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Particle motion methods for Andrews–Gordon and Stanton type identities

Jehanne Dousse * [1], Jihyeug Jang † [1], and Frédéric Jouhet ‡ [2]

[1] *Université de Genève, 7–9, rue Conseil Général, 1205 Genève, Switzerland*
[2] *Univ Lyon, Université Claude Bernard Lyon 1, UMR5208, Institut Camille Jordan, F-69622 Villeurbanne, France*

Abstract. In 2018, Stanton proved generalisations of the celebrated Andrews–Gordon and Bressoud identities (in their q-series version). In this extended abstract, we give combinatorial proofs of these identities. They are based on particle motion introduced by Warnaar and extended by the first and third authors and Konan, and use the original Andrews–Gordon and Bressoud identities as ingredients. We also find new Stanton-type generalisations of classical identities.

Keywords: integer partitions, q-series, partition identities, Andrews–Gordon identities, Bressoud's identities, bijections

1 Introduction

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of a positive integer n is a weakly decreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ such that $\sum_{i=1}^{\ell} \lambda_i = n$. Each λ_i is called a *part* of λ and $\ell(\lambda) = \ell$ is called the *length* of λ. A *frequency sequence* $(f_i)_{i \geq 1}$ is a sequence of non-negative integers. Given a partition, its frequency sequence is defined by setting f_i to be the number of parts of size i for all $i \geq 1$. This yields a one-to-one correspondence between frequency sequences and partitions. The *size* and *length* of a frequency sequence $(f_i)_{i \geq 1}$ are defined as $|f| := \sum_{i \geq 1} i f_i$ and $\ell(f) := \sum_{i \geq 1} f_i$, respectively, which coincide with the size and length of the corresponding partition. Whenever convenient, we identify a partition with its frequency sequence.

Many classical partition identities are elegantly expressed using q-series. We use standard q-series notation which can be found in [10]:

$$(a)_\infty = (a;q)_\infty := \prod_{j \geq 0} (1 - aq^j) \quad \text{and} \quad (a)_k = (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

*jehanne.dousse@unige.ch.
†jihyeug.jang@unige.ch. The first two authors are supported by the SNSF Eccellenza grant PCEFP2 202784.
‡jouhet@math.univ-lyon1.fr.

where $k \in \mathbb{Z}$, and

$$(a_1,\ldots,a_m)_k = (a_1,\ldots,a_m;q)_k := (a_1)_k \cdots (a_m)_k,$$

where k is an integer or infinity.

Gordon [11] proved the following partition identities, reformulated in terms of frequency sequences. Andrews [1] found their analytic analogue, which is now referred to as the Andrews–Gordon identities.

Theorem 1.1 (Andrews–Gordon identities [11, 1]). *Let $k \geq 1$ and $0 \leq r \leq k$ be two integers. We have*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \frac{(q^{2k+3},q^{k+1-r},q^{k+2+r};q^{2k+3})_\infty}{(q)_\infty}. \tag{1.1}$$

Equivalently, the number of frequency sequences $(f_i)_{i\geq 1}$ of n such that

$$f_i + f_{i+1} \leq k \quad \text{for all } i, \quad \text{and} \quad f_1 \leq k-r, \tag{1.2}$$

is equal to the number of partitions of n into parts $\not\equiv 0, \pm(k-r+1) \pmod{2k+3}$.

These identities generalize the celebrated Rogers–Ramanujan identities [13]. The identity (1.1) is also proved in [4] together with a similar formula [4, (3.3)], valid for all integers $k \geq 1$ and $0 \leq j \leq k$:

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+3},q^{k+2-j+2s},q^{k+1+j-2s};q^{2k+3})_\infty}{(q)_\infty}. \tag{1.3}$$

Bressoud proved an even moduli counterpart of the Andrews–Gordon identities, which was recently derived combinatorially from (1.1) in [8].

