therefore, it is clear that the conjecture holds for the leading term.*

5 Watanabe-Yoshida conjecture via zigzag posets

Now, we present another application of the convex subposet decomposition formula to derive a new formula for the order polynomial of zigzag posets. This allows us to give a new proof of the second part of the Watanabe–Yoshida conjecture.

The zigzag poset Z_n is the fence poset where the descents are all the elements of even indices; i.e., the poset with elements $\{z_1, \ldots, z_n\}$ and relations $z_1 > z_2 < z_3 > z_4 < \ldots$.

The *Euler numbers* can be defined as the number of linear extensions of the zigzag poset: $E_n := e(Z_n)$. These numbers have a well-known exponential generating function: $\sum_{n\geq 0} \frac{E_n}{n!} z^n = \tan(z) + \sec(z)$.

5.1 Hilbert–Kunz multiplicity

The Hilbert–Kunz multiplicity of a local ring is a numerical invariant defined via the iterates of the Frobenius endomorphism.

Definition 23 (Monsky [10])**.** Let $(R, \mathfrak{m})$ be a local ring of positive characteristic $p > 0$. Then the *Hilbert–Kunz multiplicity* of R is defined as the limit

$$e_{HK}(R) := \lim_{e\to\infty} \frac{\lambda(R/\mathfrak{m}^{[p^e]})}{p^{e \dim R}},$$

where $\mathfrak{m}^{[p^e]}$ is the ideal generated by all (p^e)-th powers of elements in $\mathfrak{m}$ and $\lambda(R/\mathfrak{m}^{[p^e]})$ the length of the R-module.

In 2005, Watanabe and Yoshida conjectured the following statement:

Conjecture 24 ([17])**.** *For any prime number p, $e_{HK}(\mathcal{A}_{p,n}) \geq 1 + \dfrac{E_n}{n!}$, with $\mathcal{A}_{p,n} := \mathbb{F}_p[[x_0, \cdots, x_n]]/(x_1^2 + \cdots + x_n^2)$.*

Recently, Pak, Shapiro, Smirnov, and Yoshida [12] found an explicit way to calculate the Hilbert–Kunz multiplicity of $\mathcal{A}_{p,n}$ via the order polynomial of the zigzag poset.

Theorem 25 ([12])**.** *For any odd prime p,*

$$e_{HK}(\mathcal{A}_{p,n}) \geq 1 + \frac{2^n \Omega(Z_n; p/2 - 1/2)}{p^n}. \tag{5.1}$$

Remark 26. *The authors of [12] state the result above with an equality and a denominator $p^n - L_n((p-1)/2)$, where $L_n(t)$ is the Ehrhart polynomial of a variant of $\mathcal{O}(Z_n)$. Since $L_n(t) \geq 0$ for $t \in \mathbb{Z}_{>0}$, it suffices for us to replace it by the inequality above.*

5.2 Shifted order polynomial of zigzag posets

In the zigzag case, using the convex subposet decomposition of Theorem 12, we derive a weighted sum formula where the weights are given as the coefficients of the Taylor expansion at $z = 0$ of specific functions. This allows us to analyze $\Omega(Z_n; t - 1/2)$. The result appears as a weighted sum formula. The next definition indicates how to compute the weight of a decomposition through Table 1.

Definition 27. Define a weight function w of each convex subposet in the following way: for each convex subposet of even size $w(P) = 0$, otherwise for each convex subposet of odd size $2k+1$ we have :

- If P has more upper elements than lower elements, $w(P) = w_-(2k+1)$,
- If P has more lower elements than upper elements, n **is even**, and it is the rightmost subposet of μ, $w(P) = U_{2k+1}$
- $w(P) = w_+(2k+1)$ otherwise

For a decomposition μ of Z_n into convex subposets, we define $w(\mu) = \prod_{P\in\mu} w(P)$.

The k-th coefficients of the shifted order polynomial can be expressed through this weight function.