Theorem 1.2 (Bressoud's identities, [4, (3.4)]). *Let r and k be integers with $k \geq 1$ and $0 \leq r \leq k$. Then*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(q^{2k+2},q^{k+1-r},q^{k+1+r};q^{2k+2})_\infty}{(q)_\infty}. \tag{1.4}$$

Equivalently, the number of frequency sequences $(f_i)_{i\geq 1}$ of n such that $f_i + f_{i+1} \leq k$ for all i, $f_1 \leq k-r$, and whenever $f_i + f_{i+1} = k$ for some i, then $if_i + (i+1)f_{i+1} \equiv k-r \pmod 2$, is equal to the number of partitions of n into parts $\not\equiv 0, \pm(k-r+1) \pmod{2k+2}$.

Bressoud [4, (3.5)] also proved a similar formula, which was recently derived combinatorially from (1.4) in [8]: for integers $k \geq 1$ and $0 \leq j \leq k$,

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2},q^{k+1+j-2s},q^{k+1-j+2s};q^{2k+2})_\infty}{(q)_\infty}. \tag{1.5}$$

Kurşungöz [12] considered partitions with the same frequency conditions as in Bressoud's identity, but with the opposite parity condition $if_i + (i+1)f_{i+1} \equiv k - r + 1 \pmod 2$. He proved a product side formula for their generating function. In [8], a multisum expression was given for the same generating function, and the identity of Kurşungöz was proved combinatorially using this multisum and Bressoud's identity, leading to the following identities:

Theorem 1.3 (Kurşungöz identities, [12] and [8]). *Let r, j, and k be integers with $k \geq 1$, $0 \leq r \leq k$, and $0 \leq j \leq k$. Then*

$$(1+q) \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_{k-r+1}+\cdots+s_k)+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{1}{(q)_\infty}\left((q^{2k+2},q^{k+r},q^{k-r+2};q^{2k+2})_\infty + q(q^{2k+2},q^{k+2+r},q^{k-r};q^{2k+2})_\infty\right), \quad (1.6)$$

and

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2},q^{k-j+2s},q^{k+2+j-2s};q^{2k+2})_\infty}{(q)_\infty}. \quad (1.7)$$

Stanton [14] recently generalised both the Andrews–Gordon identities (1.1) and the related identities (1.3). He also presented an even moduli version, generalising both of Bressoud's identities (1.4) and (1.5).

Theorem 1.4 ([14, Theorems 3.2 and 4.2]). *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+3},q^{k+1-r+j-2s},q^{k+2+r-j+2s};q^{2k+3})_\infty}{(q)_\infty}, \quad (1.8)$$

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2},q^{k+1-r+j-2s},q^{k+1+r-j+2s};q^{2k+2})_\infty}{(q)_\infty}. \quad (1.9)$$

As remarked by Stanton, for $j = 0$, the identity (1.8) (resp. (1.9)) reduces to (1.1) (resp. (1.4)), while for $r = 0$ it yields (1.3) (resp. (1.5)). Stanton proved his results by using some (new properties of) Laurent polynomials $H_{2n}(z, a|q)$ and some iterative process.

We prove a generalisation of Theorem 1.3, in the spirit of Stanton's results.

Theorem 1.5 (Stanton-type generalisation of the Kurşungöz identities)**.** *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$(1+q) \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_k)+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{s=0}^{j} \left(\frac{(q^{2k+2}, q^{k+2-r+j-2s}, q^{k+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty} \right. \tag{1.10}$$
$$\left. + q\, \frac{(q^{2k+2}, q^{k-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty} \right).$$

For $j = 0$, the identity (1.10) gives (1.6), while for $r = 0$ it yields (1.7).

In this extended abstract, we focus on the combinatorial aspects of Stanton's identities. Combinatorial approaches to partition identities are generally difficult. A famous example is the lack of a simple bijection for the combinatorial version of the Rogers–Ramanujan identities (the case $k = 1$ of (1.1)). Several efforts have been made to understand these identities from a combinatorial perspective, including work on bijective proofs (see, e.g., [3, 5, 6, 9]).

It is always simple to interpret the product side of the identities as partitions with congruence conditions. However, in the Andrews–Gordon identities and the other identities considered here, it is not at all obvious combinatorially that the sum side is the generating function for partitions with difference conditions. In the case of Andrews–Gordon, a more natural interpretation has been given by Andrews in terms of Durfee squares [2].

To prove combinatorially that the sum side of the Andrews–Gordon identities is the generating function for partitions with difference conditions, Warnaar [15] introduced a bijection based on particle motions. He used it to derive a finitisation of the Andrews–Gordon identities. It was formulated as a one-dimensional lattice-gas of fermionic particles using slightly different notation. Inspired by Warnaar's work, the authors of [8] generalised his approach by adding parts equal to 0 to the reasoning and introducing an explicit bijection Λ and its inverse Γ, which provided a combinatorial framework for proving several partition identities. Rather than using a finitisation, they took a different approach: by combining the infinite version with the classical Andrews–Gordon and Bressoud identities, they established several identities, among which (1.3), (1.5), (1.6), and (1.7).

Although Stanton proved Theorem 1.4 in [14], his proofs did not provide combinatorial interpretations for the sum sides. After reading [8], he asked the authors whether their techniques could be applied to his identities.

Indeed, the ideas of [15] and [8] can also be used to give combinatorial interpretations and proofs of the identities in Theorems 1.4 and 1.5. We give a suitable interpretation of the sum sides of the identities using particle motion and a combinatorial reasoning on parts equal to 0. Then we give combinatorial proofs of these identities, using this interpretation and the Andrews–Gordon and Bressoud identities (1.1) and (1.4).

Following [8], we allow partitions to have non-negative parts, not just positive ones. That is, a *partition* λ of n is a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ whose sum is n. The difference from the previous definition is that 0 is now allowed as a part, and is taken into account in the length of the partition. Accordingly, the associated frequency sequence becomes $f = (f_0, f_1, \cdots)$, rather than $(f_1, f_2, \cdots)$. With this extension, the authors in [8] provided combinatorial proofs of (1.3), (1.5), and (1.7), and gave combinatorial interpretations in terms of partitions with difference conditions. For example, they proved that the combinatorial model for (1.3) is given by

$$\text{frequency sequences } (f_i)_{i\geq 0} \text{ of } n \text{ such that } f_i + f_{i+1} \leq k \text{ for all } i, \text{ and } f_0 \leq j. \tag{1.11}$$

Following this framework, we use particle motion and Theorems 1.1 and 1.2 to give combinatorial proofs of Theorems 1.4 and 1.5, and we also provide their following partition-theoretic interpretations.

Theorem 1.6. *The left-hand sides of* (1.8), (1.9), *and* (1.10) *are the generating functions for the frequency sequences* $(f_i)_{i\geq 0}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* i, *and*
- $f_0 \leq j - \max\{f_0 + f_1 - (k-r), 0\}$,

with an additional condition distinguishing the three identities: if $f_i + f_{i+1} = k$ *for some* i, *then*

$$\begin{aligned} &\textit{for } (1.8), && \textit{no additional restriction}, \\ &\textit{for } (1.9), && if_i + (i+1)f_{i+1} \equiv k + r - j \pmod 2, \textit{ and} \\ &\textit{for } (1.10), && if_i + (i+1)f_{i+1} \equiv k + r - j + 1 \pmod 2. \end{aligned}$$

The combinatorial interpretation in Theorem 1.6 is new. When $j = 0$, the second condition becomes $f_0 = 0$ and $f_1 \leq k - r$, which yields the interpretation (1.2), and when $r = 0$, it satisfies $f_0 \leq j$, yielding (1.11). Furthermore, the same model extends naturally to the Bressoud (1.9) and Kurşungöz (1.10) cases, which shows its flexibility and confirms that it provides a natural combinatorial interpretation.

This extended abstract contains only the combinatorial part of the preprint [7]. In the full version, the Bailey pair machinery is also used to obtain another family of Stanton-type generalisations involving binomial coefficients. Furthermore, [7] contains not only generalisations of Kurşungöz identities, but also Stanton-type generalisations of the Bressoud–Göllnitz–Gordon identities.

2 Insertion map for multipartitions revisited

In this section we revisit the particle motion bijection, which was first used in [15], and generalised recently in [8], to make it more systematic and easier to apply. Warnaar's approach had a simpler combinatorial description but was less general, while the approach of [8] was more general and gave rise to useful explicit formulas but was less intuitive to understand. Our goal here is to keep the best of both worlds by reformulating the approach of [8] in the style of Warnaar, which leads both to a simple combinatorial description and explicit formulas.