Theorem 28. *Let $n \geq 2$ and $1 \leq k \leq n$, then:*

$$[t^k]\ \Omega(Z_n; t - 1/2) = \frac{n!}{k!} \cdot E_k \sum_{\mu \in \mathsf{SDecompo}_k(Z_n)} w(\mu). \tag{5.2}$$

Remark 29. *If j is an odd integer, any element of $\mathsf{SDecompo}_{n-j}(Z_n)$ has a subposet of even size. Thus, $[t^{n-j}]\ \Omega(Z_n; t-1/2) = 0$. So, if n is odd or even, $\Omega(Z_n; t-1/2)$ is an odd or even polynomial, respectively.*

We denote $g_{k,n} := \sum_{\mu \in \mathsf{SDecompo}_k(Z_n)} w(\mu)$. These numbers $g_{k,n}$ have the following simple recurrence formula:

Lemma 30. *Let n be a positive integer and $k \geq 3$. Then: $g_{k,n} = \sum_{\substack{i=2 \\ i\ \text{even}}}^{n} C_i \cdot g_{k-2,n-i}$.*

We write $G_n(t) = \sum_{k\geq 0} g_{k,n} t^k$ and $G(t; z) := \sum_{n\geq 0} G_n(t) z^n$. Using formal series manipulation, we can give a closed formula for $G(t; z)$:

Name	Closed formula	Expansion formula	Coefficient formula
$W_+(z)$	$2\arcsin(z/2)/\sqrt{1-z^2/4}$	$W_+(z) = \sum_{k\geq 0} w_+(2k+1)z^{2k+1}$	$w_+(2k+1) = \dfrac{(k!)^2}{(2k+1)!}$
$W_-(z)$	$2\arcsin(z/2)\sqrt{1-z^2/4}$	$W_-(z) = z + \sum_{k\geq 1} w_-(2k+1)z^{2k+1}$	$w_-(2k+1) = \begin{cases} 1 & k=1 \\ \dfrac{-(k!)^2}{(2k)(2k+1)!} & k \geq 1 \end{cases}$
$C(z)$	$4\arcsin^2(z/2)$	$C(z) = \sum_{k\geq 1} C_{2k}z^{2k}$	$C_{2k} = \frac{2((k-1)!)^2}{(2k)!}$
$U(z)$	$2\arcsin(\frac{z}{2})$	$U(z) = \sum_{k\geq 0} U_{2k+1}z^{2k+1}$	$U_{2k+1} = \frac{1}{(2k+1)2^{4k-1}}\binom{2k}{k}$

Table 1: Generating functions relevant to the proof of Proposition 31.

Proposition 31. *We have that*

$$G(t;z) = \sqrt{1-z^2/4} \cdot (1 + 2t\arcsin(z/2)) \,/ (1-4t^2\arcsin(z/2))$$

.

This leads to the following closed formula for the generating function of the shifted order polynomial as a Hadamard product $\odot$:

Theorem 32. *For all t:*

$$\sum_{n\geq 0} \Omega(Z_n; t-1/2)\frac{z^n}{n!} = (\tan(t) + \sec(t)) \odot_t G(t;z). \tag{5.3}$$

This can be approximated using analytic combinatorics to get the following bound.

Corollary 33. *For $n \geq 8$ and $p \geq 3$,*

$$\Omega\Big(Z_n; \frac{p}{2} - \frac{1}{2}\Big) > \frac{E_n}{n!} \cdot \Big(\frac{p}{2}\Big)^n. \tag{5.4}$$

To finish the proof of the second part of the Watanabe–Yoshida conjecture[1], we compute $e_{HK}(\mathcal{A}_{p,n})$ using (5.1) and (5.4). For the case $n < 8$, we check by hand that indeed $e_{HK}(\mathcal{A}_{p,n}) \geq 1 + E_n/n!$. Therefore:

Theorem 34. *For all $p \geq 3$ and $n \geq 2$,* $\quad e_{HK}(\mathcal{A}_{p,n}) \geq 1 + E_n/n!$.

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[1]The original conjecture is for $p \geq 2$ but the case $p = 2$, follows from a calculation (e.g., see [12, p. 2]).

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