We use the notation of a multipartition, which simply refers to a finite sequence of partitions. For an integer $k \geq 1$, a *k-multipartition* $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ is a tuple of partitions $\lambda^{(i)}$, and each of them may be empty. The *size* $|\boldsymbol{\lambda}|$ of $\boldsymbol{\lambda}$ is defined by $\sum_{i=1}^{k} |\lambda^{(i)}|$, and its *length* $\ell(\boldsymbol{\lambda})$ is $\sum_{i=1}^{k} \ell(\lambda^{(i)})$.

For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$, let $s_1, \dots, s_k$ be integers with $s_1 \geq \cdots \geq s_k \geq 0$ such that the length of $\lambda^{(i)}$ is $s_i - s_{i+1}$ for all i, where we set $s_{k+1} := 0$. We define the *frame sequence* of $\boldsymbol{\lambda}$ as the frequency sequence

$$\mathrm{fs}(\boldsymbol{\lambda}) := (\underbrace{k, 0, \dots, k, 0}_{s_k \text{ pairs}}, \dots, \underbrace{i, 0, \dots, i, 0}_{s_i - s_{i+1} \text{ pairs}}, \dots, \underbrace{1, 0, \dots, 1, 0}_{s_1 - s_2 \text{ pairs}}, 0, \dots).$$

For example, let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}) = ((3,1), \emptyset, (6,6,5,3), (19,0))$ be a 4-multipartition. Then

$$\mathrm{fs}(\boldsymbol{\lambda}) = (4,0,4,0,3,0,3,0,3,0,3,0,1,0,1,0,0,\dots).$$

Definition 2.1. For a non-negative integer k, let $\mathcal{P}_k$ denote the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is its frame sequence. Let $\mathcal{A}_k$ denote the set of all frequency sequences $(f_i)_{i \geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$.

In this section, we introduce an *insertion map* Λ for multipartitions, which defines a size-preserving bijection between the sets $\mathcal{P}_k$ and $\mathcal{A}_k$, where the size of an element $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{P}_k$ is defined as $|(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))| := |\boldsymbol{\lambda}| + |\mathrm{fs}(\boldsymbol{\lambda})|$. Although $\mathrm{fs}(\boldsymbol{\lambda})$ is uniquely determined by $\boldsymbol{\lambda}$, we regard Λ as a map from $\mathcal{P}_k$ to $\mathcal{A}_k$, rather than directly from the set of multipartitions $\boldsymbol{\lambda}$, in order to emphasize that it preserves the size.

Particle motion

Let $f = (f_0, f_1, \dots)$ be a frequency sequence. Suppose that u is a non-negative integer such that there exists $h \geq 1$ with $f_u + f_{u+1} = h$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. We now describe the procedure for applying m particle motions in f, starting from the pair

(f_u, f_{u+1}). Consider the pair (f_u, f_{u+1}). If the local condition $f_{u+1} + f_{u+2} < h$ is satisfied, we perform the following (single) *particle motion*:

$$(f_u, f_{u+1}) \mapsto (f_u - 1, f_{u+1} + 1).$$

As long as the local condition remains satisfied for the current pair (f_u, f_{u+1}), we continue to apply this particle motion repeatedly at the current pair. Once the local condition is no longer satisfied, that is, if $f_{u+1} + f_{u+2} = h$, then we increment u by 1, shift our focus to the next pair (f_{u+1}, f_{u+2}). This focus shifting is not counted as a particle motion. We then repeat the same procedure, and the process continues until exactly m particle motions are performed.

Definition 2.2. Let $f = (f_0, f_1, \dots)$ be a frequency sequence and m be a non-negative integer. Suppose that u is a non-negative integer such that there exists $h \geq 1$ with $f_u + f_{u+1} = h$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. We define $\mathsf{pm}_u^{(m)}(f)$ to be the resulting frequency sequence by applying m particle motions starting from (f_u, f_{u+1}) in f. Moreover, if $\mathsf{pm}_u^{(m)}(f) = (\overline{f}_0, \overline{f}_1, \dots)$ and the final focus is on the pair $(\overline{f}_v, \overline{f}_{v+1})$, then we say that the pair (f_u, f_{u+1}) moves to $(\overline{f}_v, \overline{f}_{v+1})$.

In [8], the authors describe the frequency sequence obtained by iteratively applying the particle motion. The resulting frequency sequence can be computed directly in a single step, without performing each individual particle motion. In the next proposition, we show that applying m particle motions as described above gives exactly the same frequency sequence.

Proposition 2.3. *Let $f = (f_0, f_1, \dots)$ be a frequency sequence. Suppose that u is a non-negative integer such that there exists $h \geq 1$ with $(f_u, f_{u+1}) = (h, 0)$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. For a non-negative integer m, the frequency sequence $(\overline{f}_i)_{i \geq 0} = \mathsf{pm}_u^{(m)}(f)$ is given explicitly by:*

$$\overline{f}_i = \begin{cases} f_i & \text{if } 0 \leq i < u, \\ f_{i+2} & \text{if } u \leq i < v-2, \\ f_v + \sum_{j=u+2}^{v} (h - (f_{j-1} + f_j)) - m & \text{if } i = v-2, \\ f_{v-1} + m - \sum_{j=u+2}^{v-1} (h - (f_{j-1} + f_j)) & \text{if } i = v-1, \\ f_i & \text{if } i \geq v, \end{cases} \tag{2.1}$$

and (f_u, f_{u+1}) moves to $(\overline{f}_{v-2}, \overline{f}_{v-1})$, where

$$v := \min \left\{ t \geq u+2 : \sum_{i=u+2}^{t} (h - (f_{i-1} + f_i)) \geq m \right\}. \tag{2.2}$$

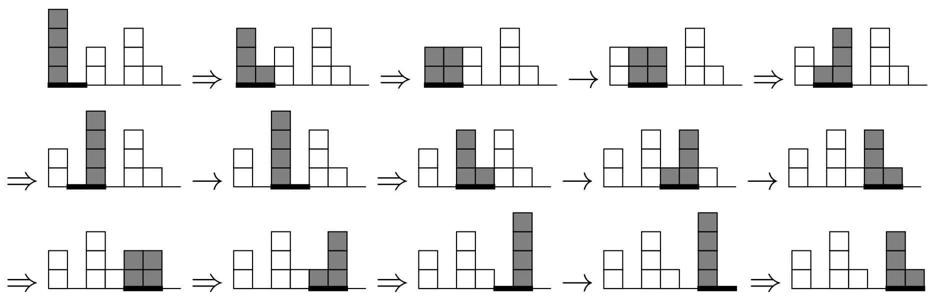

Figure 1: Illustration of applying 9 particle motions starting from (f_0, f_1) in the frequency sequence $f = (4, 0, 2, 0, 3, 1, 0, 0, \cdots)$. The symbol $\Rightarrow$ indicates a single particle motion, and $\rightarrow$ indicates a focus shift.

Example 2.4. A frequency sequence $(f_i)_{i\geq 0}$ is represented by placing boxes above the x-axis. Starting from the 0th column, the number of boxes in the ith column corresponds to f_i. The current focus is indicated by shading the corresponding boxes in gray and marking the corresponding position on the x-axis with a bold line.

Let $f = (4, 0, 2, 0, 3, 1, 0, 0, \ldots)$ be a frequency sequence. In Figure 1, we obtain $\overline{f} = \mathsf{pm}_0^{(9)}(f)$ where

$$\overline{f} = (2, 0, 3, 1, 0, 3, 1, 0, \ldots).$$

Here, $(f_0, f_1) = (4, 0)$ moves to $(\overline{f}_5, \overline{f}_6) = (3, 1)$ after 9 particle motions. The same result follows directly from the explicit formula in Proposition 2.3.

The map Λ

Now we reformulate the explicit map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ defined in [8] in terms of particle motions. Recall that to a k-multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$, we associate a sequence $(s_1, \ldots, s_k)$ of non-negative integers such that $s_1 \geq \cdots \geq s_k \geq 0$, and for each i, $\lambda^{(i)}$ has length $s_i - s_{i+1}$, where we set $s_{k+1} := 0$. Recall that the frame sequence $\mathrm{fs}(\boldsymbol{\lambda})$ consists of $s_i - s_{i+1}$ pairs equal to $(i, 0)$ for $i = 1, \ldots, k$. Let $\lambda = (\lambda_0, \ldots, \lambda_{s_1-1})$ be the sequence of non-negative integers defined by

$$(\lambda_{s_1-1}, \lambda_{s_1-2}, \ldots, \lambda_1, \lambda_0) = (\lambda_1^{(1)}, \ldots, \lambda_{s_1-s_2}^{(1)}, \ldots, \lambda_1^{(k)}, \ldots, \lambda_{s_k}^{(k)}). \tag{2.3}$$

Since $\boldsymbol{\lambda}$ can be obtained directly from $\mathrm{fs}(\boldsymbol{\lambda})$ and λ, we regard the pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ and $(\lambda, \mathrm{fs}(\boldsymbol{\lambda}))$ as essentially the same.

We define the map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ on a pair $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ to be the frequency sequence

$$\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) := \left(\mathsf{pm}_0^{(\lambda_0)} \circ \mathsf{pm}_2^{(\lambda_1)} \circ \cdots \circ \mathsf{pm}_{2(s_1-2)}^{(\lambda_{s_1-2})} \circ \mathsf{pm}_{2(s_1-1)}^{(\lambda_{s_1-1})}\right)(\mathrm{fs}(\boldsymbol{\lambda}))\,.$$

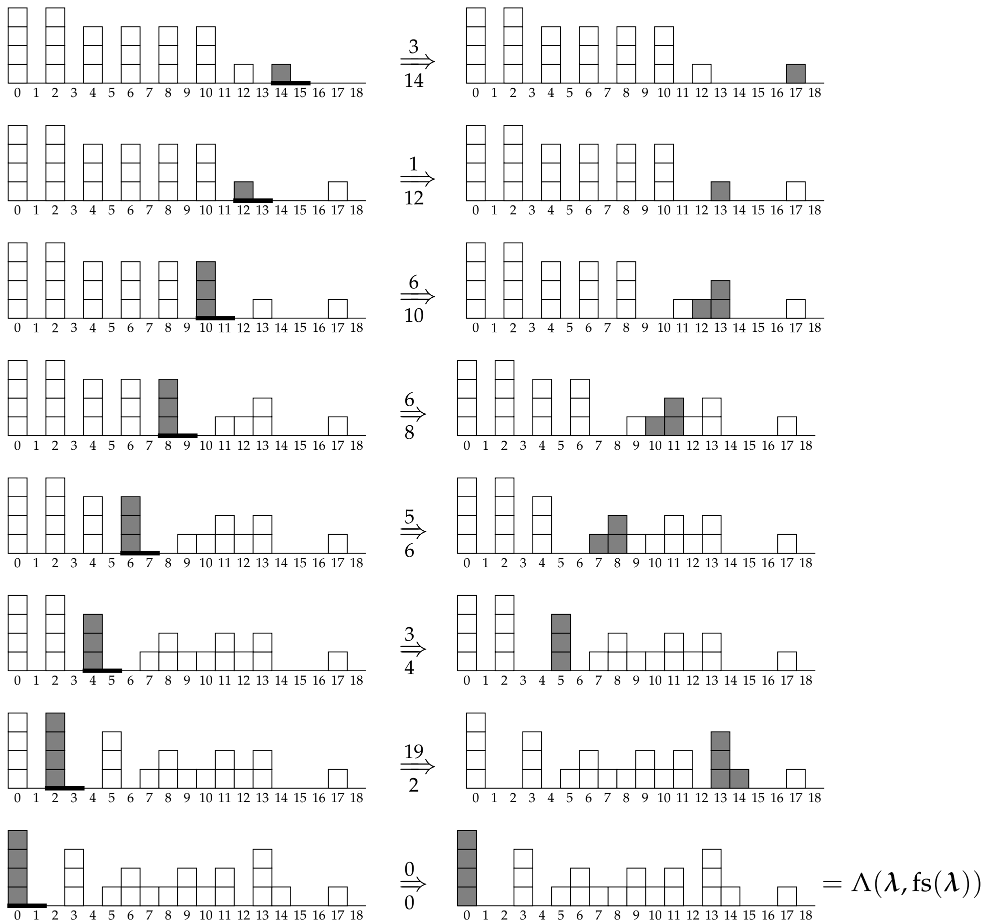

Figure 2: The process of applying the map Λ to $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ where $\boldsymbol{\lambda} = ((3,1), \emptyset, (6,6,5,3), (19,0))$. Here, the notation $f \overset{m}{\underset{u}{\Longrightarrow}} \overline{f}$ indicates that $\overline{f} = \mathsf{pm}_u^{(m)}(f)$.

Example 2.5. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}) = ((3,1), \emptyset, (6,6,5,3), (19,0))$. Figure 2 shows the process of applying the map Λ to $(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$. Hence, we have

$$\begin{aligned}\Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda}) &= \left(\mathsf{pm}_0^{(0)} \circ \mathsf{pm}_2^{(19)} \circ \mathsf{pm}_4^{(3)} \circ \mathsf{pm}_6^{(5)} \circ \mathsf{pm}_8^{(6)} \circ \mathsf{pm}_{10}^{(6)} \circ \mathsf{pm}_{12}^{(1)} \circ \mathsf{pm}_{14}^{(3)}\right)(\mathrm{fs}(\boldsymbol{\lambda})) \\ &= (4,0,0,3,0,1,2,1,1,2,1,2,0,3,1,0,0,1,0,\dots).\end{aligned}$$

The inverse map Γ of Λ is constructed in our preprint [7]; interested readers may refer to Section 5 for details. We conclude this section by stating the bijectivity of Λ. This was proved in [8], but the proof there was quite tedious. With the new formulation, the proof is only a few lines in [7].

Theorem 2.6 ([8])**.** *The map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ is a size-preserving bijection, with inverse map Γ.*

3 A combinatorial proof of (1.8)

Our strategy is as follows. We first define a subset $\mathcal{X}_{j,r,k} \subseteq \mathcal{P}_k$ whose generating function corresponds to the sum side of (1.8). On the other hand, we define a subset $\mathcal{Y}_{j,r,k} \subseteq \mathcal{A}_k$ whose generating function corresponds to the product side of (1.8), using the Andrews–Gordon identities. However, the set $\mathcal{Y}_{j,r,k}$ is an artifact introduced solely to match the desired generating function. To connect $\mathcal{X}_{j,r,k}$ and $\mathcal{Y}_{j,r,k}$, we define a new subset $\mathcal{Z}_{j,r,k} \subseteq \mathcal{A}_k$, and show that there exists a size-preserving bijection between $\mathcal{Y}_{j,r,k}$ and $\mathcal{Z}_{j,r,k}$. We then prove that the map Λ gives a bijection between $\mathcal{X}_{j,r,k}$ and $\mathcal{Z}_{j,r,k}$. Combining these results gives a partition interpretation and a combinatorial proof of (1.8).

Definition 3.1. Let j, r, and k be non-negative integers with $j + r \leq k$. Define $\mathcal{X}_{j,r,k}$ to be the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is the frame sequence corresponding to $\boldsymbol{\lambda}$, subject to the condition that each part of $\lambda^{(m)}$ is at least $m - j + \max\{m - (k - r), 0\}$, for all $m = 1, \ldots, k$.

We give a combinatorial model for the left-hand side of the equation in (1.8). We use the following facts. A simple calculation (as shown in [8, (2.15)]) shows that the weight of $\mathrm{fs}(\boldsymbol{\lambda})$ is

$$|\,\mathrm{fs}(\boldsymbol{\lambda})| = s_1^2 + \cdots + s_k^2 - (s_1 + \cdots + s_k), \tag{3.1}$$

where $s_1, \ldots, s_k$ are the integers associated with $\boldsymbol{\lambda}$. Let $P_{\ell,d}(n)$ be the number of partitions of n of length ℓ into parts at least d. Then

$$\sum_{n \geq 0} P_{\ell,d}(n) q^n = \frac{q^{d\ell}}{(q)_\ell}. \tag{3.2}$$

Using (3.1) and (3.2), we obtain a combinatorial model for the left-hand side of (1.8).

Proposition 3.2. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$\sum_{(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k}} q^{|(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - (s_1 + \cdots + s_j) + (s_{k-r+1} + \cdots + s_k)}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q)_{s_k}}.$$

Definition 3.3. For non-negative integers j, r, and k with $j + r \leq k$, we define $\mathcal{Y}_{j,r,k}$ to be the set of all frequency sequences $(f_i)_{i \geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$, subject to the condition that

$$f_0 \in \{\ell + \max\{\ell - (j - r), 0\} : 0 \leq \ell \leq j\}.$$

Using Theorem 1.1, we give a combinatorial model for the right-hand side of (1.8).

Proposition 3.4. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$\sum_{f \in \mathcal{Y}_{j,r,k}} q^{|f|} = \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

We now define the set $\mathcal{Z}_{j,r,k}$, which serves as the link between $\mathcal{X}_{j,r,k}$ and $\mathcal{Y}_{j,r,k}$, and which provides the main combinatorial model for (1.8).

Definition 3.5. For non-negative integers j, r, and k with $j + r \leq k$, we define $\mathcal{Z}_{j,r,k}$ to be the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$, subject to the condition that

$$f_0 \leq j - \max\{f_0 + f_1 - (k - r), 0\}.$$

Proposition 3.6. *There exists a size-preserving bijection from $\mathcal{Y}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$.*

The bijection proving the above proposition simply changes the 0th frequency while keeping all other entries fixed; this modification is reversible and preserves size. Details are provided in [7].

Proposition 3.7. *The map Λ is a size-preserving bijection from $\mathcal{X}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$.*

It is shown in Theorem 2.6 that the map Λ is size-preserving and has an inverse map Γ. To complete the proof, it suffices to show that $\Lambda(\mathcal{X}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$ and $\Gamma(\mathcal{Z}_{j,r,k}) \subseteq \mathcal{X}_{j,r,k}$, which we proved in Lemmas 3.8 and 3.9 of [7].

4 A combinatorial proof of (1.9) and (1.10)

We briefly describe how the identities (1.9) and (1.10) are obtained, leaving out the full details. The combinatorial proof of (1.9) follows the same approach as that of (1.8), but with Bressoud's identity in place of Andrews–Gordon's identity as key ingredient. The sum and product sides are obtained as the generating functions of new sets $\mathcal{X}'_{j,r,k}$ and $\mathcal{Y}'_{j,r,k}$, respectively. We then define a new set $\mathcal{Z}'_{j,r,k}$, and give bijections between $\mathcal{X}'_{j,r,k}$ and $\mathcal{Z}'_{j,r,k}$, and between $\mathcal{Y}'_{j,r,k}$ and $\mathcal{Z}'_{j,r,k}$. However, the case of the Kurşungöz type identities (1.10) is somewhat different. One can define $\widetilde{\mathcal{X}}'_{j,r,k}$, $\widetilde{\mathcal{Y}}'_{j,r,k}$, and $\widetilde{\mathcal{Z}}'_{j,r,k}$ each satisfying the opposite parity condition, as analogues of $\mathcal{X}'_{j,r,k}$, $\mathcal{Y}'_{j,r,k}$, and $\mathcal{Z}'_{j,r,k}$. While the sum side arises as the generating function of $\widetilde{\mathcal{X}}'_{j,r,k}$, the product side does not correspond to the generating function of $\widetilde{\mathcal{Y}}'_{j,r,k}$. Unlike in the previous cases, $\widetilde{\mathcal{Z}}'_{j,r,k}$ is not in bijection with $\widetilde{\mathcal{Y}}'_{j,r,k}$. However, by using its relation to $\mathcal{Z}'_{j,r,k}$, we can determine the generating function for $\widetilde{\mathcal{Z}}'_{j,r,k}$, which in turn yields (1.10).

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